Extending the ideal of nowhere dense subsets of rationals to a P-ideal

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Abstract. We show that the ideal of nowhere dense subsets of rationals cannot be extended to an analytic P-ideal, F_σ ideal nor maximal P-ideal. We also consider a problem of extendability to a non-meager P-ideals (in particular, to maximal P-ideals).

Keywords: P-ideal; nowhere dense set; extension; analytic ideal; maximal ideal; meager ideal; ideal convergence

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Our notation and terminology conform to that used in the most recent set-theoretic literature. The cardinality of a set X is denoted by $|X|$ and $\overline{X}$ means the closure of X. The set of all natural numbers is denoted by ω. An *ideal on* ω is a family of subsets of ω closed under taking finite unions and subsets of its elements. If not explicitly said we assume that an ideal is proper ($\neq \mathcal{P}(\omega)$) and contains all finite sets. If $\mathcal{I}$ is an ideal then by $\mathcal{I}^+$ and $\mathcal{I}^\star$ we mean a coideal and a dual filter to $\mathcal{I}$, i.e. $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$ and $\mathcal{I}^\star = \{\omega \setminus A : A \in \mathcal{I}\}$. We can talk about ideals on any other countable set by identifying this set with ω via a fixed bijection. We say that $A \subseteq \omega$ is *almost contained* in $B \subseteq \omega$ ($A \subseteq^\star B$ in symbols) if $A \setminus B$ is finite. An ideal $\mathcal{I}$ is a *P-ideal* if for every family $\{A_n : n \in \omega\}$ of sets from $\mathcal{I}$ there is an $A \in \mathcal{I}$ such that $A_n \subseteq^\star A$ for all n.

By nwd we denote the ideal of nowhere dense subsets of rationals, that is,

$$\mathsf{nwd} = \{A \subseteq \mathbb{Q} : A \text{ is nowhere dense in } \mathbb{R}\}.$$

It is not difficult to observe that nwd is not a P-ideal. Indeed, for every rational $r \in \mathbb{Q}$ let $(q_n^r)_{n\in\omega}$ be a one-to-one sequence of rationals which is convergent to r. Let $A_r = \{q_n^r : n \in \omega\}$ and suppose that nwd is a P-ideal. Then there is a set $A \in \mathsf{nwd}$ which almost contains every set A_r. But that means that the set A has points which are arbitrarily close to any rational, so it is dense in $\mathbb{Q}$ — a contradiction.

In [2], Dow proved that it is consistent with ZFC (in particular it holds under the Continuum Hypothesis) that the ideal nwd can be extended to a P-ideal (his construction of this extension is implicit in the proof of [2, Theorem 3.4].) In the same paper, Dow asked a question (see also Dow's Questions [9, Question 12]) whether the ideal nwd can be extended to a P-ideal in ZFC.

In Section 1 we show that nwd cannot be extended in ZFC to an analytic P-ideal. In Section 2 we note that nwd cannot be extended to an F_σ ideal or maximal P-ideal. In Section 3 we show that under the Continuum Hypothesis the ideal nwd can be extended to a non-meager P-ideal. In this section we also give some necessary conditions for ideals which can be extended to a non-meager P-ideal.

1. Analytic ideals

By identifying sets of naturals with their characteristic functions, we equip $\mathcal{P}(\omega)$ with the Cantor-space topology and therefore we can assign the topological complexity to the ideals of sets of integers. For instance an ideal $\mathcal{I}$ is *analytic* if it is a continuous image of a G_δ subset of the Cantor space.

A map $\phi\colon \mathcal{P}(\omega) \to [0,\infty]$ is a *submeasure on* ω if $\phi(\emptyset) = 0$ and $\phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B)$ for all $A, B \subseteq \omega$. It is *lower semicontinuous* if $\phi(A) = \lim_{n\to\infty} \phi(A \cap \{0,1,\ldots,n\})$ for all $A \subseteq \omega$. For any lower semicontinuous submeasure on ω, let $\|\cdot\|_\phi\colon \mathcal{P}(\omega) \to [0,\infty]$ be the submeasure defined by $\|A\|_\phi = \lim_{n\to\infty} \phi(A \setminus \{0,1,\ldots,n\})$. It is easy to see that $\mathrm{Exh}(\phi) = \{A \subseteq \omega : \|A\|_\phi = 0\}$ is an ideal (not necessarily proper) for an arbitrary submeasure ϕ. All analytic P-ideals are characterized by the following theorem of Solecki.

Theorem 1.1 ([7]). *$\mathcal{I}$ is an analytic P-ideal $\iff$ $\mathcal{I} = Exh(\phi)$ for some lower semicontinuous submeasure ϕ on ω.*

The following theorem will be essential to prove that nwd cannot be extended to an analytic P-ideal.

Theorem 1.2. *Let $\mathcal{I}$ be an analytic P-ideal on $\mathbb{Q}$. For every countable set $C \subseteq \mathbb{R} \setminus \mathbb{Q}$ there exists a set $X \subseteq \mathbb{Q}$ such that $X \notin \mathcal{I}$ and $|\overline{X} \cap C| \leq 1$.*

Proof: Let ϕ be a lower semicontinuous submeasure with $\mathcal{I} = \mathrm{Exh}(\phi)$. Let $\mathbb{Q} = \{q_i : i \in \omega\}$ and $C = \{c_i : i \in \omega\}$. Let

$$\alpha = \lim_n \phi(\mathbb{Q} \setminus \{q_i : i < n\}) > 0.$$

We have two cases:

(1) for each $n \in \omega$ there exists set V_n open in $\mathbb{R}$ such that $c_n \in V_n$ and $\phi(V_n \cap \mathbb{Q}) < \frac{\alpha}{2^{n+2}}$;
(2) there is $\beta > 0$ and $N \in \omega$ such that for any open set $V \ni c_N$, $\phi(V \cap \mathbb{Q}) > \beta$.

In the first case we can take $X = \mathbb{Q} \setminus \bigcup_{n\in\omega} V_n$. Clearly, $\overline{X} \cap C = \emptyset$. Since ϕ is lower semicontinuous we have

$$\phi\left(\bigcup_{n\in\omega} V_n \cap \mathbb{Q}\right) \leq \sum_{n\in\omega} \phi(V_n \cap \mathbb{Q}) \leq \sum_{n\in\omega} \frac{\alpha}{2^{n+2}} = \frac{\alpha}{2}$$

so

$$\left\| \bigcup_{n\in\omega} V_n \cap \mathbb{Q} \right\|_\phi \leq \frac{\alpha}{2}.$$

On the other hand

$$\alpha = \|\mathbb{Q}\|_\phi \leq \|X\|_\phi + \left\| \bigcup_{n\in\omega} V_n \cap \mathbb{Q} \right\|_\phi \leq \|X\|_\phi + \frac{\alpha}{2},$$

hence $\|X\|_\phi > 0$, so $X \notin \mathcal{I}$. Therefore, we will assume the second case in the sequel. Moreover, we will assume that $N = 0$. We define a sequence $(U_n)_n$ of open subsets of $\mathbb{R}$, and a sequence $(X_n)_n$ of subsets of $\mathbb{Q}$ such that for each natural number $n > 0$:

(1) $c_0 \notin \overline{U_n}$ and $\{c_1, c_2, \ldots, c_n\} \subseteq U_n$;
(2) $\bigcup_{1\leq i\leq n} X_i \cap \overline{U_n} = \emptyset$ and $X_n \cap \bigcup_{1\leq i<n} \overline{U_i} = \emptyset$;
(3) $X_n \subseteq \mathbb{Q} \setminus \{q_i : i \leq n\}$ and $\phi(X_n) > \beta$.

Let U_1 be an open set such that $c_0 \notin \overline{U_1}$ and $c_1 \in U_1$. Then by lower semicontinuity of ϕ there is a finite set $X_1 \subseteq \mathbb{Q}\setminus(\{q_0\}\cup\overline{U_1})$ such that $\phi(X_1) > \beta$. Inductively, let U_n be an open set with

$$\{c_1, c_2 \ldots, c_n\} \subseteq U_n \quad\text{and}\quad \left(\{c_0\} \cup \bigcup_{1\leq i<n} X_i\right) \cap \overline{U_n} = \emptyset.$$

Then there is a finite set

$$X_n \subseteq \mathbb{Q} \setminus \left(\{q_i : i \leq n\} \cup \bigcup_{i\leq n} \overline{U_i}\right)$$

with $\phi(X_n) > \beta$. Let $X = \bigcup_{n\in\omega} X_n$. Since $\phi(X_n) > \beta$ and $X_n \cap \{q_0, q_1, \ldots, q_n\} = \emptyset$ for each n, $\|X\| \geq \beta$ hence $X \notin \mathcal{I}$. By (2), $X \cap \bigcup_{n\in\omega} U_n = \emptyset$, and so by (1) $\overline{X} \cap C \subseteq \{c_0\}$. □

Remark. Let $c \in \mathbb{R}\setminus\mathbb{Q}$ and $(t_i)_{i\in\omega}$ be a sequence of rationals convergent to c. Let $T = \{t_i \colon i \in \omega\}$ and

$$\mathcal{I} = \{A \subseteq \mathbb{Q} : A \cap T \text{ is finite}\}.$$

$\mathcal{I}$ is an analytic P-ideal and for every C with $c \in C$, if $X \notin \mathcal{I}$ then $c \in \overline{X} \cap C$. So, in Theorem 1.2 we cannot assert that $\overline{X} \cap C = \emptyset$.

Remark. In Theorem 1.2 we can replace $\mathbb{R}$ with any Hausdorff topological space and $\mathbb{Q}$ with any countable set (not necessarily dense).

Note that if $\mathcal{I}$ is an ideal, $\mathsf{nwd} \subseteq \mathcal{I}$ and $X \notin \mathcal{I}$ then $\overline{X}$ contains an interval, and so $\overline{X} \cap C$ is infinite for every dense set C. So, we have the following corollary.

Corollary 1.3. *There is no analytic P-ideal $\mathcal{I}$ such that* $\mathsf{nwd} \subseteq \mathcal{I}$.

In [2], Dow considered the assertion Mel: "there are disjoint countable dense subsets A, B of $\mathbb{R} \setminus \mathbb{Q}$ and a P-ideal $\mathcal{I}$ on $\mathbb{Q}$ such that for each $X \in \mathcal{I}^+$ both $\overline{X} \cap A$ and $\overline{X} \cap B$ are non-empty". He proved that Mel is consistent with ZFC. The following corollary shows that an ideal $\mathcal{I}$ in Mel cannot be analytic.

Corollary 1.4. *Let $\mathcal{I}$ be an analytic P-ideal on $\mathbb{Q}$. For every disjoint countable dense sets $A, B \subseteq \mathbb{R} \setminus \mathbb{Q}$ there exists a set $X \subseteq \mathbb{Q}$ such that $X \notin \mathcal{I}$ and either $\overline{X} \cap A = \emptyset$ or $\overline{X} \cap B = \emptyset$.*

PROOF: Use Theorem 1.2 with $C = A \cup B$. $\square$

2. Maximal ideals and ideals with Bolzano-Weierstrass property

An ideal $\mathcal{I}$ satisfies FinBW (*finite Bolzano-Weierstrass property*) if for any bounded sequence $(x_n)_{n\in\omega}$ of reals there is an $A \in \mathcal{I}^+$ such that $(x_n)_{n\in A}$ is convergent ([3]). By the well-known Bolzano-Weierstrass theorem the ideal of finite subsets of ω satisfies FinBW. By a folklore argument the same is true for every maximal P-ideal. In [3] we have shown that every F_σ ideal satisfies FinBW. In the same paper we have also shown that the ideal nwd does not possess Bolzano-Weierstrass property.

Proposition 2.1 ([3, Proposition 4.1])**.** *If $\mathcal{I}, \mathcal{J}$ are ideals, $\mathcal{I} \subseteq \mathcal{J}$ and $\mathcal{J}$ satisfies FinBW then $\mathcal{I}$ satisfies FinBW.*

Hence, nwd cannot be extended to an ideal with FinBW property. In particular we get the following corollary.

Corollary 2.2. *The ideal nwd cannot be extended to any F_σ ideal or to a maximal P-ideal.*

Remark. In [10] Zapletal proved that if an analytic ideal $\mathcal{J}$ can be extended to a maximal P-ideal $\mathcal{I}$ then there is an F_σ ideal $\mathcal{K}$ such that $\mathcal{J} \subseteq \mathcal{K} \subseteq \mathcal{I}$.

3. Non-meager ideals

All ideals with Baire property are characterized by the following theorem proved independently by Jalali-Naini [4] and Talagrand [8].

Theorem 3.1 ([4], [8])**.** *The following conditions are equivalent for an ideal $\mathcal{I}$ on ω:*

(1) *$\mathcal{I}$ has the Baire property;*
(2) *$\mathcal{I}$ is meager;*
(3) *there exists $n_0 < n_1 < \dots$ such that for every $A \in \mathcal{I}$*

$$\exists_{N\in\omega} \forall_{k>N} \, \{n_k + 1, \dots, n_{k+1}\} \not\subseteq A.$$

Theorem 3.2. *Assume the Continuum Hypothesis. There exists a non-meager P-ideal $\mathcal{I}$ such that $\mathsf{nwd} \subseteq \mathcal{I}$.*

PROOF: Fix a bijection $\sigma\colon \omega \to \mathbb{Q}$ identifying $\mathbb{Q}$ with ω. Let $\mathsf{nwd} = \{A_\alpha : \alpha \in \omega_1\}$, and $\{(n_k^\alpha)_{k\in\omega} : \alpha \in \omega_1\}$ be a family of all increasing sequences of naturals. Firstly, we will construct a sequence $\{G_\alpha : \alpha \in \omega_1\}$ such that

(1) G_α is dense in $\mathbb{Q}$ for each $\alpha < \omega_1$,
(2) $G_\beta \subseteq^\star G_\alpha$ for $\alpha < \beta < \omega_1$,
(3) $G_\alpha \cap A_\alpha = \emptyset$ for each $\alpha < \omega_1$,
(4) $G_\alpha \cap \sigma[\{n_k^\alpha + 1, \dots, n_{k+1}^\alpha\}] = \emptyset$ for infinitely many k.

Let $\{B_n : n \in \omega\}$ be a basis of the topology on $\mathbb{Q}$. Suppose that we have already constructed sets G_β for $\beta < \alpha$. Let

$$\begin{cases} \{H_n : n \in \omega\} = \{G_\beta : \beta < \alpha\} & \text{if } \alpha > 0, \\ H_n = \mathbb{Q} \text{ for each } n \in \omega & \text{if } \alpha = 0. \end{cases}$$

For every $n \in \omega$ we take

$$b_n \in B_n \cap H_0 \cap H_1 \cap \dots \cap H_n \cap \sigma\left[\left\{n_{k(n)+1}^\alpha + 1, n_{k(n)+1}^\alpha + 2, \dots\right\}\right],$$

where $k(n) = \min\{k\colon \{b_0, b_1, \dots, b_{n-1}\} \subseteq \sigma[\{0, 1, \dots, n_k^\alpha\}]\}$. (Recall that since $\{H_n\}_n$ is almost-decreasing, $H_0 \cap H_1 \cap \dots \cap H_n$ is dense in $\mathbb{Q}$.) We put

$$G_\alpha = \{b_n : n \in \omega\} \setminus A_\alpha.$$

Note that, for all $n \in \omega$,

$$G_\alpha \cap \sigma\left[\left\{n_{k(n)}^\alpha + 1, \dots, n_{k(n)+1}^\alpha\right\}\right] = \emptyset.$$

Define

$$\mathcal{I} = \{A \subseteq \mathbb{Q} : |A \cap G_\alpha| < \omega \text{ for some } \alpha\}.$$

Since $\mathcal{I} \supset \mathsf{nwd}$ it is enough to show that $\mathcal{I}$ is a non-meager P-ideal. First we show that $\mathcal{I}$ is a P-ideal. Indeed, let $\{C_n : n \in \omega\}$ be a countable family of sets from $\mathcal{I}$. For every $n \in \omega$ there is $\alpha_n < \omega_1$ with $|C_n \cap G_{\alpha_n}| < \omega$. Let $\alpha = \sup_n \alpha_n$. Then $|C_n \cap G_\alpha| < \omega$ for each n, and so $C_n \subseteq^\star \omega \setminus G_\alpha \in \mathcal{I}$ for every $n \in \omega$.

Next, observe that for each increasing sequence $(n_k)_{k\in\omega} = (n_k^\alpha)_{k\in\omega}$ there exists $A = \mathbb{Q} \setminus G_\alpha \in \mathcal{I}$ such that $\sigma[\{n_k^\alpha + 1, \dots, n_{k+1}^\alpha\}] \subseteq A$ for infinitely many k. Thus, by Theorem 3.1, $\mathcal{I}$ cannot be meager. $\square$

Problem 1. The authors do not know if it is possible to prove that nwd can be extended to a meager P-ideal (under CH, for example).

Using notation of Laflamme ([6]), the game $\mathcal{G}(\mathcal{X}, [\omega]^{<\omega}, \mathcal{Y})$ is played by two players I and II as follows: at stage $k < \omega$, I chooses $X_k \in \mathcal{X}$, then II responds with finite $s_k \subseteq X_k$. At the end of the game, II is declared the winner if $\bigcup_k s_k \in \mathcal{Y}$.

Lemma 3.3 ([6, Th. 2.15])**.** *Player I has no winning strategy in $\mathcal{G}(\mathcal{I}^\star, [\omega]^{<\omega}, \mathcal{I}^\star)$ if and only if $\mathcal{I}$ is a non-meager P-ideal.*

Let

$$\mathsf{Fin} \times \mathsf{Fin} = \{A \subseteq \omega \times \omega : (\exists N \in \omega)(\forall n > N)\ \{k : (n,k) \in A\} \text{ is finite}\}.$$

We say that an ideal $\mathcal{I}$ *contains an ideal isomorphic to the ideal* $\mathsf{Fin} \times \mathsf{Fin}$ if there exists a bijection $\sigma \colon \omega \to \omega \times \omega$ such that $\sigma^{-1}[A] \in \mathcal{I}$ whenever $A \in \mathsf{Fin} \times \mathsf{Fin}$.

Lemma 3.4 ([5, Lemma 2]). *Player I has a winning strategy in $\mathcal{G}(\mathcal{I}^\star, [\omega]^{<\omega}, \mathcal{I}^+)$ if and only if $\mathcal{I}$ contains an ideal isomorphic to* $\mathsf{Fin} \times \mathsf{Fin}$.

For a given ideal $\mathcal{I}$, Debs and Saint Raymond in [1] defined the rank of $\mathcal{I}$. In particular, the $\operatorname{rank}(\mathcal{I}) \leq 1$ if and only if $\mathcal{I}$ can be separated from its dual filter by an F_σ set, i.e. if there exists an F_σ set $\mathcal{K}$ such that $\mathcal{I} \subseteq \mathcal{K}$ and $\mathcal{I}^\star \cap \mathcal{K} = \emptyset$.

Lemma 3.5 ([1, Theorem 7.5]). *If $\mathcal{I}$ is an analytic ideal then* $\operatorname{rank}(\mathcal{I}) \leq 1$ *if and only if $\mathcal{I}$ does not contain an ideal isomorphic to* $\mathsf{Fin} \times \mathsf{Fin}$.

Proposition 3.6. *If $\mathcal{I}$ is a P-ideal which is non-meager then every analytic ideal $\mathcal{J} \subseteq \mathcal{I}$ can be separated from its dual filter by an F_σ set, i.e.* $\operatorname{rank}(\mathcal{J}) \leq 1$.

PROOF: Consider two games $\mathcal{G}_1 = \mathcal{G}(\mathcal{I}^\star, [\omega]^{<\omega}, \mathcal{I}^\star)$ and $\mathcal{G}_2 = \mathcal{G}(\mathcal{I}^\star, [\omega]^{<\omega}, \mathcal{I}^+)$. By Lemma 3.3, I has no winning strategy for $\mathcal{G}_1$. Since the game $\mathcal{G}_2$ is easier for II, I has also no winning strategy for $\mathcal{G}_2$. Thus, by Lemma 3.4, $\mathcal{I}$ does not contain an ideal isomorphic to $\mathsf{Fin} \times \mathsf{Fin}$. Hence $\mathcal{J}$ does not contain an ideal isomorphic to $\mathsf{Fin} \times \mathsf{Fin}$, and thus, by Lemma 3.5, $\mathcal{J}$ can be separated from its dual filter by an F_σ set. $\square$

Recall that nwd is an analytic ideal and $\operatorname{rank}(\mathsf{nwd}) = 1$ (see [5]).

References

[1] Debs G., Saint Raymond J., *Filter descriptive classes of Borel functions*, Fund. Math. **204** (2009), no. 3, 189–213.

[2] Dow A., *The space of minimal prime ideals of $C(\beta\mathbf{N} - \mathbf{N})$ is probably not basically disconnected*, General Topology and Applications (Middletown, CT, 1988), Lecture Notes in Pure and Appl. Math., 123, Dekker, New York, 1990, pp. 81–86.

[3] Filipów R., Mrożek N., Recław I., Szuca P., *Ideal convergence of bounded sequences*, J. Symbolic Logic **72** (2007), no. 2, 501–512.

[4] Jalali-Naini S.M., *The monotone subsets of Cantor space, filters and descriptive set theory*, PhD Thesis, Oxford, 1976.

[5] Laczkovich M., Recław I., *Ideal limits of sequences of continuous functions*, Fund. Math. **203** (2009), no. 1, 39–46.

[6] Laflamme C., *Filter games and combinatorial properties of strategies*, Set theory (Boise, ID, 1992–1994), Contemp. Math., 192, Amer. Math. Soc., Providence, RI, 1996, pp. 51–67.

[7] Solecki S., *Analytic ideals and their applications*, Ann. Pure Appl. Logic **99** (1999), no. 1-3, 51–72.

[8] Talagrand M., *Compacts de fonctions mesurables et filtres non mesurables*, Studia Math. **67** (1980), no. 1, 13–43.

[9] van Mill J., Reed G.M. (eds.), *Open Problems in Topology*, North-Holland Publishing Co., Amsterdam, 1990.

[10] Zapletal J., *Preserving P-points in definable forcing*, Fund. Math. **204** (2009), no. 2, 145–154.

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