On $(4,1)^*$-choosability of toroidal graphs without chordal 7-cycles and adjacent 4-cycles

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Abstract. A graph G is called $(k,d)^*$-choosable if for every list assignment L satisfying $|L(v)| = k$ for all $v \in V(G)$, there is an L-coloring of G such that each vertex of G has at most d neighbors colored with the same color as itself. In this paper, it is proved that every toroidal graph without chordal 7-cycles and adjacent 4-cycles is $(4,1)^*$-choosable.

Keywords: toroidal graph; defective choosability; chord

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1. Introduction

Graphs considered in this paper are finite, simple and undirected. Let $G = (V, E, F)$ be a graph, where V, E and F denote the set of vertices, edges and faces of G, respectively. We use $N_G(v)$ and $d_G(v)$ to denote the set and number of vertices adjacent to a vertex v, respectively, and use $\delta(G)$ to denote the minimum degree of G. A face of an embedded graph is said to be incident with all edges and vertices on its boundary. Two faces are adjacent if they share a common edge. The degree of a face f of G, denoted also by $d_G(f)$, is the number of edges incident with it, where each cut-edge is counted twice. When no confusion may occur, we write $N(v), d(v), d(f)$ instead of $N_G(v), d_G(v), d_G(f)$. A k-vertex (or k-face) is a vertex (or face) of degree k, a k^--vertex (or k^--face) is a vertex (or face) of degree at most k, and a k^+-vertex (or k^+-face) is a vertex (or face) of degree at least k. For $f \in F(G)$, we write $f = [u_1u_2 \cdots u_n]$ if u_1, u_2, $\cdots$, u_n are the vertices clockwisely lying on the boundary of f. An n-face $[u_1u_2u_3 \cdots u_n]$ is called an $(m_1, m_2, m_3, \cdots, m_n)$-face if $d(u_i)=m_i$ for $i = 1, 2, 3, \cdots, n$. A k-cycle is a cycle with k edges. Two cycles are adjacent if they share at least one common edge. A chord of a k-cycle ($k \geq 4$) is an edge joining two nonconsecutive vertices on C and a chordal cycle is a cycle with a chord.

A list assignment of G is a function L that assigns a list $L(v)$ of colors to each vertex $v \in V(G)$. An L-coloring with impropriety d for integer $d \geq 0$, or simply $(L,d)^*$-coloring, is a mapping ϕ that assigns a color $\phi(v) \in L(v)$ to each vertex $v \in V(G)$ such that v has at most d neighbors colored with $\phi(v)$. For integers $m \geq d \geq 0$, a graph G is called $(m,d)^*$-choosable, if G admits an $(L,d)^*$-coloring

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for every list assignment L with $|L(v)| = m$ for all $v \in V(G)$. An $(m, 0)^*$-choosable graph is simply called m-choosable.

The notion of list improper coloring was introduced independently by Škrekovski [5] and Eaton and Hull [3]. They proved that every planar graph is $(3, 2)^*$-choosable and every outerplanar graph is $(2, 2)^*$-choosable. Škrekovski proved in [6] that every planar graph without 3-cycles is $(3, 1)^*$-choosable, and in [7] that every planar graph G is $(2, 1)^*$-choosable if its girth $g(G) \geq 9$, $(2, 2)^*$-choosable if $g(G) \geq 7$, $(2, 3)^*$-choosable if $g(G) \geq 6$, and $(2, d)^*$-choosable if $g(G) \geq 5$ and $d \geq 4$. Lih et al. [4] proved that every planar graph without 4-cycles and l-cycles for some $l \in \{5, 6, 7\}$ is $(3, 1)^*$-choosable, Dong and Xu [8] showed that it is also true for some $l \in \{8, 9\}$. Cushing and Kierstead [2] constructively proved that every planar graph is $(4, 1)^*$-choosable which perfectly solved the last remaining question left open in [3], [5].

A graph G is toroidal if G can be drawn on the torus so that the edges meet only at the vertices of the graph. A face f is called 2-cell if any simple closed curve inside f can be continuously contracted to a single point. An embedding of G is called a 2-cell embedding if all the faces are 2-cell. We now assume that all embeddings considered in this paper are 2-cell embeddings.

For toroidal graphs, Zhang [10] proved that every graph G without 5- and 6-cycles is $(3, 1)^*$-choosable. Xu and Zhang [9] proved that every toroidal graph without adjacent triangles is $(4, 1)^*$-choosable. Chen *et al.* [1] proved that every graph embeddable in a surface of nonnegative characteristic without a 5-cycle with a chord or a 6-cycle with a chord is $(4, 1)^*$-choosable. Equivalently, every toroidal graph without chordal k-cycles for each $k \in \{4, 5, 6\}$ is $(4, 1)^*$-choosable.

Let $\mathcal{G}$ denote the family of toroidal graphs containing no chordal 7-cycles and adjacent 4-cycles. The main result is to show that every graph in $\mathcal{G}$ is $(4, 1)^*$-choosable. In order to prove the main theorem, we use the technique of discharging to obtain several forbidden configurations for the graphs in $\mathcal{G}$ and state as a theorem below.

Theorem 1. *For every graph $G \in \mathcal{G}$, one of the following must hold:*

(1) $\delta(G) < 4$;
(2) *G contains two adjacent 4-vertices;*
(3) *G contains a $(4, 5, 5)$-face.*

As a consequence of Theorem 1, we can prove the following Theorem 2.

Theorem 2. *Every toroidal graph without chordal 7-cycles and adjacent 4-cycles is $(4, 1)^*$-choosable.*

2. Proof of the theorems

In the proof of Theorem 1, we use the technique of discharging. In the beginning, each vertex v is assigned a charge $8d_G(v) - 24$ and each face f is assigned a charge $4d_G(f) - 24$. Using the Euler-Poincare formula $|V(G)| - |E(G)| + |F(G)| = 0$

and the well-known relation $\sum_{v\in V(G)} d(v) = \sum_{f\in F(G)} d(f) = 2|E(G)|$, we have

$$\sum_{v\in V(G)} \{8d_G(v) - 24\} + \sum_{f\in F(G)} \{4d_G(f) - 24\} = 0. \tag{1}$$

By the discharging rules stated in the proof of Theorem 1, we will redistribute the charges for the vertices and faces so that the total sum of the weights is kept constant while the transferring is in progress. However, once the transferring is finished, we get the new charges are nonnegative. Moreover, there exists some $x \in V(G) \cup F(G)$ such that $w'(x) > 0$, and then

$$0 < \sum_{x\in V(G)\bigcup F(G)} w'(x) = \sum_{x\in V(G)\bigcup F(G)} w(x) = 0. \tag{2}$$

This contradiction completes the proof of Theorem 1.

Proof of Theorem 1: Assume to the contrary that the theorem does not hold. Let G be such a connected graph in $\mathcal{G}$. Let ω be a weight on $V(G) \cup F(G)$ by defining $\omega(v) = 8d_G(v) - 24$ if $v \in V(G)$, and $\omega(f) = 4d_G(f) - 24$ if $f \in F(G)$. For two elements x and y of $V(G) \cup F(G)$, we use $\tau(x \to y)$ to denote the charge transferred from x to y.

By the choice of G, we have:

(O1) $\delta(G) \geq 4$;

(O2) every 4-vertex is adjacent to only 5^+-vertices;

(O3) G contains no $(4,5,5)$-faces;

(O4) G contains no chordal 7-cycle;

(O5) G contains no adjacent 4-cycles.

Since G has neither 7-cycle with a chord nor adjacent 4-cycles, we further derive the following facts.

(O6) For any 4^+-vertex v, there are no three consecutive triangles incident with v.

(O7) Each 4-face is adjacent to at most two 3-face, each 5-face is adjacent to at most one 3-face, each 3-face f is adjacent to at most one 4-face on $b(f)$.

The proof of (O6) is got by (O5) and the proof of (O7) is got by (O4) and (O5).

Note that (O6) implies that every 4^+-vertex v is incident with at most $\lfloor \frac{2}{3}d(v)\rfloor$ 3-faces.

Let v be a k-vertex and f be an l-face incident with v. Let $m_i(v)$ be the number of i-faces incident with v. The new charge function $w'(x)$ is obtained by the discharging rules given below.

(R1) For $k \geq 4$, $\tau(v \to f) = 2$ if $l = 4$ or 5.

(R2) For $k \geq 6$, $l = 3$, $\tau(v \to f) = 6$.

(R3) For $k = 5$, $l = 3$, $\tau(v \to f) = 5$.

(R4) For $k = 4$, $l = 3$, $\tau(v \to f) = 2$.

We now verify that $w'(x) \geq 0$ for any $x \in V(G) \cup F(G)$.

Let v be a k-vertex of G.

If $k = 4$, then by (R1) and (R4), $\omega'(v) \geq \omega(v) - 2 \cdot 4 = 0$.

If $k = 5$, then v is incident with at most three 3-faces by (O6).

If $m_3(v) \leq 1$, then $\omega'(v) \geq \omega(v) - 5 \cdot m_3(v) - 2 \cdot (m_4(v) + m_5(v)) \geq 16 - 5 - 8 = 3 > 0$ by (R1) and (R3).

If $m_3(v) = 2$, $\omega'(v) \geq \omega(v) - 5 \cdot m_3(v) - 2 \cdot (m_4(v) + m_5(v)) \geq 16 - 5 \cdot 2 - 2 \cdot 3 = 0$ by (R1) and (R3).

If $m_3(v) = 3$, then $m_4(v) = m_5(v) = 0$ by (O5-O7), so $\omega'(v) = \omega(v) - 5 \cdot 3 = 16 - 15 = 1 > 0$ by (R1) and (R3).

If $k = 6$, then $m_3(v) \leq 4$ by (O5). If $m_3(v) = 4$, then $m_4(v) = m_5(v) = 0$ by (O5)–(O7), and hence $\omega'(v) = \omega(v) - 6 \cdot 4 = 0$ by (R2). If $m_3(v) = 3$, then $m_4(v) + m_5(v) \leq d(v) - m_3(v) = 3$, and hence by (R1) and (R2)

$$\omega'(v) \geq \omega(v) - 6 \cdot 3 - 2 \cdot 3 = 0. \tag{3}$$

If $m_3(v) \leq 2$, then by (R1) and (R2)

$$\omega'(v) \geq \omega(v) - 6 \cdot 2 - 2 \cdot 4 = 4 > 0. \tag{4}$$

If $k = 7$, then $w(v) = 32$ and $m_3(v) \leq 4$ by (O6). We have $w'(v) \geq 32 - 6 \cdot 4 - 2 \cdot 3 = 2 > 0$ by (R1) and (R2).

If $k \geq 8$, $m_3(v) \leq \lfloor \frac{2d(v)}{3} \rfloor$ by (O5) and $m_4(v) + m_5(v) \leq d(v) - m_3(v)$, therefore by (R1) and (R2), we have $w'(v) = w(v) - 6 \cdot m_3(v) - 2 \cdot (m_4(v) + m_5(v)) \geq w(v) - 6 \cdot m_3(v) - 2 \cdot d(v) + 2 \cdot m_3(v) = w(v) - 4 \cdot m_3(v) - 2 \cdot d(v) \geq w(v) - 4 \cdot \lfloor \frac{2d(v)}{3} \rfloor - 2 \cdot d(v) = 8d_G(v) - 24 - \frac{8}{3} \cdot d(v) - 2 \cdot d(v) = \frac{10}{3} \cdot d(v) - 24 > 0$.

Let f be an h-face of G. The proof is divided into five cases according to the value of h.

Case 1. $h \geq 7$. Then $\omega'(f) = \omega(f) = 4h - 24 > 0$.

Case 2. $h = 6$. Then $\omega'(f) = \omega(f) = 4 \cdot 6 - 24 = 0$.

Case 3. $h = 5$. Then $w(f) = -4$ and $w'(f) = -4 + 2 \cdot 5 = 6 > 0$ by (O1) and (R1).

Case 4. $h = 4$. Then $w(f) = -8$ and $\omega'(f) = -8 + 2 \cdot 4 = 0$ by (O1) and (R1).

Let $n_i(f)$ denote the number of i-vertices incident with f for $i \geq 4$ and $n_{6^+}(f)$ denote the number of 6^+-vertices incident with f.

Case 5. $h = 3$. Then $w(f) = -12$. We write $f = [v_1 v_2 v_3]$.

If $n_{6^+}(f) \geq 2$, then $w'(f) \geq -12 + 6 \cdot 2 + 2 = 2 > 0$ by (O1), (R1) and (R2).

If $n_{6^+}(f) = 0$, then $n_4(f)+n_5(f) = 3$ by (O1). By (O2), $n_4(f) \leq 1$. Moreover, we have $n_4(f) = 0$, or a contradiction to (O3). So $d(v_1) = d(v_2) = d(v_3) = 5$ and

$$w'(f) \geq -12 + 3 \cdot 5 = 3 > 0. \tag{5}$$

So we suppose that $n_{6^+}(f) = 1$, then $n_5(f) \geq 1$ by (O2), hence $w'(f) \geq -12 + 6 \cdot 1 + 5 \cdot 1 + 2 \cdot 1 = 1 > 0$.

Now, we get that $\omega'(x) \geq 0$ for each $x \in V(G) \cup F(G)$. It follows that $0 \leq \sum_{x\in V(G)\cup F(G)} \omega'(x) = \sum_{x\in V(G)\cup F(G)} \omega(x) \leq 0$.

If $\sum_{x\in V(G)\cup F(G)} \omega'(x) > 0$, we are done. Assume that $\sum_{x\in V(G)\cup F(G)} \omega'(x) = 0$, so we have no 3-face by the proof. Then for every vertex v, we have $w'(v) \geq 8d - 24 - 2d = 6d - 24$. It is obvious that every vertex should be a 4-vertex, that is to say, G is a 4-regular toroidal graph, and this contradiction completes the proof of Theorem 1. $\square$

Proof of Theorem 2: Assume to the contrary. Let G be a 'counterexample' with the fewest vertices, i.e., G is a toroidal graph without chordal 7-cycles and adjacent 4-cycles that is not $(4,1)^*$-choosable, but any proper subgraph of G is $(4,1)^*$-choosable. It is certain that we may assume that G is connected.

Let L be a list assignment of G satisfying $|L(v)| = 4$ for all $v \in V(G)$ such that G is not $(L,1)^*$-choosable.

We will show that $\delta(G) \geq 4$, and G contains neither two adjacent 4-vertices nor a (4,5,5)-face. This contradiction to Theorem 1 will complete our proof.

If $\delta(G) < 4$, let v be a 3^--vertex of G. Then, $G - v$ is $(4,1)^*$-choosable by the choice of G. Since in any $(L,1)^*$-coloring of $G - v$, there must exist a color in $L(v)$ that is not used by any neighbors of v, any $(L,1)^*$-coloring of $G - v$ can be extended to a $(L,1)^*$-coloring of G, which is a contradiction.

If G contains two adjacent 4-vertices, say u and v, then by the choice of G, $G - \{u,v\}$ is $(4,1)^*$-choosable. By the same argument as above, we get G is $(4,1)^*$-choosable, a contradiction also.

If G contains a $(4,5,5)$-face $f = [xyz]$, we may assume that $d(x) = 4$ and $d(y) = d(z) = 5$. Let $H = G - \{x,y,z\}$. By the choice of G, H admits an $(L,1)^*$-coloring ϕ. For $w \in \{x,y,z\}$, let $L'(w) = L(w) \setminus \{\phi(u) | u \in N_H(w)\}$. Then, $|L'(x)| \geq 2$, $|L'(y)| \geq 1$ and $|L'(z)| \geq 1$. If $L'(y) = L'(z)$, then color y and z with a same color γ in $L'(y)$ and color x with a color in $L'(x) \setminus \{\gamma\}$. If $L'(y) \neq L'(z)$, then color y with a color $\alpha \in L'(y) \setminus L'(z)$, color z with a color in $L'(z)$, and color x with an arbitrary color in $L'(x)$. In either case, we get an $(L,1)^*$-coloring of G. This contradiction completes the proof of Theorem 2. $\square$

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References

[1] Chen Y., Zhu W., Wang W., *Improper choosability of graphs of nonnegative characteristic*, Comput. Math. Appl. **56** (2008), 2073–2078.

[2] Cushing W., Kierstead H.A., *Planar graphs are* 1*-relaxed,* 4*-choosable*, European J. Combin. **31** (2009), 1385–1397.

[3] Eaton N., Hull T., *Defective list colorings of planar graphs*, Bull. Inst. Comb. Appl. **25** (1999), 79–87.

[4] Lih K., Song Z., Wang W., Zhang K., *A note on list improper coloring planar graphs*, Appl. Math. Lett. **14** (2001), 269–273.

[5] Škrekovski R., *List improper colorings of planar graphs*, Comb. Prob. Comp. **8** (1999), 293–299.

[6] Škrekovski R., *A Grötzsch-type theorem for list colorings with impropriety one*, Comb. Prob. Comp. **8** (1999), 493–507.

[7] Škrekovski R., *List improper colorings of planar graphs with prescribed girth*, Discrete Math. **214** (2000), 221–233.

[8] Dong W., Xu B., *A note on list improper coloring of plane graphs*, Discrete Appl. Math. **28** (2009), 433–436.

[9] Xu B., Zhang H., *Every toroidal graph without adjacent triangles is* $(4,1)^*$*-choosable*, Discrete Appl. Math. **155** (2007), 74–78.

[10] Zhang L., *A* $(3,1)^*$*-choosable theorem on toroidal graphs*, Discrete Appl. Math. **160** (2012), 332–338.

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