Smoothness properties of solutions to the nonlinear Stokes problem with nonautonomous potentials

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Abstract. We discuss regularity results concerning local minimizers $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^n$ of variational integrals like

$$\int_\Omega \{F(\cdot, \varepsilon(w)) - f \cdot w\}\, dx$$

defined on energy classes of solenoidal fields. For the potential F we assume a (p, q)-elliptic growth condition. In the situation without x-dependence it is known that minimizers are of class $C^{1,\alpha}$ on an open subset Ω_0 of Ω with full measure if $q < p\,\frac{n+2}{n}$ (for $n = 2$ we have $\Omega_0 = \Omega$). In this article we extend this to the case of nonautonomous integrands. Of course our result extends to weak solutions of the corresponding nonlinear Stokes type system.

Keywords: Stokes problem; generalized Newtonian fluids; regularity; nonautonomous functionals; slow flows

Classification: 76M30, 76D07, 49N60, 35J50

1. Introduction

In the classical formulation the Stokes problem reads as follows (see [La, p. 35]): find a velocity field $v : \Omega \to \mathbb{R}^n$ and a pressure function $\pi : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \Delta v = \nabla \pi - f & \text{in } \Omega, \\ \operatorname{div} v = 0 & \text{in } \Omega, \\ v = v_0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω denotes a domain in $\mathbb{R}^n$ $(n \geq 2)$, $f : \Omega \to \mathbb{R}^n$ is a system of volume forces and $v_0 : \partial\Omega \to \mathbb{R}^n$ represents the boundary function. For results concerning existence and regularity of solutions of (1.1) we refer to [La]. If $F(\varepsilon) = \frac{1}{2}|\varepsilon|^2$, then solutions of (1.1) are clearly minimizers of

$$J[w] := \int_\Omega \{F(\varepsilon(w)) - f \cdot w\}\, dx \tag{1.2}$$

in a suitable function class of solenoidal fields.

A natural extension of this problem is to consider minimizers of (1.2) with potentials F being of power growth (compare [La, p. 192]), i.e. we have

$$\lambda(1 + |\varepsilon|^2)^{\frac{p-2}{2}}\, |\sigma|^2 \leq D^2F(\varepsilon)(\sigma, \sigma) \leq \Lambda(1 + |\varepsilon|^2)^{\frac{p-2}{2}}\, |\sigma|^2 \tag{1.3}$$

for all $\varepsilon, \sigma \in \mathbb{S}$ with positive constants λ, Λ and an exponent $p > 1$ ($\mathbb{S}$ is the space of symmetric $n \times n$-matrices and $\varepsilon(w)$ denotes the symmetric gradient). So we gain a nonlinear variant of the first equation in (1.1):

$$\operatorname{div}\{\nabla F(\varepsilon(v))\} = \nabla \pi - f \quad \text{in } \Omega.$$

For these power law models full interior $C^{1,\alpha}$-regularity in the 2D case has been proved by Kaplický, Málek and Stará [KMS] and Wolf [Wo], whereas the higher dimensional situation is studied for example in Naumann and Wolf [NW]. For partial regularity results in dimensions $n \geq 3$ we also refer to [FuGR] and [Fu]. Bildhauer and Fuchs [BF1] consider the same problem under anisotropic growth conditions, they assume

$$\lambda(1 + |\varepsilon|^2)^{\frac{p-2}{2}} |\sigma|^2 \leq D^2 F(\varepsilon)(\sigma, \sigma) \leq \Lambda(1 + |\varepsilon|^2)^{\frac{q-2}{2}} |\sigma|^2 \tag{1.4}$$

for exponents $1 < p \leq q < \infty$. It should be remarked that such a behavior of the potential F is suggested for example in Section 5.1 of the monograph [MNRR] of Málek, Nečas, Rokyta and Růžička. The result of the paper [BF1] is (partial) $C^{1,\alpha}$-regularity provided

$$q < p\,\frac{n+2}{n}\,. \tag{A1}$$

Bildhauer and Fuchs achieve the same result in [BF3] in the framework of classical variational calculus (note that full regularity theorems are not known for our type of variational problems unlike the studies in [BF3]). In this setting it is known since the work of [ELM1] that an extension to the nonautonomous situation is problematical if we require anisotropic growth conditions. Fuchs and Bildhauer [BF2] show regularity statements by supposing the stronger hypothesis

$$q < p\,\frac{n+1}{n} \tag{1.5}$$

which is a sharp bound under the assumptions stated there (note that this bound firstly appears in [ELM2]). In [Br2] we develop conditions concerning the density F (especially for the x-dependence) to close the gap between the autonomous and the nonautonomous situation. Here we extend this arguments to the case of variational problems of the form (1.2).

Firstly, we have to assume that it holds

$$F(x, \varepsilon) = g(x, |\varepsilon|) \tag{A2}$$

for a function $g : \Omega \times [0, \infty) \to [0, \infty)$ being C^2 in the real variables in order to introduce a suitable regularization of our problem. From the physical point of view this assumption seems to be quite natural. If (A2) holds, then (1.4) reads as

$$
\text{(A3)} \qquad \begin{aligned} \lambda(1+t^2)^{\frac{p-2}{2}} &\le \frac{g'(x,t)}{t} \le \Lambda(1+t^2)^{\frac{q-2}{2}}, \\ \lambda(1+t^2)^{\frac{p-2}{2}} &\le g''(x,t) \le \Lambda(1+t^2)^{\frac{q-2}{2}}. \end{aligned}
$$

Furthermore we suppose that

$$
\text{(A4)} \qquad |\partial_\gamma g''(x,t)| \le \Lambda_2 \left[g''(x,t)(1+t^2)^{\frac{\kappa}{2}} + (1+t^2)^{\frac{p+q}{4}-1} \right]
$$

is true for all $(x,t) \in \overline{\Omega} \times [0,\infty)$ and $\gamma \in \{1,\dots,n\}$ with $0 \le \kappa \ll 1$ as well as

$$
\text{(A5)} \qquad |\partial_\gamma^2 g''(x,t)| \le \Lambda_3 (1+t^2)^{\frac{q-2}{2}}.
$$

A typical example is

$$
\text{(1.6)} \qquad \int_\Omega \left\{ (1+|\varepsilon(w)|^2)^{\frac{\mu(x)}{2}} - f\cdot w \right\} dx \;\longrightarrow\; \min
$$

for a Lipschitz-function $\mu : \Omega \to (1,\infty)$ and it is easy to show the validity of all our conditions for this density. For an extensive list of potentials we refer to [Br2, Section 6], where one can find examples with a nontrivial x-dependence and an arbitrarily wide range of anisotropy. Note that the minimizing problem (1.6) subject to the constraint $\operatorname{div}(w) = 0$ naturally appears in the study of electrorheological fluids[1] (compare [Ru]). Regularity results are proved in [AM] and for $n = 2$ in [BF5] and [DER].

Now we state our main result concerning local minimizers of

$$
\text{(1.7)} \qquad \mathbb{J}[w] := \int_\Omega \{F(\cdot,\varepsilon(w)) - f\cdot w\}\, dx
$$

in the class

$$
\mathbb{K} := \left\{ w \in W^{1,p}_{loc}(\Omega,\mathbb{R}^n) : \operatorname{div} w = 0 \right\}.
$$

Theorem 1.1. *Under the assumptions* (A1)–(A5) *where all involved derivatives are supposed to be continuous and the volume force f is assumed to be sufficiently regular we have:*

(a) *for a local minimizer $u \in \mathbb{K}$ of* (1.7) *there is an open subset Ω_0 with full Lebesgue-measure such that u belongs to the space $C^{1,\alpha}(\Omega_0,\mathbb{R}^N)$ for any $\alpha \in (0,1)$ provided $q \ge 2$;*

(b) *if $n = 2$ and $q < p + 2$ we have $\Omega_0 = \Omega$.*

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[1] These are smart materials changing their rheological properties as a reaction to an electric field.

Remark 1.1. (i) From Theorem 1.1 we obtain partial regularity in 3D (increase q if necessary, see (A1)) if $p > 6/5$. For $n = 2$ the assumption $q \geq 2$ is no restriction at all. It is possible to include the case $q < 2$ if $n = 3$. Instead of the excess function given in Section 4 one has to work with

$$E(x,r) = \fint_{B_r(x)} |\varepsilon(u) - (\varepsilon(u))_{x,r}|^2 \, dy$$

which is well defined since $\nabla u \in L^{3p}_{loc}(\Omega, \mathbb{R}^{3\times 3})$ by Lemma 3.2. The blow up arguments in this situation are similar to [FuS].

(ii) We prove our result in the case $f \equiv 0$ for a technical simplification but an extension is easy if f is located in some appropriate Morrey space.

Remark 1.2. In the paper [AM] Acerbi and Mingione prove partial regularity for electrorheological fluids (see (1.6)) under the restriction $\inf \mu > \frac{9}{5}$ in the 3D situation but with appearance of the convective term $u \otimes u$ in the equation of motion. We can extend this result in case of slow flows: Theorem 1.1 and Remark 1.1 imply partial regularity provided $\inf \mu > 1$.

2. Auxiliary results

In this section we prove regularity statements for the nonautonomous isotropic situation. The following results should not be surprising but it is hard to find a reference in literature. We consider a function $G : \widetilde{\Omega} \times \mathbb{S} \to [0, \infty)$ satisfying

$$\begin{aligned} a(1+|\varepsilon|^2)^{\frac{p-2}{2}} |\tau|^2 \leq D^2_\varepsilon G(x,\varepsilon)(\tau,\tau) \leq A(1+|\varepsilon|^2)^{\frac{p-2}{2}} |\tau|^2, \\ |\partial_\gamma D_\varepsilon G(x,\varepsilon)| \leq A(1+|\varepsilon|^2)^{\frac{p-1}{2}}, \end{aligned} \tag{2.1}$$

for all $\varepsilon, \tau \in \mathbb{S}$, all $x \in \widetilde{\Omega}$ and all $\gamma \in \{1, \ldots, n\}$. Thereby $\widetilde{\Omega}$ denotes an open set in $\mathbb{R}^n$, we suppose $p \in (1, \infty)$ and a, A are positive constants.

Lemma 2.1. *Suppose that $v \in W^{1,p}_{loc}(\widetilde{\Omega}, \mathbb{R}^n)$ is a local minimizer of the energy $w \mapsto \int_{\widetilde{\Omega}} G(\cdot, \varepsilon(w)) \, dx$ subject to the constraint $\operatorname{div} w = 0$. Then we have*

(a) *$v \in W^{2,t}_{loc}(\widetilde{\Omega}, \mathbb{R}^n)$ for $t := \min\{2, p\}$;*

(b) *$(1 + |\varepsilon(v)|^2)^{\frac{p}{4}} \in W^{1,2}_{loc}(\widetilde{\Omega})$ together with*

$$\nabla \left\{ (1 + |\varepsilon(v)|^2)^{\frac{p}{4}} \right\} = \frac{p}{2}(1 + |\varepsilon(v)|^2)^{\frac{p}{4}-1} |\varepsilon(v)| \nabla |\varepsilon(v)|;$$

(c) *$D_\varepsilon G(\cdot, \varepsilon(v)) \in W^{1,p/(p-1)}_{loc}(\widetilde{\Omega}, \mathbb{S})$ and*

$$\partial_\gamma \left\{ D_\varepsilon G(\cdot, \varepsilon(v)) \right\} = \partial_\gamma D_\varepsilon G(\cdot, \varepsilon(v)) + D^2_\varepsilon G(\cdot, \varepsilon(v))(\partial_\gamma \varepsilon(v), \cdot), \quad \gamma = 1, \ldots, n.$$

Proof: The starting point is the Euler-Lagrange equation

$$\int_{\widetilde{\Omega}} D_\varepsilon G(\cdot, \varepsilon(v)) : \varepsilon(\varphi) \, dx = 0 \tag{2.2}$$

being valid for any $\varphi \in W^{1,p}(\widetilde{\Omega}, \mathbb{R}^n)$ with $\operatorname{div} \varphi = 0$ and compact support in $\widetilde{\Omega}$. From (2.2) Bildhauer and Fuchs [BF1] deduce in the autonomous case ($\Delta_h f$ is the difference quotient from f in the γ-th direction for $h \neq 0$)

$$(2.3) \quad \int_{B_{r'}} \eta^2 B_x(\varepsilon(\Delta_h v), \varepsilon(\Delta_h v))\, dx = \int_{B_{r'}} B_x(\varepsilon(\Delta_h v), h\varepsilon(\psi) - \nabla\eta^2 \odot \Delta_h v)\, dx.$$

Thereby we have $\eta \in C_0^\infty(B_R)$ for a ball $B_R \Subset \widetilde{\Omega}$ such that $\eta \equiv 1$ on B_r, $\eta \equiv 0$ outside of $B_{r'}$, $\eta \geq 0$ and $|\nabla\eta| \leq c/(r'-r)$ where $r < r' < R$. The function ψ belongs to the space $\mathring{W}^{1,p}(B_{r'}, \mathbb{R}^n)$ such that

$$\operatorname{div} \psi = \frac{1}{h} \nabla\eta^2 \Delta_h v,$$

together with

$$(2.4) \qquad \|\nabla\psi\|_p \leq \frac{c}{h} \left\|\nabla\eta^2 \Delta_h v\right\|_p.$$

In our situation B_x stands for the bilinear form

$$B_x := \int_0^1 D_\varepsilon^2 G(x + the_\gamma, \varepsilon(v)(x) + th\varepsilon(\Delta_h v)(x))\, dt.$$

In the autonomous situation one has

$$\Delta_h \left\{DG(\varepsilon(v))\right\}(x) = B_x(\varepsilon(\Delta_h v), \cdot).$$

Here we get on account of the x-dependence

$$\begin{aligned}\Delta_h \left\{D_\varepsilon G(x, \varepsilon(v)(x))\right\} = &\int_0^1 \partial_\gamma D_\varepsilon G(x + the_\gamma, \varepsilon(v)(x) + th\varepsilon(\Delta_h v)(x))\, dt \\ &+ B_x(\varepsilon(\Delta_h v), \cdot)\end{aligned}$$

where we abbreviate the linear form defined by the first integral on the r.h.s. by L_x. As a consequence we have to add

$$\int_{B_{r'}} L_x : \left[h\varepsilon(\psi) - \nabla\eta^2 \odot \Delta_h v - \varepsilon(\Delta_h v)\eta^2\right]\, dx$$

on the r.h.s. of (2.3). This leads us to the estimation of the following three integrals (using (2.1))

$$\begin{aligned}J_1 &:= \int_{B_{r'}} \int_0^1 (1 + |\varepsilon(v)(x) + th\varepsilon(\Delta_h v)(x)|^2)^{\frac{p-1}{2}} |h\varepsilon(\psi)|\, dtdx, \\ J_2 &:= \int_{B_{r'}} \int_0^1 (1 + |\varepsilon(v)(x) + th\varepsilon(\Delta_h v)(x)|^2)^{\frac{p-1}{2}} |\nabla\eta^2 \odot \Delta_h v|\, dtdx,\end{aligned}$$

$$J_3 := \int_{B_{r'}} \eta^2 \int_0^1 (1 + |\varepsilon(v)(x) + th\varepsilon(\Delta_h v)(x)|^2)^{\frac{p-1}{2}} |\varepsilon(\Delta_h v)| \, dtdx.$$

Considering J_1 one sees by Young's inequality and (2.1)

$$\begin{aligned} J_1 \le c \int_{B_{r'}} \int_0^1 (1 + |\varepsilon(v)(x) + th\varepsilon(\Delta_h v)(x)|^2)^{\frac{p}{2}} \, dtdx \\ + ch^2 \int_{B_{r'}} |B_x||\varepsilon(\psi)|^2 \, dx. \end{aligned}$$

Following [BF1] (calculations after (3.7)) we can bound both terms by

$$c(r' - r)^{-2} \left(1 + \int_{B_{R+h}} |\nabla v|^p \, dx \right)$$

where h is chosen sufficiently small. For J_2 we obtain

$$\begin{aligned} J_2 \le c(r' - r)^{-2} \int_{B_{r'}} \int_0^1 (1 + |\varepsilon(v)(x) + th\varepsilon(\Delta_h v)(x)|^2)^{\frac{p}{2}} \, dtdx \\ + c \int_{B_{r'}} |\Delta_h v|^p \, dx. \end{aligned}$$

On account of

$$\|\Delta_h v\|_{L^p(B_{r'})} \le \|\nabla v\|_{L^p(B_R)}$$

and $v \in W^{1,p}_{loc}(\widetilde{\Omega}, \mathbb{R}^n)$ we receive for J_2 the same estimation as for J_1, hence

$$J_1 + J_2 \le c(r' - r)^{-2} \left(1 + \int_{B_{R+h}} |\nabla v|^p \, dx \right). \tag{2.5}$$

Having a look at the last integral we obtain by Young's inequality

$$\begin{aligned} J_3 \le c(\delta) \int_{B_{r'}} \int_0^1 (1 + |\varepsilon(v)(x) + th\varepsilon(\Delta_h v)(x)|^2)^{\frac{p}{2}} \, dtdx \\ + \delta \int_{B_{r'}} \eta^2 \int_0^1 (1 + |\varepsilon(v)(x) + th\varepsilon(\Delta_h v)(x)|^2)^{\frac{p-2}{2}} |\varepsilon(\Delta_h v)|^2 \, dtdx \end{aligned}$$

for an arbitrary $\delta > 0$. Whereas the first term on the r.h.s. is bounded by the r.h.s. of (2.5), the last integral can be absorbed in the l.h.s. of (2.3) on account of (2.1). Let

$$\omega(r) := \int_{B_r} B_x(\varepsilon(\Delta_h v), \varepsilon(\Delta_h v)) \, dx$$

then the authors of [BF1] prove starting from (2.3) the inequality

$$\omega(r) \le \frac{1}{2}\omega(r') + c(r'-r)^{-2}\left(1+\int_{B_{R+h}} |\nabla v|^p\,dx\right). \tag{2.6}$$

We have some additional terms but as one sees in (2.5) they can be bounded by the r.h.s. of (2.6) as well and we can get the same inequality. From (2.6) we deduce by [Gi, Lemma 3.1, p. 161])

$$\omega(r) \le c(r'-r)^{-2}\left(1+\int_{B_{R+h}} |\nabla v|^p\,dx\right), \quad 0<r<r'\le R. \tag{2.7}$$

If $p \ge 2$ we have (compare (2.1))

$$\omega(r) \ge c|\varepsilon(\Delta_h v)|^2$$

and (2.7) implies (by quoting Korn's inequality) part (a) of Lemma 2.1 in this situation. If $p<2$ then $(\dots = \varepsilon(v)(x) + th\varepsilon(\Delta_h v)(x))$

$$\begin{aligned}\int_{B_r} |\varepsilon(\Delta_h v)|^p\,dx &= \int_{B_r}\int_0^1 (1+|\dots|^2)^{\frac{p-2}{2}\frac{p}{2}} |\varepsilon(\Delta_h v)|^p (1+|\dots|^2)^{\frac{2-p}{2}\frac{p}{2}}\,dtdx\\ &\le c\omega(r) + \int_{B_r}\int_0^1 (1+|\dots|^2)^{\frac{p}{2}}\,dx\\ &\le c\omega(r) + c\left(1+\int_{B_{R+h}} |\nabla v|^p\,dx\right).\end{aligned}$$

In this case we receive Lemma 2.1(a) by (2.7), too. With a minor modification in case $p<2$ we can quote part (b) from [BF1, p. 9]. Since we know $\partial_\gamma\varepsilon(v) \in L^t_{loc}(\widetilde{\Omega}, \mathbb{S})$ we have after passing to a subsequence a.e.

$$\Delta_h\varepsilon(v) \xrightarrow{h\to 0} \partial_\gamma\varepsilon(v).$$

Therefore we get a.e.

$$\begin{aligned} B_x(\varepsilon(\Delta_h v),\cdot) &\xrightarrow{h\to 0} D^2_\varepsilon G(x,\varepsilon(v))(\partial_\gamma\varepsilon(v),\cdot),\\ L_x &\xrightarrow{h\to 0} \partial_\gamma D_\varepsilon G(x,\varepsilon(v)),\end{aligned}$$

which means we obtain a.e.

$$\Delta_h\left\{DG(\varepsilon(v))\right\}(x) \xrightarrow{h\to 0} D^2_\varepsilon G(x,\varepsilon(v))(\partial_\gamma\varepsilon(v),\cdot) + \partial_\gamma D_\varepsilon G(x,\varepsilon(v)). \tag{2.8}$$

If we are able to bound $\Delta_h\{DG(\varepsilon(v))\}$ in $L^{p/(p-1)}_{loc}(\widetilde{\Omega}, \mathbb{S})$ we get together with (2.8) the claim of part (c) using [Mo, Theorem 3.6.8 b]. In addition to the calculations from [BF1] we only have to show a uniform $L^{p/(p-1)}_{loc}$-bound on L_x. We clearly get by Jensen's inequality and the growth of $\partial_\gamma D_\varepsilon$

$$\int_{B_R} |L_x|^{p/(p-1)}\, dx \le c \left(1 + \int_{B_{R+h}} |\nabla v|^p \, dx \right)$$

and the claim follows. $\square$

3. Regularization and higher integrability

First of all we present our regularization where the main ideas arise from [CGM]. For $M \gg 1$ let

$$g_M(x,t) := \begin{cases} g(x,t), & \text{for } 0 \le t \le M \\ g(x,M) + g'(x,M)(t-M) & \\ + \int_M^t \int_M^\rho g''(x,\tau) h(x,\tau) d\tau d\rho, & \text{for } t > M \end{cases}$$

and finally $F_M(x,\varepsilon) := g_M(x,|\varepsilon|)$. As proved partly in [BF2] and partly in [Br2] this function has the following properties if we suppose (A2)–(A5) and the continuity of the involving derivatives of g:

Lemma 3.1. (i) *$F_M(x,\varepsilon) \le F(x,\varepsilon)$ for all $\varepsilon \in \mathbb{S}$.*

(ii) *For $|\varepsilon| \le M$ is $F_M(x,\varepsilon) = F(x,\varepsilon)$.*

(iii) *$F_M(x,\varepsilon)$ growths isotropically: i.e.*

$$\overline{a}\, |\varepsilon|^p - \overline{b} \le F_M(x,\varepsilon) \le A_M\, |\varepsilon|^p + B_M$$

for all $\varepsilon \in \mathbb{S}$ with uniform constants $\overline{a} > 0$, $\overline{b} \in \mathbb{R}$ and constants A_M and B_M depending on M.

(iv) *$F_M(x,\varepsilon)$ is uniformly (p,q)-elliptic, which means we have for $\varepsilon, \tau \in \mathbb{S}$ and $\gamma \in \{1,\dots,n\}$*

$$\overline{\lambda}(1+|\varepsilon|^2)^{\frac{p-2}{2}} |\tau|^2 \le D^2_\varepsilon F_M(x,\varepsilon)(\tau,\tau) \le \Lambda_3 (1+|\varepsilon|^2)^{\frac{q-2}{2}} |\tau|^2,$$
$$|\partial_\gamma D_\varepsilon F_M(x,\varepsilon)| \le \Lambda_3 (1+|\varepsilon|^2)^{\frac{q-1}{2}}$$

with constants $\overline{\lambda}, \Lambda_3 > 0$.

(v) *$F_M(x,\varepsilon)$ is p-elliptic, i.e., for $\varepsilon, \tau \in \mathbb{S}$ we have*

$$\overline{\lambda}(1+|\varepsilon|^2)^{\frac{p-2}{2}} |\tau|^2 \le D^2_\varepsilon F_M(x,\varepsilon)(\tau,\tau) \le \Lambda_M (1+|\varepsilon|^2)^{\frac{p-2}{2}} |\tau|^2,$$
$$|\partial_\gamma D_\varepsilon F_M(x,\varepsilon)| \le \Lambda_M (1+|\varepsilon|^2)^{\frac{p-1}{2}}$$

with a uniform constant $\overline{\lambda}$ and a constant Λ_M depending on M.

(vi) *For all $\varepsilon, \tau \in \mathbb{S}$ it holds*

$$\left| \partial^2_\gamma D_\varepsilon F_M(x,\varepsilon) \right| \le \Lambda_4 (1+|\varepsilon|^2)^{\frac{q-1}{2}},$$

$$\left|\partial_\gamma D_\varepsilon^2 F_M(x,\varepsilon)(\tau,\varepsilon)\right| \le \Lambda_4 \left|D_\varepsilon^2 F_M(x,\varepsilon)(\tau,\varepsilon)\right| (1+|\varepsilon|^2)^{\frac{\kappa}{2}} \\ + \Lambda_4 (1+|\varepsilon|^2)^{\frac{p+q-2}{4}} |\tau|$$

uniformly in M with $\Lambda_4 \ge 0$.

With these preparations we define the regularization u_M of the problem (1.7) as the unique minimizer of (note we assume w.l.o.g. $f \equiv 0$)

$$\mathbb{J}_M[w] = \int_B F_M(\cdot, \varepsilon(w))\,dx$$

in $u + \mathring{W}^{1,p}(B,\mathbb{R}^n)$ subject to the constraint $\operatorname{div} w = 0$ with a ball $B = B_{2R} \Subset \Omega$. This is the solution of an isotropic problem and so we get the regularity statements from Lemma 2.1 for u_M. Now we want to prove

Lemma 3.2. *Under the assumptions of Theorem* 1.1 *we get*

$$\varepsilon(u) \in \begin{cases} L_{loc}^{\frac{pn}{n-2}}(\Omega,\mathbb{S}) & \text{if } n \ge 3 \\ L_{loc}^{s}(\Omega,\mathbb{S}), & \text{for all } s<\infty, \text{ if } n=2. \end{cases}$$

Also u belongs to the space $W^{2,t}_{loc}(\Omega,\mathbb{R}^n)$ for $t := \min\{p,2\}$.

For our proof we need an inequality of Caccioppoli-type:

Lemma 3.3. *Let $\Gamma_M := 1+|\varepsilon(u_M)|^2$. Then there is a constant $c>0$ independent of M such that*

$$\int_B \eta^2 \Gamma_M^{\frac{p-2}{2}} |\nabla\varepsilon(u_M)|^2\,dx \le c\,\|\nabla\eta\|_\infty^2 \int_{\operatorname{spt}\nabla\eta} \Gamma_M^{\frac{q}{2}}\,dx + c\int_{\operatorname{spt}\eta} \Gamma_M^{\frac{q}{2}}\,dx$$

for all $\eta \in C_0^1(B,[0,1])$.

Proof: We get (compare [BF1, 4.9], which is unaffected by the x-dependence)

$$\int_B \eta^2 \Delta_h \left\{D_\varepsilon F_M(\cdot,\varepsilon(u_M))\right\} : \varepsilon(\Delta_h u_M)\,dx \\ = -2\int_B \eta\Delta_h\tau_M : (\nabla\eta \odot \Delta_h[u_M - Qx])\,dx$$

and thereby with an obvious definition for B_x and L_x

$$\int_B \eta^2 B_x(\Delta_h\varepsilon(u_M),\Delta_h\varepsilon(u_M)) = -2\int_B \eta\Delta_h\tau_M : (\nabla\eta\odot\Delta_h[u_M-Qx])\,dx \\ - \int_B \eta^2 L_x : \Delta_h\varepsilon(u_M)\,dx. \tag{3.1}$$

Here $p_M \in W^{1,p/(p-1)}(B)$ is the pressure function such that

$$\nabla p_M = \operatorname{div}\sigma_M,$$

$$\sigma_M := D_\varepsilon F_M(\cdot, \varepsilon(u_M)),$$
$$\tau_M := \sigma_M - p_M I,$$

η is a suitable cut-off function and $Q \in \mathbb{S}$ an arbitrary matrix. The l.h.s. of (3.1) is non-negative and on account of convergence a.e. we get by Fatou's lemma

$$\int_B \eta^2 D^2_\varepsilon F_M(\partial_\gamma \varepsilon(u_M), \partial_\gamma \varepsilon(u_M))\, dx \leq \liminf_{h\to 0} |\text{r.h.s. of (3.1)}|. \tag{3.2}$$

Now we have to show, that we can change limes inferior and integral in the terms on the r.h.s. of (3.1). For the first term this is already established in [BF1]. Therefore we have to find an exponent $s > 1$ such that $L_x : \Delta_h \varepsilon(u_M)$ is uniformly bounded in L^s_{loc} (than we quote Vitali's convergence theorem). We have by Jensen's inequality for $r < 2R$

$$\begin{aligned}
&\int_{B_r} |L_x : \Delta_h \varepsilon(u_M)|^s\, dx\\
&\leq c(M) \int_{B_r}\int_0^1 (1 + |\varepsilon(u_M) + th\Delta_h \varepsilon(u_M)|^2)^{\frac{sp}{4}} \times\\
&\qquad\qquad (1 + |\varepsilon(u_M) + th\Delta_h \varepsilon(u_M)|^2)^{\frac{s(p-2)}{4}} |\Delta_h \varepsilon(u_M)|^s\, dtdx\\
&\leq c(M) \int_{B_r} B_x(\Delta_h \varepsilon(u_M), \Delta_h \varepsilon(u_M))\, dx + c(M) \left(1 + \int_{B_{r+h}} |\varepsilon(u_M)|^{\frac{sp}{2-s}}\, dx\right)
\end{aligned}$$

using Lemma 3.1(v). Choosing $s > 1$ small enough the r.h.s. is bounded (independent of h, but of course depending on M) on account of Lemma 2.1(b) in combination with Sobolev's inequality. Hence we can go to the limit on the r.h.s. of (3.1). If we remember Lemma 2.1 and its proof we get

$$\begin{aligned}
\lambda \int_B \eta^2 \Gamma_M^{\frac{p-2}{2}} |\nabla \varepsilon(u_M)|^2\, dx &\leq -2 \int_B \eta \partial_\gamma \tau_M : (\nabla \eta \odot \partial_\gamma [u_M - Qx])\, dx\\
&\quad - \int_B \eta^2 \partial_\gamma D_\varepsilon F_M(\cdot, \varepsilon(u_M)) : \partial_\gamma \varepsilon(u_M)\, dx
\end{aligned} \tag{3.3}$$

which corresponds to (4.10) in [BF1]. The first integral is bounded by

$$\left(\int_B \eta^2 |\nabla \tau_M|^2 \Gamma_M^{\frac{2-q}{2}}\, dx\right)^{\frac{1}{2}} \left(\int_B |\nabla \eta|^2 \Gamma_M^{\frac{q-2}{2}} |\nabla u_M - Q|^2\, dx\right)^{\frac{1}{2}}. \tag{3.4}$$

We get (sum over $\gamma \in \{1, \ldots, n\}$)

$$\begin{aligned}
|\nabla \sigma_M|^2 \Gamma_M^{\frac{2-q}{2}} &\leq c \Gamma_M^{\frac{2-q}{2}} \big[\partial_\gamma D_\varepsilon F_M(\cdot, \varepsilon(u_M)) : \partial_\gamma \sigma_M\\
&\qquad + D^2_\varepsilon F_M(\cdot, \varepsilon(u_M))(\partial_\gamma \varepsilon(u_M), \partial_\gamma \sigma_M)\big]\\
&\leq c \Gamma_M^{\frac{2-q}{4}} \Gamma_M^{\frac{q}{4}} |\nabla \sigma_M|
\end{aligned}$$

$$+ c\Gamma_M^{\frac{2-q}{4}} D_\varepsilon^2 F_M(\cdot,\varepsilon(u_M))(\partial_\gamma\varepsilon(u_M),\partial_\gamma\varepsilon(u_M))^{\frac{1}{2}}|\nabla\sigma_M|$$

by the formula for $\partial_\gamma\sigma_M$ given in Lemma 2.1 (remember the growth estimates in Lemma 3.1(iv)). Hence we obtain on account of $|\nabla\tau_M| \le c|\nabla\sigma_M|$

$$|\nabla\tau_M|\Gamma_M^{\frac{2-q}{4}} \le c\Gamma_M^{\frac{q}{4}} + cD_\varepsilon^2 F_M(\cdot,\varepsilon(u_M))(\partial_\gamma\varepsilon(u_M),\partial_\gamma\varepsilon(u_M))^{\frac{1}{2}}.$$

As a consequence we can bound the r.h.s. of (3.4) by ($\tau > 0$ arbitrary)

$$\tag{3.5} \begin{aligned} &\tau\int_B \eta^2 D_\varepsilon^2(\partial_\gamma\varepsilon(u_M),\partial_\gamma\varepsilon(u_M))\,dx \\ &\quad + c(\tau)\left(\int_B |\nabla\eta|^2\Gamma_M^{\frac{q-2}{2}}|\nabla u_M - Q|^2\,dx + \int_B \eta^2\Gamma_M^{\frac{q}{2}}\,dx\right). \end{aligned}$$

After absorption of the τ-term (remember (3.2)) we estimate the remaining integrals by (note $q \ge 2$)

$$\tag{3.6} \begin{aligned} &c\|\nabla\eta\|_\infty^2\left[\int_{\operatorname{spt}\nabla\eta}\Gamma_M^{\frac{q}{2}}\,dx + \int_{\operatorname{spt}\nabla\eta}|\nabla u_M - Q|^q\,dx\right] + c\int_{\operatorname{spt}\eta}\Gamma_M^{\frac{q}{2}}\,dx \\ &\le c\|\nabla\eta\|_\infty^2\int_{\operatorname{spt}\nabla\eta}\Gamma_M^{\frac{q}{2}}\,dx + c\int_{\operatorname{spt}\eta}\Gamma_M^{\frac{q}{2}}\,dx \end{aligned}$$

choosing Q as a suitable skew-symmetric matrix and using Korn's inequality. Now we have to estimate

$$\begin{aligned} I &:= -\int_B \eta^2\partial_\gamma D_\varepsilon F_M(\cdot,\varepsilon(u_M)) : \partial_\gamma\varepsilon(u_M)\,dx \\ &= \int_B \partial_\gamma\left\{\eta^2\partial_\gamma D_\varepsilon F_M(\cdot,\varepsilon(u_M))\right\} : \varepsilon(u_M)\,dx \\ &= \int_B \eta^2\partial_\gamma^2 D_\varepsilon F_M(\cdot,\varepsilon(u_M)) : \varepsilon(u_M)\,dx \\ &\quad + \int_B \eta^2\partial_\gamma D_\varepsilon^2 F_M(\cdot,\varepsilon(u_M))(\partial_\gamma\varepsilon(u_M),\varepsilon(u_M))\,dx \\ &\quad + \int_B \partial_\gamma D_\varepsilon F_M(\cdot,\varepsilon(u_M)) : \varepsilon(u_M)\partial_\gamma\eta^2\,dx \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

Lemma 3.1(vi) gives

$$I_1 \le c\int_{\operatorname{spt}\eta}\Gamma_M^{\frac{q}{2}}\,dx$$

and from Lemma 3.1(iv) we deduce

$$I_3 \le c\|\nabla\eta\|_\infty\int_{\operatorname{spt}\nabla\eta}\Gamma_M^{\frac{q}{2}}\,dx \le c\|\nabla\eta\|_\infty^2\int_{\operatorname{spt}\nabla\eta}\Gamma_M^{\frac{q}{2}}dx + c\int_{\operatorname{spt}\eta}\Gamma_M^{\frac{q}{2}}\,dx.$$

For I_2 we conclude from Lemma 3.1(vi)

$$I_2 \le c \int_B \eta^2 \left| D_\varepsilon^2 F_M(\cdot, \varepsilon(u_M))(\partial_\gamma \varepsilon(u_M), \varepsilon(u_M)) \right| (1 + |\nabla u_M|^2)^{\frac{\kappa}{2}} \, dx \\ + c \int_B \eta^2 \Gamma_M^{\frac{p+q-2}{4}} \left| \nabla \varepsilon(u_M) \right| dx.$$

We can bound the first integral by

$$\tau \int_B \eta^2 D_\varepsilon^2 F_M(\cdot, \varepsilon(u_M))(\partial_\gamma \varepsilon(u_M), \partial_\gamma \varepsilon(u_M)) \, dx \\ + c(\tau) \int_B \eta^2 D_\varepsilon^2 F_M(\cdot, \varepsilon(u_M))(\varepsilon(u_M), \varepsilon(u_M))(1 + |\varepsilon(u_M)|^2)^\kappa \, dx.$$

If we know

$$\kappa < \frac{1}{2} \left(p \, \frac{n+2}{n} - q \right),$$

we can increase q to $q + 2\kappa$ w.l.o.g. Now we can absorb the first term (see (3.2)) and bound the second one by

$$c \int_{\operatorname{spt} \eta} \Gamma_M^{\frac{q}{2}} \, dx.$$

For arbitrary $\tau > 0$ we obtain by Young's inequality

$$\int_B \eta^2 \Gamma_M^{\frac{p+q-2}{4}} \left| \nabla \varepsilon(u_M) \right| \, dx \le \tau \int_B \eta^2 \Gamma_M^{\frac{p-2}{2}} \left| \nabla \varepsilon(u_M) \right|^2 \, dx + c(\tau) \int_{\operatorname{spt} \eta} \Gamma_M^{\frac{q}{2}} \, dx$$

which we handle conventionally and we finally receive the inequality from Lemma 3.3. $\square$

Proof of Lemma 3.2: If we follow the lines of [BF1] (proof of Corollary 4.2) and [Br1] (proof of Lemma 2.1) we get by Lemma 3.3

$$\varepsilon(u_M) \in \begin{cases} L_{loc}^{\frac{pn}{n-2}}(B, \mathbb{S}) & \text{if } \ n \ge 3 \\ L_{loc}^{s}(B, \mathbb{S}), & \text{for all } s < \infty, \text{ if } \ n = 2 \end{cases} \tag{3.7}$$

uniformly. Note that the integrability of $\varepsilon(u_M)$ which we need is obtained by Lemma 2.1(b) and Sobolev's inequality. To transfer the integrability to the solution u we have to show the convergence $u_M \to u$. By a combination of Lemma 3.3 and the uniform $W_{loc}^{1,q}(B, \mathbb{R}^N)$-bound of u_M (see (3.7)) we obtain

$$\nabla \varepsilon(u_M) \in L_{loc}^{t}(B, \mathbb{S}^n) \ \text{ uniformly.} \tag{3.8}$$

Since u_M is a $\mathbb{J}_M$-minimizer on boundary data u we get uniform L^p-bounds for $\varepsilon(u_M)$ using Lemma 3.1(i), (iii). As a consequence we can bound u_M in

$W^{1,p}(B,\mathbb{R}^n)$ uniformly by Korn's inequality. Using Korn's inequality for another time we obtain by (3.8) for all $\gamma \in \{1,\dots,n\}$

$$\|\partial_\gamma u_M\|_{W^{1,t}} \le c\left\{\|\partial_\gamma u_M\|_{L^t} + \|\varepsilon(\partial_\gamma u_M)\|_{L^t}\right\} \le c,$$

hence we can get after passing to a subsequence

$$u_M \rightharpoondown: v \text{ in } W^{2,t}_{loc}(B,\mathbb{R}^N) \text{ and}$$
$$\nabla u_M \to \nabla v \text{ almost everywhere on } B$$

for a function $v \in W^{2,t}_{loc}(B,\mathbb{R}^N)$. As in [Br2] (end of Section 2) we can obtain $u = v$ and thereby the claim of Lemma 3.2. □

4. Partial regularity

As in [Br2, Section 3] we get

Lemma 4.1. *Let $H_M := \Gamma_M^{\frac{p}{4}}$, $\Gamma := 1 + |\varepsilon(u)|^2$ and $H := \Gamma^{\frac{p}{4}}$. Then we have*

- *$H \in W^{1,2}_{loc}(B)$,*
- *$H_M \rightharpoondown H$ in $W^{1,2}_{loc}(B)$ for $M \to \infty$ and*
- *$\varepsilon(u_M) \to \varepsilon(u)$ almost everywhere on B for $M \to \infty$.*
- *For $\eta \in C_0^\infty(B)$ and arbitrary balls $B \Subset \Omega$ we have*

$$\int_B \eta^2 |\nabla H|^2\, dx \le c\, \|\nabla \eta\|_\infty^2 \int_{\mathrm{spt}\,\nabla\eta} \Gamma^{\frac{q}{2}}\, dx + c \int_{\mathrm{spt}\,\eta} \Gamma^{\frac{q}{2}}\, dx.$$

We define

$$E(x,r) := \fint_{B_r(x)} |\varepsilon(u) - (\varepsilon(u))_{x,r}|^q\, dy + \fint_{B_r(x)} |\varepsilon(u) - (\varepsilon(u))_{x,r}|^2\, dy$$

where $\fint \dots$ and $(\dots)_{x,r}$ denote mean values and obtain

Lemma 4.2. *Fix $L > 0$. Then there exists a constant $C^*(L)$ such that for every $\tau \in (0, 1/4)$ there is an $\varepsilon = \varepsilon(\tau, L) > 0$ satisfying: if $B_r \Subset B_R$ and we have*

$$|(\varepsilon(u))_{x,r}| \le L, \quad E(x,r) + r^{\gamma^*} \le \varepsilon$$

then

$$E(x,\tau r) \le C^* \tau^2 [E(x,r) + r^{\gamma^*}].$$

Here $\gamma^* \in (0,2)$ is an arbitrary number.

We follow the lines of [BF1] and so the only part which needs a comment is the uniform bound of $\int_{B_\rho} |\nabla \psi_m|^2\, dx$ for $\rho < 1$ with the function ψ_m defined as in [BF1]. For $\Theta(\varepsilon) := (1+|\varepsilon|^2)^{\frac{p}{4}}$ ($\varepsilon \in \mathbb{S}$) we see

$$\begin{aligned}\int_{B_\rho} |\nabla\psi_m(z)|^2\,dz &= \int_{B_\rho} |D\Theta(A_m + \lambda_m\varepsilon(u_m)(z)) : \nabla\varepsilon(u_m)(z)|^2\,dz \\ &= r_m^{-n}\frac{r_m^2}{\lambda_m^2}\int_{B_{\rho r_m}(x_m)} |\nabla H|^2\,dz \qquad (4.1)\\ &\le c(\rho)\, r_m^2\lambda_m^{-2} \fint_{B_{r_m}(x_m)} \Gamma^{\frac{q}{2}}\,dz,\end{aligned}$$

where $\lambda_m^2 := E(x_m,r_m) + r_m^2$. Furthermore we receive (note $|(\varepsilon(u))_{x_m,r_m}| \le L$)

$$\begin{aligned}\fint_{B_{r_m}(x_m)} \Gamma^{\frac{q}{2}}\,dz &\le c\Big[1 + \fint_{B_{r_m}(x_m)} |\varepsilon(u)|^q\,dz\Big] \\ &\le c\Big[1 + \fint_{B_{r_m}(x_m)} |\varepsilon(u) - (\varepsilon(u))_{x_m,r_m}|^q\,dz \\ &\quad + \fint_{B_{r_m}(x_m)} |(\varepsilon(u))_{x_m,r_m}|^q\,dz\Big] \\ &\le cE(x_m,r_m) + c(L).\end{aligned}$$

Moreover, we obtain

$$\int_{B_\rho} |\nabla\psi_m(z)|^2\,dz \le c(\rho)\left[r_m^2 + r_m^2\lambda_m^{-2}c(L)\right].$$

Recalling the choice of γ^* we have $r_m^2\lambda_m^{-2} \to 0$ and the boundedness of $\int_{B_\rho} |\nabla\psi_m|^2\,dx$ follows. Now the proof can be completed as in [BF1]. $\square$

Proof of Theorem 1.1(b): In [BFZ, 2.6], the authors establish an inequality of the form (remember (3.3))

$$\fint_{B_r(x_0)} H_M^2\,dx \le c\Big(\fint_{B_{2r}(x_0)} h_M^s H_M^s\,dx\Big)^{\frac{2}{s}} \tag{4.2}$$

for $s = 4/3$ valid for any $B_{2r}(x_0) \Subset B_{2R}$ with a constant c independent of M and r (for these calculations they need the assumption $q < p+2$). Here we have (sum over γ, $\mu := \max\{q-2, 2-p\}$)

$$H_M^2 := D_\varepsilon^2 F_M(\cdot,\varepsilon(u_M))(\partial_\gamma\varepsilon(u_M), \partial_\gamma\varepsilon(u_M)) \quad\text{and}\quad h_M := \Gamma_M^{\frac{\mu}{2}}.$$

Note that we have arbitrarily high integrability of $\varepsilon(u_M)$ uniform in M on account of (3.7). In our situation we have to add on the r.h.s. of (4.2) the term

$$-c\fint_{B_{2r}(x_0)} \eta^2\partial_\gamma D_\varepsilon F_M(\cdot,\varepsilon(u_M)) : \partial_\gamma\varepsilon(u_M)\,dx.$$

Using Lemma 3.1(iv) and Young's inequality we can estimate this integral by

$$\tau \fint_{B_r(x_0)} \eta^2 H_M^2 \, dx + c(\tau) \fint_{B_{2r}(x_0)} \Gamma_M^{\frac{2q-p}{2}} \, dx.$$

After absorption of the τ-integral in the l.h.s. of (4.2) we finally receive

$$\fint_{B_r(x_0)} H_M^2 \, dx \leq c \left(\fint_{B_{2r}(x_0)} h_M^s H_M^s \, dx \right)^{\frac{2}{s}} + c \fint_{B_{2r}(x_0)} \Gamma_M^{\frac{2q-p}{2}} \, dx. \tag{4.3}$$

Having a look at Lemma 1.2 from [BFZ], one can see that the additional term in (4.3) is no problem since we have arbitrarily high integrability of Γ_M uniform in M. Now it is possible to end up the proof as in [BFZ]. □

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