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Existence of a positive solution to a nonlocal semipositone boundary value problem on a time scale

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Abstract. We consider the existence of at least one positive solution to the dynamic boundary value problem

$$\begin{aligned} -y^{\Delta\Delta}(t) &= \lambda f(t, y(t)),\ t \in [0,T]_{\mathbb{T}} \\ y(0) &= \int_{\tau_1}^{\tau_2} F_1(s, y(s))\ \Delta s \\ y\left(\sigma^2(T)\right) &= \int_{\tau_3}^{\tau_4} F_2(s, y(s))\ \Delta s, \end{aligned}$$

where $\mathbb{T}$ is an arbitrary time scale with $0 < \tau_1 < \tau_2 < \sigma^2(T)$ and $0 < \tau_3 < \tau_4 < \sigma^2(T)$ satisfying τ_1, τ_2, τ_3, $\tau_4 \in \mathbb{T}$, and where the boundary conditions at $t = 0$ and $t = \sigma^2(T)$ can be both nonlinear and nonlocal. This extends some recent results on second-order semipositone dynamic boundary value problems, and we illustrate these extensions with some examples.

Keywords: time scales; integral boundary condition; second-order boundary value problem; cone; positive solution

Classification: Primary 34B10, 34B15, 34B18, 34N05, 39A10; Secondary 26E70, 47H07

1. Introduction

In this paper we consider the existence of at least one positive solution to the dynamic boundary value problem (BVP)

$$\begin{aligned} -y^{\Delta\Delta}(t) &= \lambda f(t, y(t)),\ t \in [0,T]_{\mathbb{T}} \\ y(0) &= \int_{\tau_1}^{\tau_2} F_1(s, y(s))\ \Delta s \\ y\left(\sigma^2(T)\right) &= \int_{\tau_3}^{\tau_4} F_2(s, y(s))\ \Delta s \end{aligned} \tag{1.1}$$

for $\lambda > 0$ a small parameter, where $\mathbb{T}$ is an arbitrary time scale with $0 < \tau_1 < \tau_2 < \sigma^2(T)$ and $0 < \tau_3 < \tau_4 < \sigma^2(T)$ satisfying τ_1, τ_2, τ_3, $\tau_4 \in \mathbb{T}$; we assume that 0, $\sigma^2(T) \in \mathbb{T}$. Since we further assume that f satisfies

$$-u(t) \leq f(t, y), \tag{1.2}$$

for some function $u : [0,T]_{\mathbb{T}} \to (0,+\infty)$, for all $(t,y) \in [0,T]_{\mathbb{T}} \times [0,+\infty)$, we consider in problem (1.1) the semipositone problem. We note that assumption (1.2), wherein the function f is bounded below by the function u rather than by a constant, is similar to the assumption used, for example, by Anderson [3]; in fact, similar to [3], we shall require that u satisfy an integrability condition — see condition (H5) in Section 2. Furthermore, since $F_1, F_2 : (0,\sigma^2(T))_{\mathbb{T}} \times [0,+\infty) \to [0,+\infty)$ need not be identically zero, it follows that the boundary condition both at $t=0$ and at $t=\sigma^2(T)$ can be both nonlocal and nonlinear. The special case in which both $F_1 \equiv F_2 \equiv 0$ and $\lambda = 1$ has already been treated in the existing literature, as will be mentioned momentarily.

In order to contextualize our result, we begin by noting, as intimated above, that semipositone problem (1.1) in the special case of both $\lambda = 1$ and $y(0) = 0 = y(\sigma^2(T))$ has been discussed by Sun and Li [34]. In that work the authors discuss the existence of at least one positive solution to the boundary value problem (1.1) under the aforementioned restrictions. A follow-up work by the same authors [35] addresses the same BVP as in [34] but with slightly more general structural assumptions on the nonlinearity f.

More generally, the study of semipositone BVPs has seen several contributions in the past few years on a variety of time scales, both in the case of second- and higher-order BVPs as well as first-order BVPs — see, for example, [2], [3], [4], [5], [8], [9], [24], [25] and the references therein. Some of these works are in the setting of an arbitrary time scale, which is the setting in which we work in this paper. For instance, in [2] Anderson studies the semipositone problem

$$\begin{aligned}\left(py^{\nabla}\right)^{\Delta} - q(t)y(t) + \lambda f(t,y(t)) = 0,\ t \in (t_1,t_n)_{\mathbb{T}}\\ \alpha y(t_1) - \beta p(t_1) y^{\nabla}(t_1) = \sum_{i=2}^{n-1} a_i y(t_i)\\ \gamma y(t_n) - \delta p(t_n) y^{\nabla}(t_n) = \sum_{i=2}^{n-1} b_i y(t_i),\end{aligned}$$

where $n \geq 3$ and $\mathbb{T}$ is a time scale. On the other hand, in [3] Anderson studies a similar problem but in the first-order setting, whereas Anderson and Zhai study in [4] a second-order delta-nabla problem but with a simpler three-point boundary condition; each of these papers is also studied in the time scales setting. The papers by Dahal [8], [9] investigate second-order semipositone problems on a time scale with two-point boundary conditions. Finally, the paper by Goodrich [24] studies the problem

$$\begin{aligned}\Delta^{\nu} y(t) = \lambda f(t+\nu-1, y(t+\nu-1)),\ t \in [0,T]_{\mathbb{Z}}\\ y(\nu-1) = y(\nu+T) + \sum_{i=1}^{N} F(t_i, y(t_i)),\end{aligned}$$

where $1 < \nu \leq 2$, which is a discrete fractional BVP.

Let us mention, furthermore, that the study of analysis on a time scale can be traced back to Hilger [27] and has attracted considerable attention in recent years. Indeed, by analyzing problem (1.1) in the time scales setting we recover here results not only for ordinary differential equations ($\mathbb{T} = \mathbb{R}$) but also difference equations ($\mathbb{T} = \mathbb{Z}$), q-difference equations ($\mathbb{T} = q^{\mathbb{Z}}$), and other even more exotic time scales.

On the other hand, the study of BVPs with either nonlocal and/or nonlinear boundary conditions has seen much study of late. In particular, the boundary conditions studied in the above mentioned papers are clearly of nonlocal-type. In addition, the special study of integral boundary conditions has been completed in a variety of settings — see, for example, [6], [11], [26], [28] and the references therein. The separate case of *nonlinear* boundary conditions has also been addressed recently by Goodrich [12], [13], [14], [16], [17], [18], [20], [22], [23], Infante [29], and Infante, et al. [30], [31], [32], [33]. Finally, Goodrich [15], [19], [21] has provided some results for nonlocal BVPs in the discrete and continuous fractional setting. The archetypical problem studied in the previously mentioned papers (at least in the continuous setting, though the discrete setting is analogous) is

$$\begin{aligned} -y''(t) &= f(t, y(t)),\, t \in (0,1) \\ y(0) &= \varphi_1(y) \\ y(1) &= \varphi_2(y), \end{aligned}$$

where $\varphi_1,\, \varphi_2 \,:\, \mathcal{C}([0,1]) \to \mathbb{R}$ are nonlinear functionals. In certain cases such as [18], [20], [22], [23], [29], [30], [31], [32], [33] this very general form is studied in a slightly specialized setting, wherein the boundary conditions are replaced by $y(0) = H_1(L_1(y))$ and $y(1) = H_2(L_2(y))$ where $H_1, H_2 \,:\, \mathbb{R} \to \mathbb{R}$ are continuous functions and $L_1, L_2 \,:\, \mathcal{C}([0,1]) \to \mathbb{R}$ are linear functionals. Depending upon the structural assumptions imposed on the functionals φ_1 and φ_2 it is certainly possible to include integral-type boundary conditions. However, we are not aware of any existing works of this sort that would include problem (1.1) as a special case. This is, in part, due to the fact that the functions F_1 and F_2 appearing in (1.1) are not necessarily linear in y.

In this brief note we combine some of the aforementioned investigations by considering problem (1.1). Of particular note, we do *not* require that either integrand F_1 or F_2 appearing in (1.1) splits in the sense that $F_1(t, y) := \alpha(t)\beta(y)$ for some suitably restricted functions α and β; this seems to be a common assumption in the literature — see certain of the aforementioned works, for instance [6], [11], [28]. Rather, we require a superlinearity-type condition on F_1 and F_2, namely that

$$\limsup_{y \to 0^+} \frac{F_i(t, y)}{y} < \mu_i, \tag{1.3}$$

uniformly for $t \in [\tau_{2i-1}, \tau_{2i}]_{\mathbb{T}} \subseteq (0, \sigma^2(T))_{\mathbb{T}}$, for each $i = 1, 2$ and for numbers μ_1 and μ_2 to be defined later — see Section 2. By utilizing (1.3) we avoid having to make growth assumptions about F_i except essentially at 0. This seems to be a new approach, and we consider this to be one of the contributions of this work. Moreover, as a second contribution of this work, since neither F_1 nor F_2 is required to be linear in y, the boundary conditions here are examples of nonlinear, nonlocal boundary conditions, and so, we also provide some new contributions that complement the existing literature on nonlocal BVPs. Our results specifically generalize Sun and Li [34], [35] since in those works only conjugate boundary conditions were considered. However, our results also complement the many papers that have recently appeared on nonlocal BVPs, as mentioned earlier in this section. Finally, the technique that we employ in this note can certainly be applied to other similar BVPs, and so, is not limited to problem (1.1).

2. Preliminaries

We assume throughout a general familiarity with the time scales calculus, and we refer the reader to the textbook by Bohner and Peterson [7] for a thorough introduction to the subject. Therefore, we begin by constructing an operator with which to study problem (1.1). To this end, define the function $q : [0, \sigma^2(T)]_{\mathbb{T}} \to [0, 1)$ by

$$q(t) := \frac{t\left(\sigma^2(T) - t\right)}{\left(\sigma^2(T)\right)^2}, \tag{2.1}$$

where we observe that $q(0) = 0 = q(\sigma^2(T))$. Note that the function q was introduced previously in [34].

We introduce next the cone in which we shall look for positive solutions of problem (1.1). In particular, let $\mathcal{C}([0, \sigma^2(T)]_{\mathbb{T}})$ represent the collection of all functions $y : [0, \sigma^2(T)]_{\mathbb{T}} \to \mathbb{R}$ such that y is continuous. The space $\mathcal{C}([0, \sigma^2(T)]_{\mathbb{T}})$ is Banach when equipped with the usual max norm, $\|\cdot\|$. We next define the cone $\mathcal{K} \subseteq \mathcal{C}([0, \sigma^2(T)]_{\mathbb{T}})$ by

$$\mathcal{K} := \left\{ y \in \mathcal{C}\left(\left[0, \sigma^2(T)\right]_{\mathbb{T}}\right) \ : \ y(t) \geq q(t)\|y\| \ \text{ for } \ t \in \left[0, \sigma^2(T)\right]_{\mathbb{T}} \right\}. \tag{2.2}$$

Furthermore, define the function $G : [0, \sigma^2(T)]_{\mathbb{T}} \times [0, \sigma^2(T)]_{\mathbb{T}} \to \mathbb{R}$ by (see either [7, Chapter 4] or [34, (2.2)])

$$G(t, s) := \frac{1}{\sigma^2(T)} \begin{cases} t\left(\sigma^2(T) - \sigma(s)\right), & t \leq s \\ \sigma(s)\left(\sigma^2(T) - t\right), & t \geq \sigma(s) \end{cases}. \tag{2.3}$$

We now provide a lemma, which is a slight modification of [34, Lemma 2.1].

Lemma 2.1. *For any $t \in [0, \sigma^2(T)]_{\mathbb{T}}$ and $s \in [0, \sigma^2(T)]_{\mathbb{T}}$ it holds that*

$$0 \leq G(t, s) \leq \sigma^2(T) q(t). \tag{2.4}$$

Furthermore, for each $t \in [0, \sigma^2(T)]_{\mathbb{T}}$ *it holds that*

$$0 \le q(t) \le \min\left\{\frac{\sigma^2(T)-t}{\sigma^2(T)}, \frac{t}{\sigma^2(T)}\right\}. \tag{2.5}$$

PROOF: The truth of inequality (2.4) follows directly from [34, Lemma 2.1]. On the other hand, to show that inequality (2.5) is true, we directly compute both that

$$\frac{t\left(\sigma^2(T)-t\right)}{\left(\sigma^2(T)\right)^2} \le \frac{\sigma^2(T)-t}{\sigma^2(T)}$$

since it holds that

$$t \le \sigma^2(T),$$

and that

$$\frac{t\left(\sigma^2(T)-t\right)}{\left(\sigma^2(T)\right)} \le \frac{t}{\sigma^2(T)}$$

since it holds that

$$\sigma^2(T) - t \le \sigma^2(T).$$

And this completes the proof. $\square$

Let us next state the assumptions that we make henceforth. As mentioned in Section 1, due to condition (H1) we make no assumptions about F_i away from $y = 0$. In particular, this means that the asymptotic behavior of F_i and f may be quite different.

H1: For each $i = 1, 2$, the function $F_i : \mathbb{T} \times [0, +\infty) \to [0, +\infty)$ is continuous, and, furthermore, there is $\mu_i \ge 0$ such that

$$\limsup_{y \to 0^+} \frac{F_i(t, y)}{y} \le \mu_i, \tag{2.6}$$

uniformly for each $t \in [\tau_{2i-1}, \tau_{2i}]_{\mathbb{T}}$.

H2: Assume that the nonlinearity $f : [0, T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is continuous.

H3: Assume that there exist numbers $\alpha_1, \alpha_2 \in (0, \sigma^2(T))_{\mathbb{T}}$, satisfying $\alpha_1 < \alpha_2$, such that $\lim_{y \to +\infty} \frac{f(t,y)}{y} = +\infty$, uniformly for $t \in [\alpha_1, \alpha_2]_{\mathbb{T}}$.

H4: Assume that $\lim_{y \to 0^+} \frac{f(t,y)}{y} = 0$, uniformly for $t \in [0, \sigma^2(T)]_{\mathbb{T}}$.

H5: Assume that there is a function $u \in \mathcal{C}((0, T)_{\mathbb{T}})$ such that $-u(t) \le f(t, y)$, for all $(t, y) \in [0, T]_{\mathbb{T}} \times \mathbb{R}$, where it holds that

$$0 < \int_0^{\sigma(T)} u(s)\, \Delta s < +\infty. \tag{2.7}$$

As is typical when studying semipositone BVPs, we shall need to consider solutions of some other problems. First we consider the unique solution of the auxiliary problem

$$
\begin{aligned}
-w^{\Delta\Delta} &= \lambda u(t),\, t \in (0,T)_{\mathbb{T}} \\
w(0) &= 0 \\
w\left(\sigma^2(T)\right) &= 0.
\end{aligned}
\tag{2.8}
$$

Henceforth, we shall denote by w the unique solution of problem (2.8). First of all, we note that w may be represented by

$$
w(t) = \lambda \int_0^{\sigma(T)} G(t,s)u(s)\ \Delta s. \tag{2.9}
$$

In addition, by Lemma 2.1 we estimate

$$
\begin{aligned}
w(t) &\le \lambda \int_0^{\sigma(T)} \sigma^2(T)q(t)u(s)\ \Delta s \\
&\le \lambda\sigma^2(T)q(t)\xi,
\end{aligned}
\tag{2.10}
$$

where in (2.10) and the sequel we put

$$
\xi := \int_0^{\sigma(T)} u(s)\ \Delta s. \tag{2.11}
$$

Note that by condition (H5) we find that

$$
0 < \xi < +\infty. \tag{2.12}
$$

Having earlier defined $\mathcal{K}$ we show next that $(y-w)(t) \ge 0$ whenever $y \in \mathcal{K}$ has sufficiently large norm.

Lemma 2.2. *Let $y \in \mathcal{K}$ be given. If $\|y\| \ge \lambda\sigma^2(T)\xi$, then $(y-w)(t) \ge 0$, for each $t \in [0,\sigma^2(T)]_{\mathbb{T}}$.*

Proof: From the bound on w given in (2.10) together with the fact that $y \in \mathcal{K}$ we estimate

$$
\begin{aligned}
y(t) - w(t) &\ge y(t) - \lambda\sigma^2(T)q(t)\xi \\
&\ge q(t)\|y\| - \lambda\sigma^2(T)q(t)\xi \\
&= q(t)\left(\|y\| - \lambda\sigma^2(T)\xi\right).
\end{aligned}
\tag{2.13}
$$

Since $q(t) \ge 0$ for each $t \in [0,\sigma^2(T)]_{\mathbb{T}}$, we deduce from (2.13) that $(y-w)(t) \ge 0$ for each $t \in [0,\sigma^2(T)]_{\mathbb{T}}$ provided that

$$
\|y\| \ge \lambda\sigma^2(T)\xi. \tag{2.14}
$$

And this completes the proof. □

Next we consider the modified problem

$$\begin{aligned} -y^{\Delta\Delta} &= \lambda \left[f\left(t, \max\left\{(y-w)(t), 0\right\}\right) + u(t)\right], \ t \in (0,T)_{\mathbb{T}} \\ y(0) &= \int_{\tau_1}^{\tau_2} F_1(t, \max\{(y-w)(t), 0\})\ \Delta t \\ y\left(\sigma^2(T)\right) &= \int_{\tau_3}^{\tau_4} F_2(t, \max\{(y-w)(t), 0\})\ \Delta t. \end{aligned} \tag{2.15}$$

Let us next show that if y solves (2.15), w solves (2.8), and $(y-w)(t) \geq 0$ for each $t \in [0, \sigma^2(T)]_{\mathbb{T}}$, then the function $x : [0, \sigma^2(T)]_{\mathbb{T}} \to \mathbb{R}$ defined by $x(t) := y(t) - w(t)$ is a positive solution of the original problem (1.1).

Lemma 2.3. *Suppose that y solves* (2.15), *w solves* (2.8), *and $(y-w)(t) \geq 0$ for each $t \in [0, \sigma^2(T)]_{\mathbb{T}}$. Then the function $x : [0, \sigma^2(T)]_{\mathbb{T}} \to \mathbb{R}$ defined by $x(t) := y(t) - w(t)$ is a positive solution of the original problem* (1.1).

Proof: Note that

$$x(0) = y(0) - w(0) = \int_{\tau_1}^{\tau_2} F_1(t, x(t))\ \Delta t \tag{2.16}$$

and that

$$x\left(\sigma^2(T)\right) = y\left(\sigma^2(T)\right) - w\left(\sigma^2(T)\right) = \int_{\tau_3}^{\tau_4} F_2(t, x(t))\ \Delta t. \tag{2.17}$$

Thus, x satisfies the boundary conditions in (1.1). On the other hand, we compute

$$-x^{\Delta\Delta}(t) = -y^{\Delta\Delta}(t) + w^{\Delta\Delta}(t) = \lambda f(t, x(t)) + \lambda u(t) - \lambda u(t) = \lambda f(t, x(t)). \tag{2.18}$$

Thus, the function x is a solution of problem (1.1). Since $x(t) \geq 0$ for each t, by assumption, the function x is a positive solution of problem (1.1), as desired. □

In light of Lemma 2.3, we see that an appropriate operator $T : \mathcal{C}_{\mathrm{rd}}([0, \sigma^2(T)]_{\mathbb{T}}) \to \mathcal{C}_{\mathrm{rd}}([0, \sigma^2(T)]_{\mathbb{T}})$ with which to study the existence of solution of the modified problem (2.15) is defined by

$$\begin{aligned} (Ty)(t) :=& \frac{\sigma^2(T) - t}{\sigma^2(T)} \int_{\tau_1}^{\tau_2} F_1\left(s, y^*(s)\right)\ \Delta s \\ &+ \frac{t}{\sigma^2(T)} \int_{\tau_3}^{\tau_4} F_2\left(s, y^*(s)\right)\ \Delta s \\ &+ \lambda \int_0^{\sigma(T)} G(t,s) \left[f\left(s, y^*(s)\right) + u(s)\right]\ \Delta s, \end{aligned} \tag{2.19}$$

where we have defined y^* by

$$y^*(t) := \max\{(y-w)(t), 0\}. \tag{2.20}$$

It then follows that fixed points of T are solutions of the modified problem (2.15). The next lemma demonstrates that $\mathcal{K}$ is invariant under T.

Lemma 2.4. *Let T be defined as in* (2.19)*. Then $T(\mathcal{K}) \subseteq \mathcal{K}$.*

PROOF: For the most part the result follows from the proof of [34, Lemma 2.4]. We need only account for the perturbation terms in (2.19) that do not appear in the corresponding operator considered in [34]. This does require a minor modification of the strategy employed in the proof of [34, Lemma 2.4].

So, to this end, let us first suppose that there exists $t_0 \in (0, \sigma^2(T))_{\mathbb{T}}$ such that $\|Ty\| = (Ty)(t_0)$. Then using both the fact that $f(t, y^*(t)) + u(t) \geq 0$, for each t, and the conclusion of Lemma 2.1 we estimate, for each $t \in [0, \sigma^2(T)]_{\mathbb{T}}$,

(2.21)

$$\begin{aligned}
(Ty)(t) &= \frac{\sigma^2(T) - t}{\sigma^2(T)} \int_{\tau_1}^{\tau_2} F_1\left(s, y^*(s)\right)\ \Delta s \\
&\quad + \frac{t}{\sigma^2(T)} \int_{\tau_3}^{\tau_4} F_2\left(s, y^*(s)\right)\ \Delta s + \lambda \int_0^{\sigma(T)} G(t,s)\left[f\left(s, y^*(s)\right) + u(s)\right]\ \Delta s \\
&\geq \frac{\sigma^2(T) - t}{\sigma^2(T)} \int_{\tau_1}^{\tau_2} F_1\left(s, y^*(s)\right)\ \Delta s + \frac{t}{\sigma^2(T)} \int_{\tau_3}^{\tau_4} F_2\left(s, y^*(s)\right)\ \Delta s \\
&\quad + q(t)\lambda \int_0^{\sigma(T)} G\left(t_0, s\right)\left[f\left(s, y^*(s)\right) + u(s)\right]\ \Delta s \\
&\geq q(t) \int_{\tau_1}^{\tau_2} F_1\left(s, y^*(s)\right)\ \Delta s + q(t) \int_{\tau_3}^{\tau_4} F_2\left(s, y^*(s)\right)\ \Delta s \\
&\quad + q(t)\lambda \int_0^{\sigma(T)} G\left(t_0, s\right)\left[f\left(s, y^*(s)\right) + u(s)\right]\ \Delta s \\
&\geq q(t)\left[\frac{\sigma^2(T) - t_0}{\sigma^2(T)} \int_{\tau_1}^{\tau_2} F_1\left(s, y^*(s)\right)\ \Delta s + \frac{t_0}{\sigma^2(T)} \int_{\tau_3}^{\tau_4} F_2\left(s, y^*(s)\right)\ \Delta s\right. \\
&\quad \left. + \lambda \int_0^{\sigma(T)} G\left(t_0, s\right)\left[f\left(s, y^*(s)\right) + u(s)\right]\ \Delta s\right] \\
&= q(t)\|Ty\|.
\end{aligned}$$

Note that in (2.21) we have used the fact that whenever $G(t_0, s) \neq 0$ it holds that

$$G(t,s) = \frac{G(t,s)}{G\left(t_0, s\right)} \cdot G\left(t_0, s\right) \geq q(t) G\left(t_0, s\right), \tag{2.22}$$

where the inequality in (2.22) follows from first part of the proof of [34, Lemma 2.4]. Moreover, observe that the desired inequality in (2.21) holds even if $G(t_0, s) = 0$ and thus (2.22) cannot be invoked. Indeed, we merely observe that if $G(t_0, s) = 0$ for some s, then it trivially follows that $G(t, s) \geq q(t)G(t_0, s) = 0$. In any case, (2.21) is valid for all admissible t and s.

Next suppose that $\|Ty\| = (Ty)(0)$. We then estimate, for each $t \in [0, \sigma^2(T)]_{\mathbb{T}}$,

$$
\begin{aligned}
(Ty)(t) \geq \frac{\sigma^2(T)-t}{\sigma^2(T)} \int_{\tau_1}^{\tau_2} F_1(s, y^*(s))\ \Delta s &\geq q(t) \int_{\tau_1}^{\tau_2} F_1(s, y^*(s))\ \Delta s \\
&= q(t)(Ty)(0) \\
&= q(t)\|Ty\|.
\end{aligned}
\tag{2.23}
$$

Finally, if $\|Ty\| = (Ty)(\sigma^2(T))$, then we estimate, for each $t \in [0, \sigma^2(T)]_{\mathbb{T}}$,

$$
\begin{aligned}
(Ty)(t) \geq \frac{t}{\sigma^2(T)} \int_{\tau_3}^{\tau_4} F_2(s, y^*(s))\ \Delta s &\geq q(t) \int_{\tau_3}^{\tau_4} F_2(s, y^*(s))\ \Delta s \\
&= q(t)(Ty)\left(\sigma^2(T)\right) \\
&= q(t)\|Ty\|.
\end{aligned}
\tag{2.24}
$$

In summary, then, no matter the value of $t_0 \in [0, \sigma^2(T)]_{\mathbb{T}}$, if $\|Ty\| = (Ty)(t_0)$, then one of (2.21), (2.23), or (2.24) implies that $(Ty)(t) \geq q(t)\|Ty\|$, for each $t \in [0, \sigma^2(T)]_{\mathbb{T}}$. And this completes the proof. $\square$

We conclude this section with a statement of Krasnosel'skiĭ's fixed point theorem (see [1]). We shall use this result to prove the existence theorem of Section 3.

Lemma 2.5. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{K} \subseteq \mathcal{B}$ be a cone. Assume that Ω_1 and Ω_2 are bounded, open sets contained in $\mathcal{B}$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$. Assume, further, that $T : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \to \mathcal{K}$ is a completely continuous operator. If either*

(1) *$\|Ty\| \leq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_1$ and $\|Ty\| \geq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_2$; or*
(2) *$\|Ty\| \geq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_1$ and $\|Ty\| \leq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_2$;*

then T has at least one fixed point in $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Main result and examples

We first state our existence result. This result asserts that problem (1.1), subject to conditions (H1)–(H5), has at least one positive solution for *small* eigenvalues λ. In order to illustrate that this result is not merely abstract, yielding no reasonable way to determine the range of eigenvalues generated by the theorem, we also provide a corollary, which gives a demonstration of how a range for λ may be reasonably calculated from the initial data. We then conclude with two examples, which shall demonstrate the use of the result in some representative situations.

Theorem 3.1. *Suppose that conditions* (H1)–(H5) *hold. In addition, suppose that*

$$
\mu_1 \int_{\tau_1}^{\tau_2} 1\ \Delta s + \mu_2 \int_{\tau_3}^{\tau_4} 1\ \Delta s < 1 \tag{3.1}
$$

holds. Then there exists a number $\lambda_1 \in (0,1]$ *such that for each* $\lambda \in (0,\lambda_1]$ *problem* (1.1) *has at least one positive solution.*

Proof: We begin by noting that standard arguments, which we thus omit, indicate that the operator T defined above is completely continuous. In light of Lemma 2.4, in order to invoke Lemma 2.5 it only remains to show that T is alternatively a cone contraction and cone expansion on appropriate sets.

To demonstrate this latter claim, we proceed as follows. First we recall that $G(t,s) \leq G(\sigma(s),s)$, for each $t \in [0,\sigma(T)]_{\mathbb{T}}$. Put

$$\vartheta_0 := \mu_1 \int_{\tau_1}^{\tau_2} 1 \; \Delta s + \mu_2 \int_{\tau_3}^{\tau_4} 1 \; \Delta s. \tag{3.2}$$

Due to condition (3.1), we may select a number $\varepsilon_1 > 0$ sufficiently small such that

$$\vartheta_0 + \varepsilon_1 < 1. \tag{3.3}$$

Define the numbers η_1, $\eta_2 > 0$ such that

$$\eta_1 \int_{\tau_1}^{\tau_2} 1 \; \Delta s + \eta_2 \int_{\tau_3}^{\tau_4} 1 \; \Delta s \leq \varepsilon_1. \tag{3.4}$$

Then by condition (H1), we may find a number $r_1 > 0$ such that

$$F_i(t,y) \leq (\eta_i + \mu_i)\, y, \tag{3.5}$$

for each $y \leq r_1$ and each $t \in [\tau_{2i-1}, \tau_{2i}]_{\mathbb{T}}$ for $i = 1, 2$. Define another number, say $\varepsilon_2 > 0$, satisfying

$$\vartheta_0 + \varepsilon_1 + \varepsilon_2 < 1. \tag{3.6}$$

Choose the number $\eta_3 > 0$ such that

$$\eta_3 \int_0^{\sigma(T)} G(\sigma(s),s) \; \Delta s \leq \varepsilon_2. \tag{3.7}$$

Then by condition (H4) we find $r_1^* > 0$, satisfying without loss of generality $r_1 > r_1^*$, such that

$$|f(t,y)| \leq \eta_3 y, \tag{3.8}$$

for each $t \in [0,\sigma^2(T)]_{\mathbb{T}}$, whenever $0 < y \leq r_1^*$. Finally, let $\lambda_1 \in \mathbb{R}$ be the number defined by

$$\lambda_1 := \min\left\{1, r_1^* \left(1 - \varepsilon_1 - \varepsilon_2 - \vartheta_0\right) \left[\int_0^{\sigma(T)} G(\sigma(s),s)u(s) \; \Delta s\right]^{-1}, \frac{r_1^*}{\sigma^2(T)\xi}\right\}, \tag{3.9}$$

which is well defined. Select an arbitrary but fixed λ such that $0 < \lambda \leq \lambda_1$. Define the set $\Omega_{r_1^*}$ by

$$\Omega_{r_1^*} := \{y \in \mathcal{B} \ : \ \|y\| < r_1^*\}, \tag{3.10}$$

and observe that for each $y \in \mathcal{K} \cap \partial\Omega_{r_1^*}$ it holds that

$$\lambda \leq \frac{r_1^*}{\sigma^2(T)\xi} = \frac{\|y\|}{\sigma^2(T)\xi}, \tag{3.11}$$

whence $\|y\| \geq \lambda\sigma^2(T)\xi$. Consequently, Lemma 2.2 implies that $(y-w)(t) \geq 0$, for each $t \in [0, \sigma^2(T)]_{\mathbb{T}}$. Finally, noting that, for each $t \in [0, \sigma(T)]_{\mathbb{T}}$,

$$(y-w)(t) \leq \|y\|, \tag{3.12}$$

it follows that for each $y \in \mathcal{K} \cap \partial\Omega_{r_1^*}$ we estimate

(3.13)

$$\begin{aligned}
(Ty)(t) &\leq \frac{\sigma^2(T)-t}{\sigma^2(T)} \int_{\tau_1}^{\tau_2} (\mu_1+\eta_1)\, y^*(s)\ \Delta s + \frac{t}{\sigma^2(T)} \int_{\tau_3}^{\tau_4} (\mu_2+\eta_2)\, y^*(s)\ \Delta s \\
&\quad + \lambda \int_0^{\sigma(T)} G(\sigma(s),s) \left[f\left(s, y^*(s)\right) + u(s)\right]\ \Delta s \\
&\leq \int_{\tau_1}^{\tau_2} (\mu_1+\eta_1)\, y^*(s)\ \Delta s + \int_{\tau_3}^{\tau_4} (\mu_2+\eta_2)\, y^*(s)\ \Delta s \\
&\quad + \eta_3\lambda \int_0^{\sigma(T)} G(\sigma(s),s) y^*(s)\ \Delta s + \lambda \int_0^{\sigma(T)} G(\sigma(s),s) u(s)\ \Delta s \\
&\leq \left[\mu_1 \int_{\tau_1}^{\tau_2} 1\ \Delta s + \mu_2 \int_{\tau_3}^{\tau_4} 1\ \Delta s\right] \|y\| + \left[\eta_1 \int_{\tau_1}^{\tau_2} 1\ \Delta s + \eta_2 \int_{\tau_3}^{\tau_4} 1\ \Delta s\right] \|y\| \\
&\quad + \varepsilon_2\|y\| + \lambda \int_0^{\sigma(T)} G(\sigma(s),s) u(s)\ \Delta s \\
&\leq \left(\vartheta_0 + \varepsilon_1 + \varepsilon_2\right) \|y\| + \left(1 - \varepsilon_1 - \varepsilon_2 - \vartheta_0\right) \|y\| \\
&= \|y\|,
\end{aligned}$$

for each $t \in [0, \sigma^2(T)]_{\mathbb{T}}$, whence T is a cone compression on $\mathcal{K} \cap \partial\Omega_{r_1^*}$.

On the other hand, first we recall that if $E \subset (0, \sigma^2(T))_{\mathbb{T}}$ is given, then it holds that there is $\gamma \in (0,1)$ such that $\min_{t\in E} G(t,s) \geq \gamma G(\sigma(s),s)$, for each s — see [7], [10]. We assume henceforth that $E := [\alpha_1, \alpha_2]_{\mathbb{T}}$, which fixes the constant γ. With this number γ henceforth fixed, we proceed as follows. Define the number q_0 by

$$q_0 := \min_{t\in[\alpha_1,\alpha_2]_{\mathbb{T}}} q(t). \tag{3.14}$$

Observe that since the open set $(\alpha_1, \alpha_2)_{\mathbb{T}}$ is assumed to be contained in $(0, \sigma^2(T))_{\mathbb{T}}$ — cf., condition (H3) —, it follows from the definition of q that $q_0 > 0$. Select a number $\eta_4 > 0$ such that

$$\eta_4 \int_0^{\sigma(T)} \frac{1}{2}\lambda\gamma q_0 G(\sigma(s), s)\ \Delta s \geq 1, \tag{3.15}$$

where λ is fixed from the first part of the proof. Condition (H3) then implies the existence of a number $r_2 > 0$ such that

$$f(t, y) \geq \eta_4 y, \tag{3.16}$$

for each $t \in [\alpha_1, \alpha_2]_{\mathbb{T}}$, whenever $y \geq r_2$. Next note that if $\|y\| > 2\lambda\sigma^2(T)\xi$, then it follows that

$$\frac{1}{2} < 1 - \frac{\lambda\sigma^2(T)\xi}{\|y\|}. \tag{3.17}$$

Consequently, in observance of the proof of Lemma 2.2, especially estimate (2.13), for each $y \in \mathcal{K}$ satisfying $\|y\| > 2\lambda\sigma^2(T)\xi$ it holds that

$$\begin{aligned} (y - w)(t) &\geq q(t)\left[\|y\| - \lambda\sigma^2(T)\xi\right] \\ &\geq q_0\left[\|y\| - \lambda\sigma^2(T)\xi\right] \\ &= \|y\| \cdot q_0\left[1 - \frac{\lambda\sigma^2(T)\xi}{\|y\|}\right] \\ &\geq \frac{1}{2}q_0\|y\|, \end{aligned} \tag{3.18}$$

for each $t \in [\alpha_1, \alpha_2]_{\mathbb{T}}$. In addition, observe that if it also holds that $\|y\| \geq \frac{r_2}{q_0} + \lambda\sigma^2(T)\xi$, then it follows that

$$(y - w)(t) \geq q_0\left[\|y\| - \lambda\sigma^2(T)\xi\right] \geq r_2, \tag{3.19}$$

for each $t \in [\alpha_1, \alpha_2]_{\mathbb{T}}$. Consequently, let $y \in \mathcal{K} \cap \partial\Omega_{r_2^*}$, where the number r_2^* defined by

$$r_2^* := \max\left\{2r_1^*, 2\lambda\sigma^2(T)\xi, \frac{r_2}{q_0} + \lambda\sigma^2(T)\xi\right\}, \tag{3.20}$$

and the open set $\Omega_{r_2^*}$ is defined as suggested by (3.10). Let $t_0 \in \mathbb{T}$ be selected such that $\alpha_1 \leq t_0 \leq \alpha_2$. Then for each such y we estimate

$$
\begin{aligned}
(Ty)(t_0) &\geq \lambda \int_0^{\sigma(T)} G(t_0, s)\left[f(s, y^*(s)) + u(s)\right]\ \Delta s \\
&\geq \lambda \int_{\alpha_1}^{\alpha_2} \gamma G(\sigma(s), s)\left[f(s, (y-w)(s)) + u(s)\right]\ \Delta s \\
&\geq \lambda \int_{\alpha_1}^{\alpha_2} \gamma G(\sigma(s), s)\left[\eta_4\left(y(s) - w(s)\right) + u(s)\right]\ \Delta s \\
&\geq \lambda \int_{\alpha_1}^{\alpha_2} \gamma G(\sigma(s), s)\eta_4 (y-w)(s)\ \Delta s \\
&\geq \lambda \int_{\alpha_1}^{\alpha_2} \eta_4 \gamma G(\sigma(s), s) q(s)\left[\|y\| - \lambda\sigma^2(T)\xi\right]\ \Delta s \\
&\geq \|y\| \cdot \eta_4 \left[\frac{1}{2}\lambda \int_{\alpha_1}^{\alpha_2} \gamma G(\sigma(s), s) q_0\ \Delta s\right] \\
&\geq \|y\|,
\end{aligned}
\tag{3.21}
$$

where we use the nonnegativity of F_1 and F_2 in the first inequality and estimate (2.13) in the fifth inequality. Consequently, since (3.21) holds for the particular t_0 chosen above, we conclude that $\|Ty\| \geq \|y\|$, for each $y \in \mathcal{K} \cap \partial\Omega_{r_2^*}$. By Lemma 2.5 we deduce the existence of a function

$$
y_0 \in \mathcal{K} \cap \left(\overline{\Omega}_{r_2^*} \setminus \Omega_{r_1^*}\right)
\tag{3.22}
$$

such that y_0 is a solution of the modified problem (2.15).

It remains to argue that the original problem (1.1) has a positive solution. To this end, define the function $x : [0, \sigma^2(T)]_{\mathbb{T}} \to \mathbb{R}$ by $x(t) := y_0(t) - w(t)$. Since $\|y_0\| \geq r_1^*$, it follows from the preceding analysis — in particular, inequality (3.11) — that $x(t) \geq 0$, for each $t \in [0, \sigma^2(T)]_{\mathbb{T}}$. Consequently, an invocation of Lemma 2.3 provides the desired conclusion. And this completes the proof. □

Corollary 3.2. *Define* ϑ_0 *as in* (3.2) *in the proof of Theorem* 3.1. *Let the numbers* ε_1, ε_2, η_1, η_2, $\eta_3 \geq 0$ *be defined as follows.*

$$
\begin{aligned}
\varepsilon_1 &:= \frac{1}{2}(1 - \vartheta_0) \\
\varepsilon_2 &:= \frac{1}{4}(1 - \vartheta_0) \\
\eta_1 &:= \frac{1}{4}\left[\int_{\tau_1}^{\tau_2} 1\ \Delta s + \int_{\tau_3}^{\tau_4} 1\ \Delta s\right]^{-1}(1 - \vartheta_0) \\
\eta_2 &:= \frac{1}{4}\left[\int_{\tau_1}^{\tau_2} 1\ \Delta s + \int_{\tau_3}^{\tau_4} 1\ \Delta s\right]^{-1}(1 - \vartheta_0)
\end{aligned}
$$

$$\eta_3 := \frac{1}{4}\left[\int_0^{\sigma(T)} G(\sigma(s), s)\ \Delta s\right]^{-1}(1-\vartheta_0)$$

Finally, define the number r_1^* by

$$\min\left\{\sup_{y\in[0,+\infty)}\max_{t\in[\tau_1,\tau_2]_{\mathbb{T}}} F_1(t,y) \le (\eta_1+\mu_1)\,y,\ \sup_{y\in[0,+\infty)}\max_{t\in[\tau_3,\tau_4]_{\mathbb{T}}} F_2(t,y)\right.$$
$$\left.\le (\eta_2+\mu_2)\,y,\ \sup_{y\in[0,+\infty)}\max_{t\in[0,\sigma^2(T)]_{\mathbb{T}}} |f(t,y)| \le \eta_3 y\right\}.$$

Then defining the number λ_1 as in (3.9) *in the proof of Theorem* 3.1 *it follows that problem* (1.1) *has at least one positive solution for each $\lambda \in (0, \lambda_1]$.*

Proof: Omitted since it follows from the first half of the proof of Theorem 3.1. □

Example 3.3. We consider first an example on the time scale $\mathbb{T} = \mathbb{R}$. Note that in this case hypothesis (3.1) reduces to

$$\mu_1(\tau_2-\tau_1)+\mu_2(\tau_4-\tau_3) < 1. \tag{3.23}$$

Consequently, it follows that whenever (3.23) together with hypotheses (H1)–(H5) hold, problem (1.1) will have at least one positive solution for all $\lambda \in (0, \lambda_1]$ for some $\lambda_1 > 0$ sufficiently small.

In fact, remaining on the time scale $\mathbb{T} = \mathbb{R}$ let us give an explicit illustration of the use of Corollary 3.2. Let us specifically consider the problem

$$\begin{aligned} -y^{\Delta\Delta}(t) &= \lambda f(t, y(t)),\ t\in(0,1)\\ y(0) &= \int_{\frac{1}{4}}^{\frac{3}{10}} \frac{1}{10} s[y(s)]^2\ ds\\ y(1) &= \int_{\frac{1}{2}}^{\frac{3}{5}} \frac{1}{5}[y(s)]^3\ ds, \end{aligned} \tag{3.24}$$

where the function f is defined by

$$f(t,y) := \begin{cases} -\frac{1}{2}t^2y^3, & 0 \le y \le 1 \text{ and } t\in[0,1]\\ t^2\left[-\frac{1}{2}+(y-1)^2\right], & 1\le y<+\infty \text{ and } t\in[0,1]\end{cases}.$$

Note that we may set

$$u(t) \equiv \frac{1}{2}.$$

Furthermore, straightforward calculations demonstrate that the constants in Corollary 3.2 assume the following values.

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{2}\\
\varepsilon_2 &= \frac{1}{4}\\
\eta_1 &= \frac{5}{3}\\
\eta_2 &= \frac{5}{3}\\
\eta_3 &= \frac{3}{2}\\
\xi &= \frac{1}{2}\\
\left[\int_0^1 G(s,s)u(s)\ ds\right]^{-1} &= 12\\
r_1^* &= \min\left\{\sqrt{3}, \frac{50}{3}, \sqrt{\frac{15}{2}}\right\}
\end{aligned}$$

Then (3.9) implies that

$$\lambda_1 = 1.$$

Thus, in this example we would conclude that problem (3.24) has a positive solution whenever $\lambda \in (0,1]$. In particular, this implies that the nonlocal BVP

$$\begin{aligned}
-y^{\Delta\Delta}(t) &= f(t, y(t)),\ t \in (0,1)\\
y(0) &= \int_{\frac{1}{4}}^{\frac{3}{10}} \frac{1}{10} s[y(s)]^2\ ds\\
y(1) &= \int_{\frac{1}{2}}^{\frac{3}{5}} \frac{1}{5}[y(s)]^3\ ds,
\end{aligned} \tag{3.25}$$

has at least one positive solution — i.e., problem (3.24) in case $\lambda = 1$.

Example 3.4. Let us now consider the time scale $\mathbb{T} = \mathbb{Z}$. Note that in this case hypothesis (3.1) reduces to

$$\mu_1 \sum_{s=\tau_1}^{\tau_2 - 1} 1 + \mu_2 \sum_{s=\tau_3}^{\tau_4 - 1} 1 < 1. \tag{3.26}$$

Consequently, it follows that whenever (3.26) together with hypotheses (H1)–(H5) hold, problem (1.1) will have at least one positive solution for all $\lambda \in (0, \lambda_1]$ for some $\lambda_1 > 0$ sufficiently small. Note that to obtain (3.26) we use the fact — see

[7] — that on time scale $\mathbb{T} = \mathbb{Z}$ it holds that

$$\int_a^b f(t)\ \Delta t = \sum_{t=a}^{b-1} f(t).$$

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