M-weak and L-weak compactness of b-weakly compact operators

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Abstract. We characterize Banach lattices under which each b-weakly compact (resp. b-AM-compact, strong type (B)) operator is L-weakly compact (resp. M-weakly compact).

Keywords: b-weakly compact operator; b-AM-compact operator; strong type (B) operator; order continuous norm; positive Schur property

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1. Introduction

The class of b-weakly compact operators was introduced by Alpay, Altin and Tonyali in [4] on vector lattices. After that, a series of papers, which gave different characterizations of this class of operators, were published [2], [3], [5], [6], [7].

Many relations between this class and other classes of operators was studied in [13], [14], [16]. In fact, in [14] the authors studied the b-weak compactness of semi-compact operators, and in [13] the authors studied the b-weak compactness of order weakly compact (resp. AM-compact) operators. Also, the compactness of b-weakly compact operator was studied in [16]. On the other hand, the M-weak compactness and the L-weak compactness of weakly compact operator was investigated in [17]. Also, Aqzzouz, Elbour and H'Michane [9] characterize Banach lattices on which each Dunford-Pettis operator is M-weakly compact (resp. L-weakly compact). After that, in [12] the authors characterize Banach lattices on which each semi compact operator is M-weakly compact (resp. L-weakly compact).

Our aim in this paper is to study the M-weak compactness and the L-weak compactness of b-weakly compact (resp. strong type (B), resp. b-AM-compact) operators. The article is organized as follows: we give in preliminaries all common notations and definitions of Banach lattice theory. In main results section, we study in the first subsection the L-weak compactness of b-weakly compact (resp. b-AM-compact, strong type (B)) operators and in the second subsection the M-weak compactness of b-weakly compact (resp. b-AM-compact, strong type (B)) operators.

2. Preliminaries

Let us recall from [4] that an operator T from a Banach lattice E into a Banach space X is said to be b-weakly compact if it carries each b-order bounded subset of E (i.e., order bounded in E'') into a relatively weakly compact subset of X. Recall from [10] that an operator defined from a Banach lattice E into a Banach space X is said to be b-AM-compact if it carries b-order bounded set of E into norm relatively compact set of X.

Note that each b-AM-compact operator from a Banach lattice E into a Banach space X is b-weakly compact but the converse is not true in general. In fact, the identity operator of the Banach lattice $L^1[0,1]$ is b-weakly compact (because $L^1[0,1]$ is a KB-space, see [2, Proposition 2.1]) but it is not b-AM-compact (because $L^1[0,1]$ is not a discrete KB-space, see [10, Proposition 2.3]). Moreover, if E' is discrete then the class of b-weakly compact operators coincides with that of b-AM-compact operators (see [18, Theorem 3]).

An operator T defined from a Banach lattice E into a Banach space X is said to be strong type (B) if $T''(B) \subset X$ where B is the band generated by E in E''.

Since E'' is Dedekind complete, every band in E'' is a projection band and in particular there is a projection of E'' onto B. Thus, strong type (B) operators extend to E''. It is easy to see that each strong type (B) operator is a b-weakly compact operator but the converse is not true in general. Indeed, for $p > 1$ the operator $T_p : X_p \longrightarrow c_0$ mentioned in [19] does not preserve any copy of c_0 and it follows from Proposition 2.10 of [15] that the operator T_p is b-weakly compact. On the other hand, T_p is not a strong type (B) operator. Otherwise, since the Banach lattice X_p does not contain a complemented copy of ℓ^1 then, the norm of $(X_p)'$ is order continuous and hence it follows from [8, Proposition 3.2] that the operator T_p is weakly compact, which is impossible. For more details on strong type (B) operators, we refer the reader to [8], [19], [20].

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. Note that if E is a Banach lattice, its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice.

A Banach lattice E is said to have the positive Schur property if every weakly convergent sequence to 0 in E^+ is norm convergent to zero. For example, the Banach space ℓ^1 has the positive Schur property. A Banach lattice E is called a KB-space whenever every increasing norm bounded sequence of E^+ is norm convergent. As an example, each reflexive Banach lattice is a KB-space. A nonzero element x of a vector lattice E is discrete if the order ideal generated by x equals the lattice subspace generated by x. The vector lattice E is discrete, if it admits a complete disjoint system of discrete elements. A subset A of a vector lattice E is called order bounded, if it is included in an order interval in E. A linear mapping T from a vector lattice E into another F is order bounded if it carries an order bounded set of E into an order bounded set of F. We will use the term operator $T : E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. The operator T is regular if $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F. Note that each positive linear mapping on a Banach lattice is continuous. If an operator $T : E \longrightarrow F$ between two Banach lattices is positive, then its adjoint $T' : F' \longrightarrow E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$. For terminology concerning Banach lattice theory and positive operators we refer the reader to the excellent book of Aliprantis-Burkinshaw [1].

3. Main results

3.1 L-weak compactness of b-weakly compact operator. Recall that a non-empty bounded subset A of a Banach lattice E is said to be L-weakly compact if for every disjoint sequence (x_n) in the solid hull of A, we have $\lim_{n\to\infty} \|x_n\| = 0$. An operator T from a Banach space X into E is L-weakly compact if $T(B_X)$ is L-weakly compact in E, where B_X denotes the closed unit ball of X.

Note that any L-weakly compact operator from a Banach space into a Banach lattice is weakly compact ([1, Theorem 5.61]) and any weakly compact operator is clearly b-weakly compact, but there exists a b-weakly compact (resp. b-AM-compact, resp. strong type (B)) operator which is not L-weakly compact. In fact, the identity operator of the Banach lattice ℓ^2 is b-weakly compact (resp. b-AM-compact, resp. of strong type(B)), but it is not L-weakly compact. Also, the operator $T : C([0,1]) \longrightarrow c_0$ defined by:

$$T(f) = (\int_0^1 f r_n dt)_1^\infty \quad \text{for each} \quad f \in C([0,1]),$$

is weakly compact ([17, Example 4.4]) and hence is b-weakly compact, where r_n is the n-th Rademacher function on $[0,1]$, but T is not L-weakly compact ([17, Example 4.4]).

In the following result, we give the necessary conditions under which each b-weakly compact operator is L-weakly compact:

Theorem 3.1. *Let E and F be two Banach lattices. If each b-weakly compact operator $T : E \longrightarrow F$ is L-weakly compact, then one of the following assertions is valid:*

(1) $E = \{0\}$,
(2) *F is finite dimensional,*
(3) *the norms of E' and F are order continuous.*

Proof: The proof follows along the lines of the proof of Theorem 3.3 of [9]. We prove separately the two following assertions.

(a) If the norm of E' is not order continuous then F is finite-dimensional.
(b) If the norm of F is not order continuous, then $E = \{0\}$.

Assume that (a) is false. i.e., the norm of E' is not order continuous and F is infinite dimensional. It follows from Theorem 3.1 of [9] that there exists a disjoint norm bounded sequence (y_n) of F^+ which does not converge in norm to zero. And since the norm of E' is not order continuous, then it follows from Theorem 2.4.14 and Proposition 2.3.11 of [22] that E contains a sub-lattice isomorphic to ℓ^1 and there exists a positive projection $P : E \longrightarrow \ell^1$.

To finish the proof, we have to construct a b-weakly compact operator which is not L-weakly compact.

Consider the operator $S : \ell^1 \longrightarrow F$ defined by

$$S((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n y_n \quad \text{for each} \quad (\lambda_n) \in \ell^1.$$

The operator S is well defined and it is b-weakly compact because ℓ^1 is a KB-space (resp. ℓ^1 is a discrete KB-space, resp. S is b-weakly compact and ℓ^1 has an order continuous norm (see Proposition 2.11 of [4])). But S is not L-weakly compact. Otherwise, since $S(e_n) = y_n$ for all $n \geq 1$ where (e_n) is the canonical basis of ℓ^1 and (y_n) is a disjoint sequence, then (y_n) is norm convergent to zero and this is false.

On the other hand, since the identity operator of the Banach lattice ℓ^1 is b-weakly compact then the composed operator $T = S \circ P : E \longrightarrow \ell^1 \longrightarrow F$ is b-weakly compact because $S \circ P = S \circ Id_{\ell^1} \circ P$. But T is not L-weakly compact. Otherwise, $T \circ i = S$ is L-weakly compact where $\imath : \ell^1 \longrightarrow E$ is the canonical injection of ℓ^1 into E, and this is a contradiction.

Now, assume that (b) is false, i.e., the norm of F is not order continuous and $E \neq \{0\}$. Choose $z \in E^+$ such that $\|z\| = 1$. Hence, it follows from Theorem 39.3 of [21] that there exists $\phi \in (E')^+$ such that $\|\phi\| = 1$ and $\phi(z) = \|\phi\| = 1$.

On the other hand, since the norm of F is not order continuous, there exists some $y \in F^+$ and there exists a disjoint sequence $(y_n) \subset [0, y]$ which does not converge to zero in norm.

We consider the operator $T : E \to F$ defined by

$$T(x) = \phi(x) \cdot y \quad \text{for each} \quad x \in E.$$

It is clear that T is positive and compact (because its rank is one) and hence T is b-weakly compact. But T is not L-weakly compact. In fact, since $\|z\| = 1$ and $T(z) = \phi(z) \cdot y = y$ then $y \in T(B_E)$. As $(y_n) \subset [0, y]$, we conclude that (y_n) is a disjoint sequence in the solid hull of $T(B_E)$. Hence, if T is L-weakly compact then $\lim_{n\to\infty} \|y_n\| \to 0$, which is a contradiction. $\square$

Remark 1. The two necessary conditions (1) and (2) in Theorem 3.1 are sufficient, but the condition (3) is not. In fact, the identity operator of the Banach lattice ℓ^2 is b-weakly compact, but it is not L-weakly compact. However the norm of $(\ell^2)' = \ell^2$ is order continuous.

Remark 2. Since any strong type (B) operator is b-weakly compact and any b-AM-compact operator is b-weakly compact then the tree necessary conditions in Theorem 3.1 are also necessary if each strong type (B) operator $T : E \longrightarrow F$

is L-weakly compact or each b-AM-compact operator $T : E \longrightarrow F$ is L-weakly compact.

Now, we give sufficient conditions under which each strong type (B) operator is L-weakly compact:

Theorem 3.2. *Let E and F be two Banach lattices. Each strong type (B) operator T from E into F is L-weakly compact, if one of the following statements is valid:*

(1) $E = \{0\}$,
(2) *F is finite dimensional,*
(3) *E' has an order continuous norm and F has the positive Schur property.*

PROOF: (1) Obvious.

(2) Since F is finite dimensional, then it follows from Corollary 3.2 of [9] that T is L-weakly compact.

(3) Let $T : E \longrightarrow F$ be a strong type (B) operator then $T''(B) \subset F$ where B is the band generated by E in E''. As the norm of E' is order continuous, then it follows from Theorem 2.4.14 of [22] that $B = E''$ and hence T is weakly compact.

Now, since F has the positive Schur property, then by Theorem 3.4 of [17] T is L-weakly compact. □

Let us remark that if the norm of the Banach lattice E is order continuous then it follows from [4, Proposition 2.11] that the strong type (B) operators defined from E into an arbitrary Banach space coincide with b-weakly compact operators. On the other hand, all b-AM-compact operators are b-weakly compact.

As a consequence of Theorem 3.2, we give the following result:

Proposition 3.3. *Let E and F be two Banach lattices. Then each b-weakly compact (resp, b-AM-compact) operator $T : E \longrightarrow F$ is L-weakly compact, if one of the following statements is valid:*

(1) $E = \{0\}$,
(2) *F is finite dimensional,*
(3) *the norms of E' and E are order continuous and F has the positive Schur property.*

As a consequence of Theorem 3.1 and Proposition 3.3, we obtain the following characterization:

Corollary 3.4. *Let E be a Banach lattice with order continuous norm and F a Banach lattice with the positive Schur property. Then the following statements are equivalent.*

(1) *Each b-weakly compact operator $T : E \longrightarrow F$ is L-weakly compact.*
(2) *Each positive b-weakly compact operator $T : E \longrightarrow F$ is L-weakly compact.*
(3) *One of the following conditions is valid:*
   (a) $E = \{0\}$,

(b) *E' has an order continuous norm,*
(c) *F is finite dimensional.*

As another consequence of Theorem 3.1 and Theorem 3.2, we obtain the following characterization:

Corollary 3.5. *Let E and F be two Banach lattices such that F has the positive Schur property. Then the following statements are equivalent.*

(1) *Each strong type (B) operator T from E into F is L-weakly compact.*
(2) *One of the following conditions is valid:*
   (a) $E = \{0\}$,
   (b) *E' has an order continuous norm,*
   (c) *F is finite dimensional.*

Remark 3. As a particular case of Corollary 3.4 and Corollary 3.5, we have the following characterizations.

(1) Let E be a non-void Banach lattice with order continuous norm and F an infinite-dimensional Banach lattice with the positive Schur property. Each b-weakly compact operator $T : E \longrightarrow F$ is L-weakly compact, if and only if each positive b-weakly compact operator $T : E \longrightarrow F$ is L-weakly compact, if and only if E' has an order continuous norm.
(2) Let E be a non-void Banach lattice and F an infinite-dimensional Banach lattice with the positive Schur property. Then, each strong type (B) operator T from E into F is L-weakly compact, if and only if E' has an order continuous norm.

3.2 M-weak compactness of b-weakly compact operator. An operator $T : E \longrightarrow X$ from a Banach lattice E into a Banach space X is said to be M-weakly compact if for every disjoint sequence (x_n) in B_E we have $\lim_{n\to\infty} \|T(x_n)\| = 0$, where B_E denotes the closed unit ball of E.

Note that every M-weakly compact operator from a Banach lattice into a Banach space is weakly compact ([1, Theorem 5.61]) and any weakly compact operator is clearly b-weakly compact. But there exists a b-weakly compact (resp. b-AM-compact, resp. strong type (B)) operator which is not M-weakly compact. In fact, Id_{ℓ^1} is b-weakly compact (resp. b-AM-compact, resp. strong type (B)) but it is not M-weakly compact.

Our following result gives necessary conditions under which each b-weakly compact (resp. b-AM-compact, resp. strong type (B)) operator is M-weakly compact:

Theorem 3.6. *Let E and F be two Banach lattices. If each b-weakly compact (resp. b-AM-compact, resp. strong type (B)) operator $T : E \longrightarrow F$ is M-weakly compact, then one of the following assertions is valid:*

(1) $F = \{0\}$,
(2) *E' has an order continuous norm.*

Proof: Assume by way of contradiction that the norm of E' is not order continuous norm and $F \neq \{0\}$. To finish the proof, we have to construct a positive b-weakly compact operator $T : E \longrightarrow F$ (resp. b-AM-compact, resp. strong type (B)) operator which is not M-weakly compact. Since the norm of E' is not order continuous norm, it follows from Theorem 2.4.14 and Proposition 2.3.11 of Meyer-Nieberg [22] that E contains a closed sub-lattice which is isomorphic to ℓ^1 and there exists a positive projection $P : E \longrightarrow \ell^1$. On the other hand, as $F \neq \{0\}$, there exists a non-null element $y \in F^+$.

Now, we consider the operator $S : \ell^1 \longrightarrow F$ defined by

$$S((\lambda_n)) = \Big(\sum_{n=1}^{\infty} \lambda_n\Big) y \quad \text{for each} \quad (\lambda_n) \in \ell^1.$$

It is clear that S is well defined and positive. Also, S is compact (because its rank is one). Hence the positive operator

$$T = S \circ P : E \longrightarrow \ell^1 \longrightarrow F$$

is compact and T is b-weakly compact (resp. b-AM-compact; resp. strong type (B)) but it is not M-weakly compact. In fact, if we denote by (e_n) the canonical basis of $\ell^1 \subset E$, the sequence (e_n) is disjoint and bounded in E, moreover we have $T((e_n)) = y$ for each $n \geq 1$. Then $\|T((e_n))\| \nrightarrow 0$ (because $y \neq 0$). So, T is not M-weakly compact and this proves the result. □

Remark 4. The necessary condition (1) in Theorem 3.6 is sufficient, but the condition (2) is not. In fact, the identity operator of the Banach lattice ℓ^2 is b-weakly compact (resp. b-AM-compact, resp. strong type (B)) but is not M-weakly compact. However the norm of $(\ell^2)' = \ell^2$ is order continuous.

In the following result, we give sufficient conditions under which each b-weakly compact operator is M-weakly compact:

Theorem 3.7. *Let E and F be two Banach lattices.*

(1) *If $F = \{0\}$ or the norm of E is order continuous and E' has the positive Schur property then each b-weakly compact operator $T : E \longrightarrow F$ is M-weakly compact.*

(2) *If the norms of E and E' are order continuous and F has the positive Schur property then each regular b-weakly compact operator $T : E \longrightarrow F$ is M-weakly compact.*

Proof: (1) If $F = \{0\}$, clearly each operator is M-weakly compact. In the latter case, let $T : E \longrightarrow F$ be a b-weakly compact operator. Since the norm of E is order continuous and the norm of E' is order continuous (because E' has the positive Schur property), then it follows from the proof of Proposition 3.3 that T is weakly compact.

Now, since E' has the positive Schur property, it follows from [17, Theorem 3.3] that T is M-weakly compact.

(2) Let $T : E \longrightarrow F$ be an order bounded b-weakly compact operator. Since the norms of E and E' are order continuous and F has the positive Schur property,

then by Proposition 3.3 T is L-weakly compact. Therefore, by [1, Theorem 5.67] T is M-weakly compact. □

Now, we give sufficient conditions under which each operator of strong type (B) is M-weakly compact:

Theorem 3.8. *Let E and F be two Banach lattices.*

(1) *If $F = \{0\}$ or E' has the positive Schur property then each strong type (B) operator $T : E \longrightarrow F$ is M-weakly compact.*
(2) *If the norm of E' is order continuous and F has the positive Schur property then each regular strong type (B) operator $T : E \longrightarrow F$ is M-weakly compact.*

Proof: (1) If $F = \{0\}$, clearly each operator is M-weakly compact. In the latter case, let $T : E \longrightarrow F$ be a strong type (B) operator. Since the norm of E' is order continuous, then it follows from [8, Proposition 3.2] that T is weakly compact. Now, since E' has the positive Schur property, then by [17, Theorem 3.3] T is M-weakly compact.

(2) It follows from Theorem 3.6 of [17]. □

As a consequence of Theorem 3.6 and Theorem 3.8, we have the following characterization:

Corollary 3.9. *Let E and F be two Banach lattices such that F has the positive Schur property. Then the following statements are equivalent.*

(1) *Each regular operator T from E into F of strong type (B) is M-weakly compact.*
(2) *One of the following conditions is valid:*
   (a) $F = \{0\}$,
   (b) *E' has an order continuous norm.*

Remark 5. As a particular case of Corollary 3.9, we have the following characterization: Let E be a Banach lattice and F a non-void Banach lattice with the positive Schur property. Then, each regular strong type (B) operator $T : E \longrightarrow F$ is M-weakly compact if and only if E' has an order continuous norm.

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