Comment.Math.Univ.Carolin. 54,3 (2013) 407–428

Some remarks on distributive semilattices

Sergio Celani, Ismael Calomino

Abstract. In this paper we shall give a survey of the most important characterizations of the notion of distributivity in semilattices with greatest element and we will present some new ones through annihilators and relative maximal filters. We shall also simplify the topological representation for distributive semilattices given in Celani S.A., *Topological representation of distributive semilattices*, Sci. Math. Japonicae online **8** (2003), 41–51, and show that the meet-relations are closed under composition. So, we obtain that the DS-spaces with meet-relations is a category dual to the category of distributive semilattices with homomorphisms. These results complete the topological representation presented in Celani S.A., *Topological representation of distributive semilattices*, Sci. Math. Japonicae online **8** (2003), 41–51, without the use of ordered topological spaces. Finally, following the work of G. Bezhanishvili and R. Jansana in *Generalized Priestley quasi-orders*, Order **28** (2011), 201–220, we will prove a characterization of homomorphic images of a distributive semilattice A by means of family of closed subsets of the dual space endowed with a lower Vietoris topology.

Keywords: distributive semilattices; topological representation; meet-relations

Classification: Primary 03G10, 06A12; Secondary 06D50

1. Introduction

It is well know that the notion of distributivity in a lattice can be characterized in several equivalent ways. In semilattices we have a different situation, because there are already several different notions of distributivity. For example, a meet-semilattice $\langle A, \wedge \rangle$ is called *weakly distributive* if whenever $a_1 \vee \cdots \vee a_n$ exists in A then $(b \wedge a_1) \vee \cdots \vee (b \wedge a_n)$ exists and $(b \wedge a_1) \vee \cdots \vee (b \wedge a_n) = b \wedge (a_1 \vee \cdots \vee a_n)$. This class of semilattices was introduced by R. Balbes in [1] with the name of *prime semilattices*, and were intensively studied in [6]. The distributivity implies the 0-*distributivity*, which is a concept that was introduced and studied by J. Varlet in [14]. Another interesting class is the class of *mildly distributive semilattices* introduced and studied by E.C. Hickman in [9]. A meet-semilattice $\langle A, \wedge \rangle$ is mildly distributive if the set of all strong ideals is a distributive lattice. Perhaps, the most studied class is the class of *distributive semilattices*. A meet-semilattice $\langle A, \wedge \rangle$ is distributive if for all $a, b, c \in A$ with $a \wedge b \leq c$ there exist $a_1, b_1 \in A$ such that $a \leq a_1$, $b \leq b_1$ and $c = a_1 \wedge b_1$. This class of semilattices was studied by several authors in [5], [7], [12], [15] and [17].

This research was supported by the CONICET under grant PIP 112-201101-00636.

In the study of distributive lattices, their topological representation plays an important role. The first to develop a topological representation for bounded distributive lattices was M. Stone in [13]. In this paper Stone establishes that the category of bounded distributive lattices with homomorphisms is dually equivalent to the category of spectral spaces with continuous maps. Later, H. Priestley in [11] proves that there is a duality between certain ordered topological spaces, called *Priestley spaces*, and bounded distributive lattices. Through both versions we can have a duality for Boolean algebras. In [7] George Grätzer develops a topological representation for distributive semilattices and extends the representation given by Stone. On the other hand, a full duality between distributive meet-semilattices with greatest element and certain ordered topological spaces was developed in [4]. The main novelty of [4] was the characterization of meet-semilattice homomorphisms preserving top by means of certain binary relations defined between certain spectral spaces. Recently in [2], G. Bezhanishvili and R. Jansana develop a "Priestley-like" duality for the category of bounded distributive meet-semilattices and bounded meet-semilattice homomorphisms.

This paper has two main objectives. First, we shall survey all known characterizations of distributive semilattices and we will present some new ones. The second one is to complete and simplify the duality given in [4]. Also we will show that the homomorphic images of a distributive semilattice A can be characterized in terms of families of closed sets of the DS-space $\langle X(A), \tau \rangle$ endowed with a *lower Vietoris topology*. These results are motivated by similar results given in [3] on the dual characterization of homomorphic images of a bounded distributive meet-semilattice by means of Vietoris families.

The paper is organized as follows. In Section 2 we shall provide all the needed information to make the paper self-contained. In Section 3 we will present the most important characterizations of distributive semilattices and we will develop new characterizations through annihilators and relative maximal filters. In Section 4 we shall give a simplification of the topological representation developed in [4] by means of sober spaces. In Section 5 we shall study the structure of the lattice of filters of a distributive meet-semilattice through the dual space. In Section 6 is proved that the composition of meet-relations is a meet-relation and we will show that the class of DS-spaces with meet-relations form a category. Finally, in Section 7 we will study the dual characterization of homomorphic images by means of a family of closed subsets of a DS-space endowed with a lower Vitories topology.

2. Preliminaries

In this section we will give some necessary notations and definitions. Let us consider the poset $\langle X, \leq \rangle$. A subset $U \subseteq X$ is said to be *increasing* (*decreasing*) if for all $x \in X$ such that $x \in U$ ($y \in U$) and $x \leq y$, we have $y \in U$ ($x \in U$). The set of all subsets of X is denoted by $\mathcal{P}(X)$ and the set of all increasing subsets of X is denoted by $\mathcal{P}_i(X)$. A subset $K \subseteq \mathcal{P}(X)$ is called *dually directed* if for any $U, V \in K$ there exists $W \in K$ such that $W \subseteq U \cap V$. The set complement of subset $Y \subseteq X$ will be denoted by Y^c or $X - Y$. For each $Y \subseteq X$, the increasing (decreasing) set generated by Y is $[Y) = \{x \in X : \exists y \in Y : y \leq x\}$ ($(Y] = \{x \in X : \exists y \in Y : x \leq y\}$). If $Y = \{y\}$, then we will write $[y)$ and $(y]$ instead of $[\{y\})$ and $(\{y\}]$, respectively.

Let us recall that a *meet-semilattice with greatest element* is an algebra $\langle A, \wedge, 1\rangle$ of type $(2, 0)$ such that the operation $\wedge$ is idempotent, commutative, associative and $a \wedge 1 = a$ for all $a \in A$. As usual, the binary relations $\leq$ defined by $a \leq b$ if and only if $a \wedge b = a$ is a partial order. In what follows we will say *semilattice* instead of meet-semilattice with greatest element. A *bounded* semilattice is an algebra $\langle A, \wedge, 0, 1\rangle$ of type $(2, 0, 0)$ such that $\langle A, \wedge, 1\rangle$ is a semilattice and $a \wedge 0 = 0$ for all $a \in A$.

A *filter* of a semilattice A is a subset $F \subseteq A$ such that $1 \in F$, if $a \leq b$ and $a \in F$, then $b \in F$ and if $a, b \in F$, then $a \wedge b \in F$. The *filter generated by a subset* $H \subseteq A$, in symbols $F(H)$, is the set $F(H) = \{x \in A : \exists\{h_0, \ldots, h_n\} \subseteq H$ and $h_0 \wedge \cdots \wedge h_n \leq x\}$. A filter F is said to be *finitely generated* if $F = F(H)$ for some finite non-empty subset H of A. Note that if $H = \{a\}$ then $F(\{a\}) = [a)$. We will denote by $\mathrm{Fi}(A)$ and $\mathrm{Fi}_f(A)$ the set of all filters and finitely generated filters of A, respectively.

Theorem 1. *Let A be a semilattice. Then $\mathrm{Fi}(A)$ is a lattice if and only if any pair of elements of A has an upper bound in common.*

A proper filter P of A is *irreducible* if for all $F_1, F_2 \in \mathrm{Fi}(A)$ such that $P = F_1 \cap F_2$, then $P = F_1$ or $P = F_2$. The set of all irreducible filters of A will be denoted by $X(A)$. A subset I of A is called an *order-ideal* of A if I is decreasing and for all $a, b \in I$ there exists an element $c \in I$ such that $a \leq c$ and $b \leq c$. A proper filter F of A is *weakly irreducible* if $I = F^c = \{a \in A : a \notin F\}$ is an order-ideal. We note that in all semilattices, every weakly irreducible filter is an irreducible filter. We will denote by $X_\omega(A)$ and $\mathrm{Id}(A)$ the set of all weakly irreducible filters and proper order-ideals of A, respectively. Finally, we will say that a proper filter M of A is *maximal* if for any $F \in \mathrm{Fi}(A)$ such that $M \subseteq F$, we have $F = M$ or $F = A$.

The following result, analogue of the Prime Filter theorem, was proved in [4] for semilattices in general.

Theorem 2. *Let A be a semilattice. Let $F \in \mathrm{Fi}(A)$ and $I \in \mathrm{Id}(A)$ such that $F \cap I = \emptyset$. Then there exists $P \in X(A)$ such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Corollary 3. *Let A be a semilattice. Then every proper filter is the intersection of irreducible filters.*

Lemma 4. *Let A be a semilattice and let $F \in \mathrm{Fi}(A)$. Then F is irreducible if and only if for every $a, b \notin F$ there exists $c \notin F$ and $f \in F$ such that $a \wedge f \leq c$ and $b \wedge f \leq c$.*

Lemma 5. *Let A be a semilattice and let $F \in \mathrm{Fi}(A)$. Then F is weakly irreducible if and only if for all $F_1, F_2 \in \mathrm{Fi}(A)$ such that $F_1 \cap F_2 \subseteq F$, then $F_1 \subseteq F$ or $F_2 \subseteq F$.*

3. Distributive semilattices

In this section we present several characterizations of distributive semilattices.

Definition 6. A semilattice A is *distributive* if for all $a, b, c \in A$ such that $a \wedge b \leq c$ there exist $a_1, b_1 \in A$ such that $a \leq a_1$, $b \leq b_1$ and $c = a_1 \wedge b_1$.

A lattice is distributive if and only if it is distributive as a semilattice (see [7] or [5]). We will denote by $\mathcal{DS}$ the class of distributive semilattices.

Lemma 7. *Let $A \in \mathcal{DS}$. Thus,* $\mathrm{Fi}(A)$ *is a lattice.*

Proof: Since $1 \in A$, then any pair of elements of A has an upper bound in commmon. Thus, $\mathrm{Fi}(A)$ is a lattice. □

In the following theorem we collect different results obtained by different authors that characterize the distributivity in a semilattice.

Theorem 8. *Let A be a semilattice. Then the following conditions are equivalent.*

(1) *A is distributive.*
(2) *The set* $\mathrm{Fi}(A)$*, considered as a lattice, is distributive.*
(3) $X(A) = X_\omega(A)$.
(4) *Let $F \in \mathrm{Fi}(A)$ and $I \in \mathrm{Id}(A)$ such that $F \cap I = \emptyset$. Then there exists $P \in X_\omega(A)$ such that $F \subseteq P$ and $P \cap I = \emptyset$.*

The equivalence between the condition (1) and (4) of Theorem 8 was given by J. Varlet in [15]. This result provides a characterization of distributivity of a semilattice through a separation property and generalize the Stone's theorem for distributive lattices. Later, the equivalence between (1) and (2) was proved by G. Grätzer in [7]. Finally, the equivalence of the conditions (1) and (3) was proved by S. Celani in [4]. Compared with the theory of lattices, this equivalence is similar to the well-known result which states that a lattice is distributive if and only if every irreducible filter is prime.

Now, we will focus on the notion of *annihilator*. Let A be a semilattice. For $a, b \in A$, the *annihilator $\langle a, b\rangle$ of a relative to b* is defined by

$$\langle a, b\rangle = \{x \in A : x \wedge a \leq b\}.$$

In [10] Mandelker studied the properties of relative annihilator and characterized distributive lattices in terms of their relative annihilators. Later, Varlet in [16] gave a similar characterization for distributive semilattices. Here we present a slight generalization of the Varlet's characterization.

Let $A \in \mathcal{DS}$ and $X, Y \subseteq A$. We denote by $\langle X, Y\rangle$ the set

$$\langle X, Y\rangle = \bigcup\{\langle a, b\rangle : (a, b) \in X \times Y\}.$$

Remark 9. Note that $\langle [a), (b]\rangle = \langle a, b\rangle$ for all $a, b \in A$.

The following theorem extends the results given in [16] adding the condition (3).

Theorem 10. *Let A be a semilattice. Then the following conditions are equivalent.*

(1) *A is distributive.*
(2) *The set $\langle a, b\rangle \in \mathrm{Id}(A)$, for all $a, b \in A$.*
(3) *The set $\langle F, I\rangle \in \mathrm{Id}(A)$, for all $F \in \mathrm{Fi}(A)$ and $I \in \mathrm{Id}(A)$.*

Proof: (1)⇒(2) Let $a, b \in A$. It is easy to prove that $\langle a, b\rangle$ is decreasing. Let $x, y \in \langle a, b\rangle$, then $x \wedge a \leq b$ and $y \wedge a \leq b$. By hypothesis, there exist $x_1, a_1 \in A$ such that $x \leq x_1, a \leq a_1$ and $b = x_1 \wedge a_1$. In particular, we have $b \leq x_1$. Then $y \wedge a \leq x_1$, and again by hypothesis there exist $y_1, a_2 \in A$ such that $y \leq y_1, a \leq a_2$ and $x_1 = y_1 \wedge a_2$. Later $x \leq y_1, y \leq y_1$ and

$$y_1 \wedge a \leq (y_1 \wedge a_2) \wedge a_1 = x_1 \wedge a_1 = b$$

i.e., $y_1 \in \langle a, b\rangle$. Therefore $\langle a, b\rangle \in \mathrm{Id}(A)$.

(2)⇒(3) Let $F \in \mathrm{Fi}(A)$ and $I \in \mathrm{Id}(A)$. We prove that $\langle F, I\rangle$ is an order-ideal. It is clear that it is decreasing. Let $a, b \in \langle F, I\rangle$. Then there exist $(f_1, i_1), (f_2, i_2) \in F \times I$ such that $a \in \langle f_1, i_1\rangle$ and $b \in \langle f_2, i_2\rangle$, i.e., $a \wedge f_1 \leq i_1$ and $b \wedge f_2 \leq i_2$. Since F is a filter, then $f = f_1 \wedge f_2 \in F$ and as I is an order-ideal there exists $y \in I$ such that $i_1 \leq y$ and $i_2 \leq y$. Then $a \wedge f \leq y$ and $b \wedge f \leq y$. So, $a, b \in \langle f, y\rangle$ and by hypothesis there exists $c \in \langle f, y\rangle$ such that $a \leq c$ and $b \leq c$, where $\langle f, y\rangle \subseteq \langle F, I\rangle$. Thus, $\langle F, I\rangle$ is an order-ideal.

(3)⇒(1) Let $a, b, c \in A$ such that $a \wedge b \leq c$. Since $\langle a, c\rangle = \langle [a), (c]\rangle$ and $b, c \in \langle a, c\rangle$ by hypothesis we have there exists $b_1 \in \langle a, c\rangle$ such that $c \leq b_1, b \leq b_1$. Then $b_1 \wedge a \leq c$ and $c, a \in \langle b_1, c\rangle$. Again, as $\langle b_1, c\rangle$ is an order-ideal, there exists $a_1 \in \langle b_1, c\rangle$ such that $c \leq a_1, a \leq a_1$ and $a_1 \wedge b_1 \leq c$. But $c \leq b_1$ and $c \leq a_1$, so $c \leq a_1 \wedge b_1$. Therefore $c = a_1 \wedge b_1$ and A is distributive. □

Now we present a new characterization of distributive semilattices in terms of the notion of relative maximal filter with respect to a set.

Definition 11. Let A be a semilattice and let S be a subset of A closed under meet. A filter F of A shall be called a *relative maximal filter with respect to S*, when F is maximal among filters which are disjoint to S.

If $S = (a]$, for some $a \in A$, then F is called *relative maximal filter with respect to a*.

Lemma 12. *Let A be a semilattice. Let $F \in \mathrm{Fi}(A)$ and $I \in \mathrm{Id}(A)$. Then F is relative maximal filter with respect to I if and only if $\langle H, I\rangle \cap F \neq \emptyset$, for all $H \in \mathrm{Fi}(A)$ such that $H \nsubseteq F$.*

Proof: Suppose that F is relative maximal filter with respect to I. Let $H \in \mathrm{Fi}(A)$ such that $H \nsubseteq F$. Consider $F \vee H$. Since F is relative maximal filter with respect to I and $F \subseteq F \vee H$ then $(F \vee H) \cap I \neq \emptyset$, i.e., there exist $i \in I, f \in F$ and $h \in H$ such that $f \wedge h \leq i$. Moreover, we have $f \in \langle h, i\rangle \subseteq \langle H, I\rangle$ and $\langle H, I\rangle \cap F \neq \emptyset$.

Assume that $\langle H, I\rangle \cap F \neq \emptyset$ for all $H \in \mathrm{Fi}(A)$ such that $H \nsubseteq F$. Suppose that F is not relative maximal filter with respect to I. Then there exists $H \in \mathrm{Fi}(A)$ such that $F \subset H$ and $H \cap I = \emptyset$. Since $H \nsubseteq F$, by hypothesis we get $\langle H, I\rangle \cap F \neq \emptyset$. So, there exist $f \in F$ and $(h, i) \in H \times I$ such that $f \in \langle h, i\rangle$, i.e., $f \wedge h \leq i$. But as $f \wedge h \in H$, we get $i \in H$ which is a contradiction. □

Theorem 13. *Let A be a semilattice. Then the following conditions are equivalent.*

(1) *A is distributive.*
(2) *Every relative maximal filter F with respect to an order-ideal I of A is weakly irreducible.*

Proof: (1)⇒(2) Let $I \in \mathrm{Id}(A)$ and $F \in \mathrm{Fi}(A)$ be such that it is relative maximal filter with respect to I. Let us prove that F is weakly irreducible using Lemma 5. Let $F_1, F_2 \in \mathrm{Fi}(A)$ and $F_1 \cap F_2 \subseteq F$. Suppose that $F_1 \nsubseteq F$ and $F_2 \nsubseteq F$, then there exist $a \in F_1 - F$ and $b \in F_2 - F$. Consider the filters $F_a = F \vee [a)$ and $F_b = F \vee [b)$. Since F is relative maximal filter with respect to I, $F_a \cap I \neq \emptyset$ and $F_b \cap I \neq \emptyset$. Then, there are elements $f_1, f_2 \in F$ and $i_1, i_2 \in I$ such that $f_1 \wedge a \leq i_1$ and $f_2 \wedge b \leq i_2$. As F is a filter, then $f = f_1 \wedge f_2 \in F$. Also, as I is an order-ideal, there exists $i \in I$ such that $i_1, i_2 \leq i$. Then $f \wedge a \leq i$ and $f \wedge b \leq i$. By distributivity of $\mathrm{Fi}(A)$, we have

$$\begin{array}{rcccl} i \in F_a \cap F_b & = & (F \vee [a)) \cap (F \vee [b)) & = & F \vee ([a) \cap [b)) \\ & \subseteq & F \vee (F_1 \cap F_2) & = & F \end{array}$$

and this implies that $i \in F$, i.e., $F \cap I \neq \emptyset$, which is a contradiction. Therefore, F is weakly irreducible.

(2)⇒(1) By Lemma 7, $\mathrm{Fi}(A)$ is a lattice. We prove that $\mathrm{Fi}(A)$ is a distributive lattice. Let $F_1, F_2, F_3 \in \mathrm{Fi}(A)$. We know that it is always $(F_1 \cap F_2) \vee (F_1 \cap F_3) \subseteq F_1 \cap (F_2 \vee F_3)$. Suppose that $F_1 \cap (F_2 \vee F_3) \nsubseteq (F_1 \cap F_2) \vee (F_1 \cap F_3)$, then there exists $a \in F_1 \cap (F_2 \vee F_3)$ such that $a \notin (F_1 \cap F_2) \vee (F_1 \cap F_3)$. Consider the following family

$$\mathcal{F} = \{F \in \mathrm{Fi}(A) : (F_1 \cap F_2) \vee (F_1 \cap F_3) \subseteq F \ \text{ and } \ F \cap (a] = \emptyset\}.$$

This family is non-empty. By Zorn's Lemma there exists maximal element $M \in \mathcal{F}$. It is not difficult to show that M is relative maximal filter with respect to a. Then, by hypothesis, M is weakly irreducible. As $F_1 \cap F_2 \subseteq M$, $F_1 \cap F_3 \subseteq M$, $a \in F_1$ and $a \notin M$, we deduce that $F_1 \nsubseteq M$ and that $F_2 \subseteq M$ and $F_3 \subseteq M$. So, $F_2 \vee F_3 \subseteq M$, but $a \in (F_2 \vee F_3)$, which is a contradiction. Thus, $\mathrm{Fi}(A)$ is distributive and by Theorem 8 A is distributive. □

A known result of lattice theory states that a lattice is distributive if and only if every filter is intersection of prime filters. Here we present a generalization of this characterization.

Let A be a semilattice. For each filter F of A we consider the family

$$\widehat{F} = \{P \in X_\omega(A) : F \subseteq P\}.$$

Theorem 14. *Let A be a semilattice. Then the following conditions are equivalent.*

(1) *A is distributive.*
(2) *$F = \bigcap \widehat{F}$ for all $F \in \mathrm{Fi}(A)$.*

Proof: (1)$\Rightarrow$(2) By Theorem 8 and Corollary 3, we have that $F = \bigcap \widehat{F}$ for all $F \in \mathrm{Fi}(A)$.

(2)$\Rightarrow$(1) Let $a, b \in A$. We prove that $\langle a, b\rangle$ is an order-ideal of A. It is easy to show that $\langle a, b\rangle$ is increasing. Let $x, y \in \langle a, b\rangle$ and suppose that $[x) \cap [y) \cap \langle a, b\rangle = \emptyset$. Let $F = [x) \cap [y)$. Since A is a distributive semilattice, then $F \neq \emptyset$. Consider the filter $[F \cup \{a\})$. Then $b \notin [F \cup \{a\})$, because if $b \in [F \cup \{a\})$ then there exists $f \in F$ such that $f \wedge a \leq b$, i.e., $f \in \langle a, b\rangle$. Therefore $[x) \cap [y) \cap \langle a, b\rangle \neq \emptyset$, which is a contradiction. Moreover, as $[F \cup \{a\}) \in \mathrm{Fi}(A)$, by hypothesis,

$$[F \cup \{a\}) = \bigcap \widehat{[F \cup \{a\})}$$

and there exists $Q \in X_\omega(A)$ such that $F \subseteq Q, a \in Q$ and $b \notin Q$. Since $[x) \cap [y) \subseteq Q$ and Q is weakly irreducible, by Lemma 5 then $x \in Q$ or $y \in Q$. In any case $Q \cap \langle a, b\rangle \neq \emptyset$. So, there exists $q \in Q$ such that $q \wedge a \leq b$, but as $q \wedge a \in Q$ and Q is a filter, we have $b \in Q$ which is a contradiction. Thus, $[x) \cap [y) \cap \langle a, b\rangle \neq \emptyset$ and therefore there exists $z \in \langle a, b\rangle$ such that $x \leq z$ and $y \leq z$, i.e., $\langle a, b\rangle$ is an order-ideal. □

4. *DS*-spaces

The purpose of this section is to simplify the topological representation of distributive semilattices given in [4]. We note that this simplification lies in a Stone style duality which differs from the results given in [2], where a full Priestley style duality is developed for *bounded* distributive semilattices.

We will recall some topological notions. The following definitions can be found in [8]. Let $\langle X, \tau\rangle$ be a topological space. The *closure* of a set $Y \subseteq X$ is denoted by $\mathrm{cl}(Y)$. If $Y = \{y\}$, then we will write $\mathrm{cl}(\{y\}) = \mathrm{cl}(y)$. An arbitrary non-empty subset $Y \subseteq X$ is *irreducible* if $Y \subseteq Z \cup W$ for closed subsets Z and W implies $Y \subseteq Z$ or $Y \subseteq W$. Remark that for each $x \in X$ the set $\mathrm{cl}(x)$ is irreducible.

We recall that the *specialization order* of X is defined by

$$x \preceq y \quad \text{iff} \quad x \in \mathrm{cl}(y).$$

The dual order of $\preceq$ is denoted by $\leq$, i.e., $x \leq y$ if and only if $y \in \mathrm{cl}(x)$. Note that the relation $\preceq$ is reflexive and transitive, but not necessarily antisymmetric.

Definition 15. A topological space $\langle X, \tau\rangle$ is *sober* if for every irreducible closed set Y of X, there exists a unique $x \in X$ such that $\mathrm{cl}(x) = Y$.

We have the following result, which indicates the necessary and sufficient conditions for the relation $\preceq$ to be an order.

Lemma 16. *Let $\mathcal{X} = \langle X, \tau \rangle$ be a topological space. Then:*

(1) *if $\mathcal{X}$ is sober, then $\preceq$ is an order;*
(2) *the relation $\preceq$ is an order if and only if $\mathcal{X}$ is T_0.*

PROOF: (1) It is clear that $\preceq$ is reflexive and transitive. We need to show that $\preceq$ is antisymmetric. Let $x, y \in X$ such that $x \preceq y$ and $y \preceq x$. Then $x \in \mathrm{cl}(y)$ and $y \in \mathrm{cl}(x)$. So, $\mathrm{cl}(y) = \mathrm{cl}(x)$, but as $\mathrm{cl}(x)$ and $\mathrm{cl}(y)$ are closed and irreducible, by hypothesis we have $x = y$.

(2) Let $x, y \in X$ such that $x \neq y$. Since $\preceq$ is an order, we have $x \not\preceq y$ or $y \not\preceq x$. Suppose $x \not\preceq y$. The other case is analogous. Then $x \not\preceq y$ if and only if $x \notin \mathrm{cl}(y)$. Later, if we take the open set $U = \mathrm{cl}(y)^c$, we have that $x \in U$ and $y \notin U$. So, $\mathcal{X}$ is T_0.

Conversely, it is easy to check that $\preceq$ is reflexive and transitive. Let $x, y \in X$ such that $x \preceq y$ and $y \preceq x$. Suppose $x \neq y$. Since $\mathcal{X}$ is T_0, then there exists an open U such that $x \in U$ and $y \notin U$. So, $x \notin U^c$ and $y \in U^c$, i.e., $\mathrm{cl}(y) \subseteq U^c$. Therefore, we have that $x \notin \mathrm{cl}(y)$, which is a contradiction because $x \preceq y$. □

Remark 17. By Lemma 16, observe that a sober space is automatically T_0.

In [4] the dual space of a distributive semilattice was defined as an ordered topological space $\langle X, \leq, \tau \rangle$ satisfying certain additional conditions. To be more precise:

Definition 18. An *ordered DS-space* is an ordered topological space $\langle X, \leq, \tau \rangle$ such that:

(1) the set of all open and compact subsets $\mathcal{K}$ forms a basis for the topology τ;
(2) all closed subsets are increasing;
(3) for every $x, y \in X$, if $x \not\leq y$, then there exists $U \in \mathcal{K}$ such that $x \in U$ and $y \notin U$;
(4) if Y is a closed subset of X, and $L \subseteq \mathcal{K}$ is a dually directed set such that $Y \cap U \neq \emptyset$ for all $U \in L$, then $Y \cap \bigcap\{U : U \in L\} \neq \emptyset$.

Let $\langle X, \tau \rangle$ be a topological space with a basis $\mathcal{K}$ of open and compact subsets. Consider the set

$$D_{\mathcal{K}}(X) = \{U \subseteq X : U^c \in \mathcal{K}\}.$$

It is clear that $\langle D_{\mathcal{K}}(X), \cap, X \rangle$ is a semilattice. If $\langle X, \leq, \tau \rangle$ is an ordered DS-space, then $\langle D_{\mathcal{K}}(X), \cap, X \rangle$ is a distributive semilattice (see [7]).

Remark 19. We note that an ordered DS-space $\langle X, \leq, \tau \rangle$ is compact if and only if $D_{\mathcal{K}}(X)$ is a bounded distributive semilattice.

Now we will give an equivalent definition of ordered DS-space without using the order. This allows us to give a simplified topological representation. Moreover, we can dispense with the condition (2). In fact, any closed subset in a sober space X is increasing respect to the dual specialization order of X.

We are able to show the equivalence between the last two conditions of the Definition 18 and sober spaces.

Theorem 20. *Let $\mathcal{X} = \langle X, \tau \rangle$ be a topological space with a basis $\mathcal{K}$ of open and compact subsets for τ. Then the following conditions are equivalent.*

(1) *$\mathcal{X}$ is T_0 and, for each closed subset Y and each dually directed subset $L \subseteq \mathcal{K}$ such that $Y \cap U \neq \emptyset$ for all $U \in L$, we have that $Y \cap \bigcap\{U : U \in L\} \neq \emptyset$.*

(2) *$\mathcal{X}$ is T_0, and the application $H_X : X \to X_\omega(D_\mathcal{K}(X))$, defined by*

$$H_X(x) = \{U \in D_\mathcal{K}(X) : x \in U\}$$

for each $x \in X$, is onto.

(3) *$\mathcal{X}$ is sober.*

Proof: (1)$\Rightarrow$(2) First, we prove that H_X is well-defined. Let $x \in X$. It is clear that $H_X(x)$ is a filter of $D_\mathcal{K}(X)$. We prove that $H_X(x)$ is weakly irreducible. Let $F_1, F_2 \in \mathrm{Fi}(D_\mathcal{K}(X))$ such that $F_1 \cap F_2 \subseteq H_X(x)$ and suppose that $F_1 \nsubseteq H_X(x)$ and $F_2 \nsubseteq H_X(x)$. Then there exist $U_1 \in F_1$ and $U_2 \in F_2$ such that $U_1 \notin H_X(x)$ and $U_2 \notin H_X(x)$, i.e., $x \notin U_1 \cup U_2$. Since $U_1^c \cap U_2^c$ is open and $\mathcal{K}$ is a basis, there exists $O^c \in \mathcal{K}$ such that $x \in O^c \subseteq U_1^c \cap U_2^c$. Later, $O \notin H_X(x)$. On the other hand, as $U_1 \subseteq O$ we have that $O \in F_1$. Similarly $O \in F_2$. Therefore, $O \in F_1 \cap F_2$ and $O \in H_X(x)$, which is a contradiction. So, $H_X(x)$ is a weakly irreducible filter. Thus, H_X is well-defined.

Let $P \in X_\omega(D_\mathcal{K}(X))$. Let us consider the set $L = \{U_j^c : U_j \notin P\} \subseteq \mathcal{K}$. We prove that L is dually directed. Let $U_i^c, U_j^c \in L$. Since P is a weakly irreducible filter and $U_i, U_j \notin P$ then there exists $U_k \notin P$ such that $U_i \subseteq U_k$ and $U_j \subseteq U_k$. Then we have $U_k^c \subseteq U_i^c \cap U_j^c$ and thus L is dually directed. The set $Y = \bigcap\{V_i : V_i \in P\}$ is closed and $Y \cap U_j^c \neq \emptyset$ for each $U_j^c \in L$ because, otherwise, there exists $U_j^c \in L$ such that $U_j^c \subseteq \bigcup\{V_i^c : V_i \in P\}$. Since U_j^c is compact, $U_j^c \subseteq V_1^c \cup V_2^c \cup \cdots \cup V_n^c$, i.e., $V_1 \cap V_2 \cap \cdots \cap V_n \subseteq U_j$. It follows that $U_j \in P$, which is a contradiction. Then $Y \cap \bigcap\{U_j^c : U_j \notin P\} \neq \emptyset$, i.e., there exists $x \in \bigcap\{V_i : V_i \in P\} \cap \bigcap\{U_j^c : U_j \notin P\}$, which implies that $P = H_X(x)$.

(2)$\Rightarrow$(3) Let Y be an irreducible closed subset of X. Let us consider the set $P_Y = \{U \in D_\mathcal{K}(X) : Y \subseteq U\}$. It is easy to see that P_Y is a filter of $D_\mathcal{K}(X)$. We prove that P_Y is weakly irreducible. Let $F_1, F_2 \in \mathrm{Fi}(D_\mathcal{K}(X))$ be such that $F_1 \cap F_2 \subseteq P_Y$. Suppose that $F_1 \nsubseteq P_Y$ and $F_2 \nsubseteq P_Y$, then there exist $U \in F_1 - P_Y$ and $V \in F_2 - P_Y$. Hence, $Y \nsubseteq U$ and $Y \nsubseteq V$, and as Y is irreducible, $Y \nsubseteq U \cup V$. So, there exists $x \in Y$ such that $x \in U^c \cap V^c$. As $U^c, V^c \in \mathcal{K}$ and $\mathcal{K}$ is a basis of open and compact subsets for the topology τ, there exists $W \in D_\mathcal{K}(X)$ such that $x \in W^c \subseteq U^c \cap V^c$. So, $Y \nsubseteq W$. Thus, $W \notin P_Y$. On the other hand $U \subseteq W$, but as F_1 is a filter, $W \in F_1$. Analogously $W \in F_2$. Later $W \in P_Y$, which is a contradiction. Then $F_1 \subseteq P_Y$ or $F_2 \subseteq P_Y$ and by Lemma 5 P_Y is a weakly irreducible filter of $D_\mathcal{K}(X)$. Since $\mathcal{X}$ is T_0, H_X is one-to-one, and as H_X is onto, there exists a unique $y \in X$ such that $H_X(y) = P_Y$. It is easy to check that $Y = \mathrm{cl}(y)$. Thus, $\mathcal{X}$ is sober.

(3)⇒(1) Let Y be a closed subset of X and let $L = \{U_i : i \in I\}$ be a dually directed subfamily of $\mathcal{K}$ such that $Y \cap U_i \neq \emptyset$ for all $i \in I$. Since Y^c is an open subset and $\mathcal{K}$ is a basis, $Y = \bigcap\{V : V \in B \subseteq D_{\mathcal{K}}(X)\}$. Let us consider the set $H = \{U_i^c : U_i \in L\} \subseteq D_{\mathcal{K}}(X)$. Since L is dually directed, the subset $(H] = \{W \in D_{\mathcal{K}}(X) : W \subseteq U_i^c$, for some $U_i^c \in H\}$ is an order-ideal of $D_{\mathcal{K}}(X)$. Let $F(B)$ be the filter generated by B. We prove that

$$F(B) \cap (H] = \emptyset.$$

Suppose the contrary. Then, there exists $U_k^c \in H$ and there exist $V_1, \ldots, V_n \in B$ such that $V_1 \cap \cdots \cap V_n \subseteq U_k^c$. Since $Y \cap U_k \neq \emptyset$, there exists $x \in X$ such that $x \in Y$ and $x \in U_k$. As $x \in V_1, \ldots, V_n$ and $V_1 \cap \cdots \cap V_n \subseteq U_k^c$, we deduce that $x \in U_k^c$, which is a contradiction. Thus, there exists $P \in X(D_{\mathcal{K}}(X))$ such that $F(B) \subseteq P$ and $P \cap (H] = \emptyset$. Consider the set

$$Z = \bigcap\{V : V \in P\}.$$

Then, $Z \subseteq Y$. It is easy to see that Z is an irreducible set. As $\mathcal{X}$ is sober, there exists a unique $x \in X$ such that $Z = \mathrm{cl}(x)$. We prove that $x \in \bigcap\{U : U \in L\}$. If there exists $U \in L$ such that $x \notin U$, then $Z = \mathrm{cl}(x) \subseteq U^c$. So, $U \subseteq \bigcup\{V^c : V \in P\}$ and as U is compact, there exists a finite subset $\{V_1, \ldots, V_n\}$ of P such that $U \subseteq V_1^c \cup \cdots \cup V_n^c$, i.e., $V_1 \cap \cdots \cap V_n \subseteq U^c$, but as P is a filter, $U^c \in P$, which is a contradiction. Thus, $x \in \bigcap\{U : U \in L\} \cap Y$. □

From Theorem 20, we can simplify the Definition 18 saying that an ordered DS-space can be characterized as a topological space $\langle X, \tau\rangle$ by the following conditions:

(1) the set of all open and compact subsets $\mathcal{K}$ forms a basis for the topology τ;
(2) $\langle X, \tau\rangle$ is sober.

In this case, the order $\leq$ is the dual of the specialization order. Consequently from now on an ordered DS-spaces will be called a DS-space. Note that from Theorem 20 we have a new characterization of DS-spaces different of the definition given by G. Grätzer in [7].

Let $A \in \mathcal{DS}$. Let us consider the set $X(A)$ and let us consider the mapping

$$\beta : A \to \mathcal{P}_i(X(A))$$

defined by $\beta(a) = \{P \in X(A) : a \in P\}$. Let $\beta[A] = \{\beta(a) : a \in A\}$. In [4] the following theorem was obtained.

Theorem 21 (Representation theorem)**.** *Let $A \in \mathcal{DS}$. Then, A is isomorphic to the subalgebra $\beta[A]$ of $\mathcal{P}_i(X(A))$.*

With the results of this section and [4], we define the *dual space of* A (without using the order), as the structure $\mathcal{F}(A) = \langle X(A), \beta[A]^c \rangle$ where $\beta[A]^c$ is a basis for a topology on $X(A)$. Then from the results given in [4] we have that:

Theorem 22. *Let $A \in \mathcal{DS}$. Then $\mathcal{F}(A) = \langle X(A), \beta[A]^c \rangle$ is a DS-space.*

5. The lattice of filters

The purpose of this section is to study the structure of the lattice of filters of a distributive semilattice A. For each $F \in \mathrm{Fi}(A)$, let us consider the set

$$\Phi(F) = \{P \in X(A) : F \subseteq P\}.$$

Note that

$$\Phi(F) = \bigcap\{\beta(a) : a \in F\},$$

for each $F \in \mathrm{Fi}(A)$. If $F \in \mathrm{Fi}_f(A)$ then there exists a finite subset $\{a_1, \ldots, a_n\} \subseteq A$ such that $F = F(\{a_1, \ldots, a_n\})$. It is not difficult to show that

$$F = F(\{a_1, \ldots, a_n\}) \text{ iff } \Phi(F) = \beta(a_1) \cap \cdots \cap \beta(a_n).$$

Remark that if $F = [a)$, then $\Phi(F) = \beta(a)$. For the proof of the following result see [4].

Theorem 23. *Let $A \in \mathcal{DS}$. Let $\mathcal{F}(A)$ be the dual space of A. Then:*

(1) *a subset $U \subseteq X(A)$ is open in $\mathcal{F}(A)$ if and only if there exists $F \in \mathrm{Fi}(A)$ such that $U = \Phi(F)^c$;*
(2) *a subset $U \subseteq X(A)$ is open-compact in $\mathcal{F}(A)$ if and only if there exists $a \in A$ such that $U = \Phi([a))^c$.*

Let $\langle X, \tau \rangle$ be a topological space. We will denote by $\mathcal{C}(X)$ (resp. $\mathcal{O}(X)$) the set of all non-empty closed subsets (resp. open) of X. Let us denote by $\mathcal{KO}(X)$ the set of all compact and open subsets of X. Note that $\mathcal{C}(X)$ and $\mathcal{O}(X)$ are lattices under the inclusion relation.

From the result developed in [4], we have the following theorem.

Theorem 24. *Let $A \in \mathcal{DS}$. Let $\mathcal{F}(A)$ be the dual space of A. Then:*

(1) *the lattices $\mathrm{Fi}(A)$ and $\mathcal{C}(X(A))$ are dually isomorphic under the map*

$$\Phi : \mathrm{Fi}(A) \to \mathcal{C}(X(A));$$

(2) *the poset $\mathcal{KO}(X(A))$ is isomorphic to the poset $\mathrm{Fi}_f(A)$ under the map*

$$\Psi : \mathrm{Fi}_f(A) \to \mathcal{KO}(X(A))$$

defined by

$$\Psi(F) = (\Phi(F))^c.$$

Recall that a *Heyting algebra* is an algebra $\langle A, \vee, \wedge, \Rightarrow, 0, 1\rangle$ of type $(2, 2, 2, 0, 0)$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and the operation $\Rightarrow$ satisfies the condition: $a \wedge b \leq c$ if and only if $a \leq b \Rightarrow c$, for all $a, b, c \in A$.

For each pair $F, H \in \mathrm{Fi}(A)$ let us define the subset $F \mapsto H$ of A as follows:

$$F \mapsto H = \{a \in A : [a) \cap F \subseteq H\}.$$

Theorem 25. *Let $A \in \mathcal{DS}$. Let $F, H \in \mathrm{Fi}(A)$. Then:*

(1) *$F \mapsto H \in \mathrm{Fi}(A)$;*

(2) *$\langle \mathrm{Fi}(A), \vee, \wedge, \mapsto, \{1\}, A\rangle$ is a Heyting algebra.*

PROOF: (1) Let $a, b \in A$ such that $a \leq b$ and $a \in F \mapsto H$. Then $b \in [a)$ and $[a) \cap F \subseteq H$. Since $[b) \subseteq [a)$, we have $[b) \cap F \subseteq H$, i.e., $b \in F \mapsto H$. Let $a, b \in F \mapsto H$. We prove that $a \wedge b \in F \mapsto H$. As $\mathrm{Fi}(A)$ is distributive,

$$\begin{aligned} [a \wedge b) \cap F &= \{[a) \vee [b)\} \cap F \\ &= \{[a) \cap F\} \vee \{[b) \cap F\} \\ &\subseteq H. \end{aligned}$$

Therefore, $a \wedge b \in F \mapsto H$ and $F \mapsto H \in \mathrm{Fi}(A)$.

(2) Let $F_1, F_2, F_3 \in \mathrm{Fi}(A)$. We prove that $F_1 \cap F_2 \subseteq F_3$ if and only if $F_1 \subseteq F_2 \mapsto F_3$. Suppose that $F_1 \cap F_2 \subseteq F_3$ and let $a \in F_1$. Take $x \in [a) \cap F_2$, then $a \leq x$ and $x \in F_2$. Therefore $x \in F_1 \cap F_2$ and by hypothesis $x \in F_3$, i.e., $F_1 \subseteq F_2 \mapsto F_3$.

Conversely, let $x \in F_1 \cap F_2$. By hypothesis $F_1 \subseteq F_2 \mapsto F_3$, then $x \in F_2 \mapsto F_3$. As $[x) \cap F_2 \subseteq F_3$, we have $x \in F_3$. $\square$

The following result is known for the lattice of all open subsets of a topological space. Here we need the version of this result for the lattice of all closed subsets of a topological space.

Lemma 26. *Let $\langle X, \tau\rangle$ be a topological space. Then $\langle \mathcal{C}(X), \cup, \cap, \rightsquigarrow, \emptyset, X\rangle$ where*

$$U \rightsquigarrow V = \mathrm{cl}(U^c \cap V)$$

is a Heyting algebra.

Now we will prove that the isomorphism given in Theorem 24 is an isomorphism of Heyting algebras.

Theorem 27. *Let $A \in \mathcal{DS}$. Let $\mathcal{F}(A)$ be the dual space of A. Then the application Φ is an isomorphism between Heyting algebras $\langle \mathrm{Fi}(A), \vee, \wedge, \mapsto, \{1\}, A\rangle$ and $\langle \mathcal{C}(X(A)), \cup, \cap, \rightsquigarrow, \emptyset, X\rangle$.*

PROOF: We only have to prove that if $F, H \in \mathrm{Fi}(A)$ then

$$\Phi(F \mapsto H) = \mathrm{cl}(\Phi(F)^c \cap \Phi(H)).$$

If we prove first that $\Phi(F)^c \cap \Phi(H) \subseteq \Phi(F \mapsto H)$, then as $\Phi(F \mapsto H)$ is a closed subset of $X(A)$, we get $\mathrm{cl}(\Phi(F)^c \cap \Phi(H)) \subseteq \Phi(F \mapsto H)$. Let $P \in \Phi(F)^c \cap \Phi(H)$.

Then $F \nsubseteq P$ and $H \subseteq P$. Let $f \in F$ and $f \notin P$. If $a \in F \mapsto H$, then $[a) \cap F \subseteq H$. So, $[a) \cap F \subseteq P$, and since P is weakly irreducible and $f \notin P$ we have by Lemma 5 that $[a) \subseteq P$, i.e., $a \in P$. Later, $F \mapsto H \subseteq P$ and $\Phi(F)^c \cap \Phi(H) \subseteq \Phi(F \mapsto H)$.

Let $P \in \Phi(F \mapsto H)$. Suppose that

$$P \notin \mathrm{cl}(\Phi(F)^c \cap \Phi(H)) = \bigcap\{\Phi(D) : D \in \mathrm{Fi}(A) \text{ and } \Phi(F)^c \cap \Phi(H) \subseteq \Phi(D)\}.$$

Then there exists $D \in \mathrm{Fi}(A)$ such that

$$\Phi(F)^c \cap \Phi(H) \subseteq \Phi(D) \text{ and } D \nsubseteq P.$$

Then, there exists $d \in D$ and $d \notin P$. By hypothesis, since $F \mapsto H \subseteq P$, then $d \notin F \mapsto H$, i.e., there exists $x \in A$ such that $d \leq x$, $x \in F$ and $x \notin H$. From Theorem 2 there exists $Q \in X(A)$ such that $H \subseteq Q$ and $x \notin Q$. So, as $F \nsubseteq Q$, we have that

$$Q \in \Phi(F)^c \cap \Phi(H) \subseteq \Phi(D).$$

Then $D \subseteq Q$ and $d \in Q$. Therefore, $x \in Q$ which is a contradiction. Thus, $P \in \mathrm{cl}(\Phi(F)^c \cap \Phi(H))$. □

6. Meet-relations

In this section we will focus in the representation of homomorphisms of distributive semilattices. In [4] it was shown that there exists a duality between homomorphisms of distributive semilattices and certain binary relations, called *meet-relations*. It was mentioned that the DS-spaces with meet-relations form a category, but this result was not given. Our purpose is to show that the usual composition of two meet-relations is a meet-relation and that matches with the composition introduced in [2].

Definition 28. Let $A, B \in \mathcal{DS}$. A mapping $h : A \to B$ is a *homomorphism* if for every $a, b \in A$:

(1) $h(a \wedge b) = h(a) \wedge h(b)$,
(2) $h(1) = 1$.

Let us define a binary relation $R_h \subseteq X(B) \times X(A)$ by

$$(P, Q) \in R_h \text{ iff } h^{-1}(P) \subseteq Q$$

where $h^{-1}(P) = \{x \in A : h(x) \in P\}$.

Let X_1 and X_2 be two sets and let $R \subseteq X_1 \times X_2$ be a binary relation. For each $x \in X_1$, let $R(x) = \{y \in X_2 : (x, y) \in R\}$. Define the mapping $h_R : \mathcal{P}(X_2) \to \mathcal{P}(X_1)$ by

$$h_R(U) = \{x \in X_1 : R(x) \subseteq U\}.$$

It is easy to verify that $h_R(U \cap V) = h_R(U) \cap h_R(V)$ and $h_R(X_2) = X_1$. If $S \subseteq X_2 \times X_3$ is another relation, then the composition of the relations R and S

is the relation $S \circ R \subseteq X_1 \times X_3$ defined by:

$$S \circ R = \{(x, z) \in X_1 \times X_3 : \exists y \in X_2 \, [(x, y) \in R \;\text{ and }\; (y, z) \in S]\}.$$

Note that $h_{(S \circ R)}(U) = (h_R \circ h_S)(U)$ for all $U \in \mathcal{P}(X_3)$.

Recall that in [4] a *meet-relation* was defined as a subset $R \subseteq X_1 \times X_2$ such that:

(1) for every $U \in D_{\mathcal{K}_2}(X_2)$, $h_R(U) \in D_{\mathcal{K}_1}(X_1)$, and
(2) $R(x) = \bigcap\{U \in D_{\mathcal{K}_2}(X_2) : R(x) \subseteq U\}$, for all $x \in X_1$.

The last item can be formulated in two equivalent forms.

Lemma 29. *Let X_1 and X_2 be two DS-spaces. Let $R \subseteq X_1 \times X_2$ be a binary relation. Suppose that for every $U \in D_{\mathcal{K}_2}(X_2)$, $h_R(U) \in D_{\mathcal{K}_1}(X_1)$. Then the following conditions are equivalent.*

(1) *For every $(x, y) \notin R$, there exists $U \in D_{\mathcal{K}_2}(X_2)$ such that $R(x) \subseteq U$ and $y \notin U$.*
(2) *$R(x) = \bigcap\{U \in D_{\mathcal{K}_2}(X_2) : R(x) \subseteq U\}$, for all $x \in X_1$.*
(3) *For any $(x, y) \in X_1 \times X_2$,*

$$(x, y) \in R \;\text{ iff }\; (H_{X_1}(x), H_{X_2}(y)) \in R_{h_R}.$$

Proof: (1)⇒(2) It is clear that $R(x) \subseteq \bigcap\{U \in D_{\mathcal{K}_2}(X_2) : R(x) \subseteq U\}$. Let us prove the other inclusion. Suppose that $\bigcap\{U \in D_{\mathcal{K}_2}(X_2) : R(x) \subseteq U\} \nsubseteq R(x)$. So there exists $y \in \bigcap\{U \in D_{\mathcal{K}_2}(X_2) : R(x) \subseteq U\}$ such that $(x, y) \notin R$. By hypothesis, there exists $U_0 \in D_{\mathcal{K}_2}(X_2)$ such that $R(x) \subseteq U_0$ and $y \notin U_0$, which is a contradiction. Therefore, $R(x) = \bigcap\{U \in D_{\mathcal{K}_2}(X_2) : R(x) \subseteq U\}$.

(2)⇒(3) Let $(x, y) \notin R$, i.e., $y \notin R(x) = \bigcap\{U \in D_{\mathcal{K}_2}(X_2) : R(x) \subseteq U\}$. Then there exists $U_0 \in D_{\mathcal{K}_2}(X_2)$ such that $R(x) \subseteq U_0$ and $y \notin U_0$. As $x \in h_R(U_0)$, we have $h_R(U_0) \in H_{X_1}(x)$. It follows that $U_0 \in h_R^{-1}(H_{X_1}(x))$ and $U_0 \notin H_{X_2}(y)$, and therefore, $h_R^{-1}(H_{X_1}(x)) \nsubseteq H_{X_2}(y)$, or equivalently, $(H_{X_1}(x), H_{X_2}(y)) \notin R_{h_R}$. The other direction is similar.

(3)⇒(1) Suppose that $(x, y) \notin R$. Then, by hypothesis, $(H_{X_1}(x), H_{X_2}(y)) \notin R_{h_R}$, i.e., $h_R(H_{X_1}(x))^{-1} \nsubseteq H_{X_2}(y)$. Therefore there exists $U \in D_{\mathcal{K}_2}(X_2)$ such that $U \in h_R(H_{X_1}(x))^{-1}$ and $U \notin H_{X_2}(y)$. Since $h_R(U) \in H_{X_1}(x)$, we have $x \in h_R(U)$, i.e., $R(x) \subseteq U$. So, there exists $U \in D_{\mathcal{K}_2}(X_2)$ such that $R(x) \subseteq U$ and $y \notin U$. ☐

Remark 30. Let $R \subseteq X_1 \times X_2$ be a meet-relation. Then $R(y) \subseteq R(x)$ for all $x, y \in X$ such that $x \leq y$.

In [2] the following definition is introduced.

Definition 31. Let X_1, X_2 and X_3 are DS-spaces and $R \subseteq X_1 \times X_2$, $S \subseteq X_2 \times X_3$ are meet-relations. Define a new binary relation $S * R \subseteq X_1 \times X_3$ by

$$(x, z) \in S * R \;\text{ iff }\; (\forall U \in D_{\mathcal{K}_3}(X_3))((S \circ R)(x) \subseteq U \Rightarrow z \in U),$$

where $S \circ R$ denotes the usual composition.

We note that

$$\begin{array}{rcl}(S * R)(x) & = & \bigcap\{U \in D_{\mathcal{K}_3}(X_3) : (S \circ R)(x) \subseteq U\} \\ & = & \bigcap\{U \in D_{\mathcal{K}_3}(X_3) : R(x) \subseteq h_S(U)\} \\ & = & \bigcap\{U \in D_{\mathcal{K}_3}(X_3) : x \in h_R(h_S(U))\} \\ & = & \bigcap\{U \in D_{\mathcal{K}_3}(X_3) : x \in (h_R \circ h_S)(U)\} \\ & = & \mathrm{cl}((S \circ R)(x)).\end{array}$$

As an immediate consequence, we have $S \circ R \subseteq S * R$.

Lemma 32. *Let X_1 and X_2 be two DS-spaces. Let $R \subseteq X_1 \times X_2$ be a meet-relation. For each closed subset C of X_1,*

$$R[C] = \{y \in X_2 : \exists x \in C\,((x, y) \in R)\}$$

is a closed subset of X_2.

Proof: Let $C \in \mathcal{C}(X_1)$. By Lemma 29, it suffices to prove that for any $y \notin R[C]$ there exists $U \in \mathcal{K}_2$ such that $R[C] \subseteq U^c$ and $y \in U$. So, let $y \notin R[C]$, i.e., $y \notin R(x)$ for all $x \in C$. Since $R(x)$ is closed, for each $x \in C$ there exists $U_x \in \mathcal{K}_2$ such that $y \in U_x$ and $R(x) \subseteq (U_x)^c$, i.e., $x \in h_R((U_x)^c)$. Then we have that $C \subseteq \bigcup\{h_R(U_x^c) : x \in C\}$, or equivalently, $C \cap \bigcap\{h_R(U_x^c)^c : x \in C\} = \emptyset$. Now consider the set $H = \{h_R(U_x^c)^c : x \in C\}$. Since R is a meet-relation, $H \subseteq \mathcal{K}_1$. Moreover, it is easy to prove that H is a dually directed family. So by Theorem 20, there exists $h_R(U^c)^c \in H$ such that $C \cap h_R(U^c)^c = \emptyset$, i.e., $C \subseteq h_R(U^c)$. Thus there exists $U \in \mathcal{K}_2$ such that $R[C] \subseteq U^c$ and $y \in U$. It follows that $R[C] \in \mathcal{C}(X_2)$. □

Remark 33. We note that in the DS-spaces the usual composition of meet-relations coincides with the composition $*$ introduced in [2].

Theorem 34. *Let X_1, X_2 and X_3 be DS-spaces and $R \subseteq X_1 \times X_2, S \subseteq X_2 \times X_3$ are meet-relations. Then:*

(1) *the relation $\leq_2 \subseteq X_2 \times X_2$ satisfies*

$$R \circ \leq_2 = R \ \text{ and } \ \leq_2 \circ\, S = S;$$

(2) *$S \circ R \subseteq X_1 \times X_3$ is a meet-relation.*

Proof: (1) We prove that $R \circ \leq_2 = R$ and $\leq_2 \circ S = S$. It is clear that $R \subseteq R \circ \leq_2$ and $S \subseteq \leq_2 \circ S$. Let $(x, z) \in R \circ \leq_2$, then there exists $y \in X_2$ such that $(x, y) \in \leq_2$ and $(y, z) \in R$, i.e., $x \leq_2 y$ and $z \in R(y)$. Later, by Remark 30, $R(y) \subseteq R(x)$ and $z \in R(x)$. So, $(x, z) \in R$ and $R \circ \leq_2 = R$. Similarly, let $(x, z) \in \leq_2 \circ S$, then there exists $y \in X_2$ such that $(x, y) \in S$ and $(y, z) \in \leq_2$. Thus, $y \in S(x)$ and $y \leq_2 z$. As $S(x)$ is closed, and therefore increasing, we have $z \in S(x)$. Then $(x, z) \in S$ and $\leq_2 \circ S = S$.

(2) It is easy to prove that $(h_R \circ h_S)(U) = h_{(S \circ R)}(U) \in D_{\mathcal{K}_1}(X_1)$ for all $U \in D_{\mathcal{K}_3}(X_3)$. By Lemma 32 it follows that $(S \circ R)(x)$ is a closed subset of X_3 for all $x \in X_1$. Therefore, $S \circ R \subseteq X_1 \times X_3$ is a meet-relation. □

By Theorem 34 we conclude that the DS-spaces with meet-relations form a category where the identity morphism of an DS-space X is $\leq_X$. Note that the composition of relations reverses the order of the actual composition in the category.

Now we study the characterization of onto and one-to-one homomorphisms between distributive semilattices using special meet-relations. These results are similar to those given by G. Bezhanishvili and R. Jansana in [2] for generalized Priestley morphisms between generalized Priestley spaces, and we will use them to characterize the homomorphic images of a distributive semilattice.

Definition 35. Let X_1 and X_2 be two DS-spaces. Let $R \subseteq X_1 \times X_2$ be a meet-relation. We shall say that R is *one-to-one* if for each $x \in X_1$ and $U \in D_{\mathcal{K}_1}(X_1)$ with $x \notin U$, there exists $V \in D_{\mathcal{K}_2}(X_2)$ such that $U \subseteq h_R(V)$ and $x \notin h_R(V)$.

Theorem 36. *Let X_1 and X_2 be two DS-spaces. Let $R \subseteq X_1 \times X_2$ be a meet-relation. Then:*

(1) *the homomorphism $h_R : D_{\mathcal{K}_2}(X_2) \to D_{\mathcal{K}_1}(X_1)$ is one-to-one if and only if for each $y \in X_2$ there exists $x \in X_1$ such that $R(x) = \mathrm{cl}(y)$;*

(2) *the homomorphism $h_R : D_{\mathcal{K}_2}(X_2) \to D_{\mathcal{K}_1}(X_1)$ is onto if and only if R is one-to-one.*

PROOF: (1) Suppose that $h_R : D_{\mathcal{K}_2}(X_2) \to D_{\mathcal{K}_1}(X_1)$ is one-to-one. Let $y \in X_2$. Consider the set $L = \{h_R(V)^c : y \notin V\}$. Let $Z = \bigcap\{h_R(U) : y \in U\}$. As R is a meet-relation, $h_R(U) \in D_{\mathcal{K}_1}(X_1)$. Then Z is a closed subset of X_1. We prove that $Z \cap h_R(V)^c \neq \emptyset$, for each $V \in D_{\mathcal{K}_2}(X_2)$ such that $y \notin V$. Suppose that there exists $V \in D_{\mathcal{K}_2}(X_2)$ with $y \notin V$ such that $Z \cap h_R(V)^c = \emptyset$. Then $h_R(V)^c \subseteq \bigcup h_R(U)^c : y \in U\}$ and as $h_R(V)$ is compact, there exists a finite set $\{U_1, \ldots, U_n\}$ such that $h_R(V)^c \subseteq h_R(U_1)^c \cup \cdots \cup h_R(U_n)^c = h_R(U_1 \cap \cdots \cap U_n)^c = h_R(U)^c$. So, $h_R(U) \subseteq h_R(V)$ and as h_R is one-to-one, we get $U \subseteq V$, but this implies that $y \in V$, which is a contradiction. Thus, $Z \cap h_R(V)^c \neq \emptyset$, for each $V \in D_{\mathcal{K}_2}(X_2)$ such that $y \notin V$. It is easy to see that the family $L = \{h_R(V)^c : y \notin V\}$ is dually directed. By Theorem 20, $Z \cap \bigcap\{h_R(V)^c : y \notin V\} \neq \emptyset$. Then there exists $x \in X_1$ such that $x \in \bigcap\{h_R(U) : y \in U\} \cap \bigcap\{h_R(V)^c : y \notin V\}$. As $x \in \bigcap\{h_R(U) : y \in U\}$, we have that $R(x) \subseteq U$ for all U such that $y \in U$. Then $R(x) \subseteq \mathrm{cl}(y)$. If there exists $z \in \mathrm{cl}(y)$ such that $z \notin R(x)$, then as R is a meet-relation, there exists $V \in D_{\mathcal{K}_2}(X_2)$ such that $R(x) \subseteq V$ and $z \notin V$. So, $x \notin h_R(V)^c$ and since $z \in \mathrm{cl}(y)$ and $z \notin V$, we have $y \notin V$. But $x \in \bigcap\{h_R(V)^c : y \notin V\}$, which is a contradiction. Thus, $R(x) = \mathrm{cl}(y)$.

Conversely, we assume that for each $y \in X_2$ there is $x \in X_1$ such that $R(x) = \mathrm{cl}(y)$. Let $U, V \in D_{\mathcal{K}_2}(X_2)$ such that $h_R(U) = h_R(V)$. Suppose that $U \neq V$. Then there exists $y \in U - V$. So, $\mathrm{cl}(y) \subseteq U$ and by hypothesis, there exists $x \in X_1$ such that $R(x) = \mathrm{cl}(y)$. Then $x \in h_R(U) \subseteq h_R(V)$. So, $R(x) = \mathrm{cl}(y) \subseteq V$, which is a contradiction.

(2) Assume that $h_R : D_{\mathcal{K}_2}(X_2) \to D_{\mathcal{K}_1}(X_1)$ is onto. Let $x \in X_1$ and $U \in D_{\mathcal{K}_1}(X_1)$ with $x \notin U$. Since h_R is onto, there exists $V \in D_{\mathcal{K}_2}(X_2)$ such that $h_R(V) = U$. So, $x \notin h_R(V)$.

Suppose that for each $x \in X_1$ and $U \in D_{\mathcal{K}_1}(X_1)$ with $x \notin U$, there is $V \in D_{\mathcal{K}_2}(X_2)$ such that $U \subseteq h_R(V)$ and $x \notin h_R(V)$. Let $U \in D_{\mathcal{K}_1}(X_1)$. For each $x \notin U$ there exists $V_x \in D_{\mathcal{K}_2}(X_2)$ such that $U \subseteq h_R(V_x)$ and $x \notin h_R(V_x)$. So, $U^c \subseteq \bigcup\{h_R(V_x)^c : x \notin U\}$ and as U^c is compact, there exists $V_{x_1}, \ldots, V_{x_n} \in D_{\mathcal{K}_2}(X_2)$ such that $h_R(V_{x_1}) \cap \cdots \cap h_R(V_{x_2}) = h_R(V) \subseteq U$. Thus $U = h_R(V)$, and consequently h_R is onto. $\square$

Example 37. Let $A \in \mathcal{DS}$ and $\theta \subseteq A \times A$ be a semilattice congruence. As the class of distributive semilattice is not a variety, the quotient semilattice A/θ cannot be a distributive semilattice. We say that θ is a *distributive congruence* when A/θ is a distributive semilattice. If θ is a distributive congruence of A, then we have a natural homomorphism $q_\theta : A \to A/\theta$ assigning to $a \in A$ the equivalence class $q_\theta(a) \in A/\theta$. Since q_θ is onto, we have that $R_{q_\theta} \subseteq X(A/\theta) \times X(A)$ is a meet-relation one-to-one.

7. Homomorphic images of a distributive semilattice

Let $\mathcal{C}(X)$ be the family of all non-empty closed subsets of a topological space $\langle X, \tau\rangle$. Let $\mathcal{F}$ be a subfamily of $\mathcal{C}(X)$. The *lower Vietoris topology* τ_L is the topology generated by the collection of all sets of the form

$$H_U = \{Y \in \mathcal{F} : Y \cap U \neq \emptyset\}$$

as a sub-basis where U is an open set of $\langle X, \tau\rangle$. For each open subset U we consider the set

$$M_U = \{Y \in \mathcal{F} : Y \cap U = \emptyset\}.$$

We note that $M_U = \mathcal{F} - H_U = H_U^c$. The *Vietoris topology* τ_V defined on $\mathcal{F}$ is the topology generated by the collection of sets

$$\{H_U : U \in \mathcal{O}(X)\} \cup \{M_V : V \in \mathcal{O}(X)\}$$

as a sub-basis.

In [3] it was shown that if $X = \langle X, \tau, \leq, X_0\rangle$ is a generalized Priestley space, then there exists a duality between homomorphic images of the bounded distributive semilattice X^* and generalized Priestley spaces $\langle \mathcal{F}, \tau_V, \supseteq, \mathcal{F}_0\rangle$, where τ_V is the Vietoris topology defined on a family of closed subsets $\mathcal{F}$ of X, $\mathcal{F}_0 \subseteq \mathcal{F}$ and each element $Y \in \mathcal{F}$ is intersection of elements of X^*. Here we will show that the homomorphic images can be described by means of DS-spaces of the form $\langle \mathcal{F}, \tau_L\rangle$, where τ_L is a lower Vietoris topology and $\mathcal{F}$ is a family of closed subsets of a DS-space $\langle X, \tau\rangle$.

Let $A \in \mathcal{DS}$ and let $\langle X, \tau\rangle$ be a DS-space. Consider a one-to-one meet-relation $R \subseteq X \times X(A)$. Let $\mathcal{F}_R = \{R(x) : x \in X\}$. It is clear that $\mathcal{F}_R \subseteq \mathcal{C}(X(A))$. Define the lower Vietoris topology τ_L on $\mathcal{F}_R$ as follows. For $a \in A$, we consider the set

$$H_a = \{R(x) : R(x) \cap \beta(a)^c \neq \emptyset\}.$$

Then $\mathfrak{B} = \{H_a : a \in A\}$ is a sub-basis for τ_L. Indeed, let $R(x) \in \mathcal{F}_R$. Since $\mathcal{K}$ is a basis of X, there exists $U \in D_{\mathcal{K}}(X)$ such that $x \notin U$. Then, as R is a one-to-one meet-relation, there exists $V \in D_{\mathcal{K}_A}(X(A))$ such that $U \subseteq h_R(V)$ and $x \notin h_R(V)$. By Theorem 21, there exists $a \in A$ such that $V = \beta(a)$. So, $x \notin h_R(\beta(a))$ and $x \in H_a$. Therefore, $\mathcal{F}_R = \bigcup\{H_a : a \in A\}$ and $\mathfrak{B}$ is a sub-basis for τ_L.

Let $M_a = \{R(x) : R(x) \subseteq \beta(a)\}$. Note that $M_a = \mathcal{F}_R - H_a = H_a^c$. It is easy to see that $M_a \cap M_b = M_{a\wedge b}$, and $M_1 = \mathcal{F}_R$. Thus $D_{\mathfrak{B}}(\mathcal{F}_R) = \{M_a : a \in A\}$ is a meet-semilattice.

Lemma 38. *Let $A \in \mathcal{DS}$ and $\langle X, \tau\rangle$ be a DS-space. Let $R \subseteq X \times X(A)$ be a one-to-one meet-relation. Then the family*

$$\mathfrak{B} = \{H_a : a \in A\}$$

is a basis for a topology τ_L defined on $\mathcal{F}_R$.

Proof: Let $a, b \in A$ such that $H_a \cap H_b \neq \emptyset$. Then there exists $x \in X$ such that $R(x) \cap \beta(a)^c \neq \emptyset$ and $R(x) \cap \beta(b)^c \neq \emptyset$. So, $x \notin h_R(\beta(a))$ and $x \notin h_R(\beta(b))$, i.e., $h_R(\beta(a))$, $h_R(\beta(b)) \notin H_X(x)$. As $H_X(x)$ is irreducible, by Lemma 4, there exists $U_1 \notin H_X(x)$ and $U_2 \in H_X(x)$ such that

$$h_R(\beta(a)) \cap U_2 \subseteq U_1 \text{ and } h_R(\beta(b)) \cap U_2 \subseteq U_1.$$

Since the map $h_R : D_{\mathcal{K}_A}(X(A)) \to D_{\mathcal{K}}(X)$ is onto, by Theorem 36, there exists $c \in A$ such that $h_R(\beta(c)) = U_1$. Later, $x \notin h_R(\beta(c))$, i.e., $R(x) \in H_c$. On the other hand, $H_c \subseteq H_a \cap H_b$. In fact, if $R(x) \in H_c$ then $x \notin h_R(\beta(c)) = U_1$ and $x \notin h_R(\beta(a)) \cap U_2$. As $x \in U_2$, then it must be $x \notin h_R(\beta(a))$, i.e., $R(x) \in H_a$. Similarly, we have that $R(x) \in H_b$. So, $R(x) \in H_c \subseteq H_a \cap H_b$ and $\mathfrak{B}$ is a basis for τ_L. □

Lemma 39. *Let $A \in \mathcal{DS}$ and $\langle X, \tau\rangle$ be a DS-space. Let $R \subseteq X \times X(A)$ be a one-to-one meet-relation. Then the following statements hold.*

(1) *Let $a \in A$ and consider the set H_a. We have $H_a \subseteq \bigcup\{H_b : b \in B \subseteq A\}$ if and only if $h_R(\beta(a))^c \subseteq \bigcup\{h_R(\beta(b))^c : b \in B \subseteq A\}$.*
(2) *A subset $U \subseteq \mathcal{F}_R$ is open-compact in $\langle \mathcal{F}_R, \tau_L\rangle$ if and only if $U \in \mathfrak{B}$.*
(3) *A subset $Y \subseteq \mathcal{F}_R$ is closed in $\langle \mathcal{F}_R, \tau_L\rangle$ if and only if there exists a filter $F \in \mathrm{Fi}(A)$ such that $Y = \{R(x) : R(x) \subseteq \Phi(F)\}$.*

Proof: (1) We consider H_a with $a \in A$. Assume that $H_a \subseteq \bigcup\{H_b : b \in B \subseteq A\}$. If $x \in h_R(\beta(a))^c$ then $R(x) \nsubseteq \beta(a)$. Later, $R(x) \cap \beta(a)^c \neq \emptyset$ and $R(x) \in H_a \subseteq \bigcup\{H_b : b \in B \subseteq A\}$. So, there exists $b \in B$ such that $R(x) \in H_b$, i.e., $R(x) \cap \beta(b)^c \neq \emptyset$. Thus, $R(x) \in \bigcup\{h_R(\beta(b))^c : b \in B \subseteq A\}$. The converse is similar.

(2) Suppose that $U \subseteq \mathcal{F}_R$ is an open-compact of $\langle \mathcal{F}_R, \tau_L\rangle$. Since $\mathfrak{B}$ is a basis, we have $U = \bigcup\{H_b : b \in B \subseteq A\}$. As U is compact, there exist $b_1, \dots, b_n \in B$ such that $U = H_{b_1} \cup \dots \cup H_{b_n} = H_{b_1\wedge\dots\wedge b_n}$. Thus, $U \in \mathfrak{B}$. The other direction is immediate.

(3) Let $Y \subseteq \mathcal{F}_R$ be a closed set of $\langle \mathcal{F}_R, \tau_L \rangle$. Then $Y^c = \mathcal{F}_R - Y$ is open. Since $\mathfrak{B}$ is a basis, then $Y^c = \bigcup\{H_b : b \in B \subseteq A\}$, or equivalently, $Y = \bigcap\{M_b : b \in B \subseteq A\}$. Take the filter $F = F(B)$. Let $R(x) \in Y$, then for all $b \in B$, $R(x) \in M_b$, i.e., $R(x) \subseteq \beta(b)$. So, $R(x) \subseteq \bigcap\{\beta(b) : b \in B \subseteq A\} = \Phi(F)$. The reciprocal is analogous. □

The following result motivates our next definition.

Lemma 40. *Let $A, B \in \mathcal{DS}$. Let $h : A \to B$ be an onto homomorphism. Then $\langle \mathcal{F}_{R_h}, \tau_L \rangle$ is a DS-space which is homeomorphic to $X(B)$.*

Proof: By Lemma 39, $\mathfrak{B} = \{H_a : a \in A\}$ is the set of all open-compact subsets of $\langle \mathcal{F}_{R_h}, \tau_V \rangle$. It is clear that $\langle \mathcal{F}_{R_h}, \tau_V \rangle$ is T_0. We prove that if $Y \subseteq \mathcal{F}_{R_h}$ is a closed subset of $\langle \mathcal{F}_{R_h}, \tau_L \rangle$ and $L = \{H_a : a \in D\}$ is a dually directed family such that $Y \cap H_a \neq \emptyset$ for all $H_a \in L$, then $Y \cap \bigcap\{H_a : H_a \in L\} \neq \emptyset$.

As Y is closed, by Lemma 39 there exists a filter $F \in \mathrm{Fi}(A)$ such that

$$Y = \{R_h(P) : R_h(P) \subseteq \Phi(F)\}.$$

Consider the filter F' generated by the set $h[F] = \{h(f) : f \in F\}$ and also the set $I = \{h(c) : \exists H_a \in L(c \leq a)\}$. Since L is a dually directed family, we have that I is an order-ideal of B. We prove $F' \cap I = \emptyset$. Suppose that there exists $e \in F' \cap I$. Then there exist $f_1, \ldots, f_n \in F$ and $a \in D$ such that $h(f_1 \wedge \cdots \wedge f_n) \leq h(e) \leq h(a)$. As $Y \cap H_a \neq \emptyset$, there exists $P \in X(B)$ such that $R_h(P) \cap \beta(a)^c \neq \emptyset$ and $R_h(P) \subseteq \Phi(F)$. Later, $P \notin h_{R_h}(\beta(a))$. Since $f_1, \ldots, f_n \in F$, we get $R_h(P) \subseteq \beta(f_1 \wedge \cdots \wedge f_n)$, i.e., $P \in h_{R_h}(\beta(f_1 \wedge \cdots \wedge f_n)) \subseteq h_{R_h}(\beta(e)) \subseteq h_{R_h}(\beta(a))$ which is a contradiction. Thus, $F \cap I = \emptyset$. Then there exists $Q \in X(B)$ such that $h[F] \subseteq Q$ and $Q \cap I = \emptyset$. So, $F \subseteq h^{-1}(Q)$ and $a \notin h^{-1}(Q)$, for any $a \in D$. Then $R_h(Q) \subseteq \Phi(F)$ and $R_h(Q) \in \bigcap\{H_a : H_a \in L\}$. This implies that $Y \cap \bigcap\{H_a : H_a \in L\} \neq \emptyset$. Thus, we obtain that $\langle \mathcal{F}_{R_h}, \tau_L \rangle$ is a DS-space.

We show that $\langle \mathcal{F}_{R_h}, \tau_V \rangle$ is homeomorphic to $\langle X(B), \tau \rangle$. We define the application

$$f : X(B) \to \mathcal{F}_{R_h}$$

by

$$f(P) = R_h(P).$$

Let $P, Q \in X(B)$ be such that $R_h(P) = R_h(Q)$ but $P \neq Q$. We can suppose that $P \nsubseteq Q$, i.e., $Q \notin \mathrm{cl}(P)$. Then there exists $a \in B$ such that $P \in \beta(a)$ and $Q \notin \beta(a)$. Since h is onto, by Theorem 36, there exists $b \in B$ such that $\beta(a) \subseteq h_R(\beta(b))$ and $R_h(Q) \nsubseteq \beta(b)$. As $P \in \beta(a) \subseteq h_R(\beta(b))$, we get that $R_h(P) = R_h(Q) \subseteq \beta(b)$, which is a contradiction. Thus $P = Q$. It is clear that f is onto. Thus f is a bijection. Moreover, for $a \in A$ and $P \in X(B)$ we have $P \in f^{-1}[H_a]$ if and only if $f(P) \in H_a$, i.e., $R_h(P) \cap \beta(a)^c \neq \emptyset$. Then $R_h(P) \nsubseteq \beta(a)$ and $P \notin h_{R_h}(\beta(a)) = \beta(h(a))$. Thus, $P \in \beta(h(a))^c$. Consequently,

$f^{-1}[H_a] = \beta(h(a))^c$ and so f is continuous. We prove that f is an open map. Let $b \in B$. As h is onto, there exists $a \in A$ such that $h(a) = b$. Then

$$\begin{array}{rclcl} f\left[\beta(b)^c\right] & = & \{f(P) : P \in \beta(b)^c\} & = & \{R_h(P) : P \in \beta(h(a))^c\} \\ & = & \{R_h(P) : P \notin h_R(\beta(a))\} & = & \{R_h(P) : R_h(P) \nsubseteq \beta(a)\} \\ & = & \{R_h(P) : R_h(P) \cap \beta(a) \neq \emptyset\} & = & H_a. \end{array}$$

So, f is open. Therefore, f is a homeomorphism. □

Definition 41. Let $\langle X, \tau\rangle$ be a DS-space. We say that a family $\mathcal{F}$ of non-empty closed sets of $\langle X, \tau\rangle$ is a *lower Vietoris family* if $\langle \mathcal{F}, \tau_L\rangle$ is a DS-space.

Let $A \in \mathcal{DS}$. If $\mathcal{F} \subseteq \mathcal{C}(X(A))$ is a lower Vietoris family, then by Lemma 39 the family $\mathfrak{B} = \{H_a : a \in A\}$ is the set of all open-compact subsets of $\langle \mathcal{F}, \tau_L\rangle$ and $D_{\mathfrak{B}}(\mathcal{F}) = \{M_a : a \in A\}$ is the dual distributive semilattice.

For a lower Vietoris family $\mathcal{F} \subseteq \mathcal{C}(X(A))$ we define a relation $R_{\mathcal{F}} \subseteq \mathcal{F} \times X(A)$ by

$$(Y, P) \in R_{\mathcal{F}} \text{ iff } P \in Y.$$

Lemma 42. *Let $A \in \mathcal{DS}$ and let $\mathcal{F}$ be a lower Vietoris family of $\langle X(A), \beta[A]^c\rangle$. Then $R_{\mathcal{F}} \subseteq \mathcal{F} \times X(A)$ is a one-to-one meet-relation.*

Proof: First we show that $R_{\mathcal{F}} \subseteq \mathcal{F} \times X(A)$ is a meet-relation. Suppose that $Y \in \mathcal{F}$, $P \in X(A)$ and $(Y, P) \notin R_{\mathcal{F}}$. Then $P \notin Y$, and as $Y = \bigcap\{\beta(a) : Y \subseteq \beta(a)\}$, there exists $a \in A$ such that $P \notin \beta(a)$ and $Y \subseteq \beta(a)$. Thus, there is $a \in A$ such that $P \notin \beta(a)$ and $R_{\mathcal{F}}(Y) \subseteq \beta(a)$. Then $R_{\mathcal{F}}$ is a closed relation.

Now, let $a \in A$ and $Y \in \mathcal{F}$. We prove that $h_{R_{\mathcal{F}}}(\beta(a)) = M_a \in D_{\mathfrak{B}}(\mathcal{F})$ for each $a \in A$. Let $a \in A$ and $Y \in h_{R_{\mathcal{F}}}(\beta(a))$. Then $R_{\mathcal{F}}(Y) \subseteq \beta(a)$. Later $\{P \in X(A) : P \in R_{\mathcal{F}}(Y)\} \subseteq \beta(a)$ and by definition of $R_{\mathcal{F}}$ we have $\{P \in X(A) : P \in Y\} \subseteq \beta(a)$. So, $Y \subseteq \beta(a)$, which implies $Y \in M_a$. By backward reasoning, we have that $h_{R_{\mathcal{F}}}(\beta(a)) = M_a$. Therefore, $R_{\mathcal{F}}$ is a meet-relation.

We show that $R_{\mathcal{F}}$ is one-to-one. Let $Y \in \mathcal{F}$ such that $Y \notin M_a$. As $R_{\mathcal{F}}(Y) = Y$, we have that $R_{\mathcal{F}}(Y) \nsubseteq \beta(a)$, i.e., $Y \notin h_{R_{\mathcal{F}}}(\beta(a))$. On the other hand, it is clear that $M_a \subseteq h_{R_{\mathcal{F}}}(\beta(a))$. Thus, $R_{\mathcal{F}}$ is one-to-one. □

Lemma 43. *Let $A \in \mathcal{DS}$ and $\langle X, \tau\rangle$ be a DS-space. Then:*

(1) *if $R \subseteq X \times X(A)$ is a one-to-one meet-relation, then for each $x \in X$ and $P \in X(A)$ we have*

$$(x, P) \in R \text{ iff } (R(x), P) \in R_{\mathcal{F}_R};$$

(2) *if $\mathcal{F} \subseteq \mathcal{C}(X(A))$ is a lower Vietoris family, then $\mathcal{F} = \mathcal{F}_{R_{\mathcal{F}}}$.*

Proof: (1) Let $x \in X$ and $P \in X(A)$. Then $(R(x), P) \in R_{\mathcal{F}_R}$ if and only if $P \in R(x)$, i.e., $(x, P) \in R$.

(2) Let $F \in \mathcal{F}_{R_{\mathcal{F}}}$. Then there exists $G \in \mathcal{F}$ such that $F = R_{\mathcal{F}}(G)$, but as $R_{\mathcal{F}}(G) = G$, we have $F \in \mathcal{F}$ and $\mathcal{F}_{R_{\mathcal{F}}} = \mathcal{F}$. □

Since homomorphic images of a distributive semilattice A are dually characterized by one-to-one meet-relations of $X(A)$ then, by Theorem 36 and Lemmas 40, 42 and 43 together, we obtain:

Theorem 44. *Let $A \in \mathcal{DS}$. Then the homomorphic images of an A are dually characterized by lower Vietoris families on $X(A)$.*

Remark 45. Let $A, B \in \mathcal{DS}$ and $h : A \to B$ be a homomorphism. We say that B is a *distributive homomorphic image* of A if h is onto. Suppose that B is a distributive homomorphic image of A. We consider the set

$$\ker h = \{(a, b) \in A^2 : h(a) = h(b)\}.$$

It is easy to see that $\ker h$ is a semilattice congruence of A. By the Homomorphism Theorem of Universal Algebra we have that the quotient semilattice $A/\ker h$ is isomorphic to B. Then $A/\ker h$ is a distributive semilattice and thus $\ker h$ is a distributive congruence of A. Conversely, let θ be a distributive congruence of A. By Example 37, we have that $q_\theta : A \to A/\theta$ is onto and the quotient semilattice A/θ is a distributive homomorphic image of A. Then there is an isomorphism between distributive congruences and distributive homomorphic images. Also, by Theorem 44, there exists a isomorphism between distributive congruences and lower Vietoris families.

Acknowledgments. We would like to thank the referee for the comments and suggestions on the paper.

References

[1] Balbes R., *A representation theory for prime and implicative semilattices*, Trans. Amer. Math. Soc. **136** (1969), 261–267.

[2] Bezhanishvili G., Jansana R., *Priestley style duality for distributive meet-semilattices*, Studia Logica **98** (2011), 83–122.

[3] Bezhanishvili G., Jansana R., *Generalized Priestley quasi-orders*, Order **28** (2011), 201–220.

[4] Celani S.A., *Topological representation of distributive semilattices*, Sci. Math. Japonicae online **8** (2003), 41–51.

[5] Chajda I., Halaš R., Kühr J., *Semilattice Structures*, Research and Exposition in Mathematics, 30, Heldermann Verlag, Lemgo, 2007.

[6] Cornish W.H., Hickman R.C., *Weakly distributive semilattices*, Acta Math. Acad. Sci. Hungar. **32** (1978), 5–16.

[7] Grätzer G., *General Lattice Theory*, Birkhäuser, Basel, 1998.

[8] Gierz G., Hofmann K.H., Keimel K., Lawson J.D., Mislove M.W., Scott D.S., *Continuous Lattices and Domains*, Cambridge University Press, Cambridge, 2003.

[9] Hickman R.C., *Mildly distributive semilattices*, J. Austral. Math. Soc. Ser. A **36** (1984), 287–315.

[10] Mandelker M., *Relative annihilators in lattices*, Duke Math. J. **37** (1970), 377–386.

[11] Priestley H.A., *Representation of distributive lattices by means of ordered Stone spaces*, Bull. London Math. Soc. **2** (1970), 186–190.

[12] Rhodes J.B., *Modular and distributive semilattices*, Trans. Amer. Math. Soc. **201** (1975), 31–41.

[13] Stone M., *Topological representation of distributive lattices and Brouwerian logics*, Časopis pěst. mat. fys. **67** (1937), 1–25.
[14] Varlet J.C., *A generalization of the notion of pseudo-complementedness*, Bull. Soc. Roy. Sci. Liège **37** (1968), 149–158.
[15] Varlet J.C., *Distributive semilattices and Boolean lattices*, Bull. Soc. Roy. Liège **41** (1972), 5–10.
[16] Varlet J.C., *Relative annihilators in semilattices*, Bull. Austral. Math. Soc. **9** (1973), 169–185.
[17] Varlet J.C., *On separation properties in semilattices*, Semigroup Forum **10** (1975), 220–228.

CONICET and Departamento de Matemáticas, Facultad de Ciencias Exactas, Univ. Nac. del Centro, Pinto 399, 7000 Tandil, Argentina

E-mail: scelani@exa.unicen.edu.ar
calomino@exa.unicen.edu.ar

(*Received* October 30, 2012*, revised* February 27, 2013)