# Commutativity theorems for rings with differential identities on Jordan ideals

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*Abstract.* In this paper we investigate commutativity of ring $R$ with involution $'*'$ which admits a derivation satisfying certain algebraic identities on Jordan ideals of $R$. Some related results for prime rings are also discussed. Finally, we provide examples to show that various restrictions imposed in the hypotheses of our theorems are not superfluous.

*Keywords:* derivation; generalized derivation; $*$-Jordan ideal

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## 1. Introduction

Throughout this paper, $R$ will denote an associative ring with center $Z(R)$. For any $x, y \in R$ the symbol $[x, y]$ will denote the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. $R$ is 2-torsion free if $2x = 0$ with $x \in R$ implies $x = 0$. $R$ is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. An additive mapping $x \mapsto x^*$ on a ring $R$ is said to be an involution if $(xy)^* = y^*x^*$ and $(x^*)^* = x$ hold for all $x, y \in R$. A ring equipped with an involution $'*'$ is called a ring with involution or a $*$-ring. A ring $R$ with involution $*$ is said to be $*$-prime if $aRb = aRb^* = 0$ implies that either $a = 0$ or $b = 0$, equivalently, $aRb = a^*Rb = 0$ implies that either $a = 0$ or $b = 0$. Note that every prime ring having an involution $*$ is $*$-prime but the converse is not true in general. Indeed, if $R^o$ denotes the opposite ring of a prime ring $R$, then $R \times R^o$ equipped with the exchange involution $*_{ex}$, defined by $*_{ex}(x, y) = (y, x)$, is $*_{ex}$-prime but not prime. This example shows that every prime ring can be injected in a $*$-prime ring and from this point of view $*$-prime rings constitute a more general class of prime rings.

Let $R$ be a $*$-prime ring. The set of symmetric and skew symmetric elements of $R$ will be denoted by $Sa_*(R)$ i.e., $Sa_*(R) = \{x \in R \mid x^* = \pm x\}$. An additive subgroup $J$ of $R$ is said to be a Jordan ideal of $R$ if $u \circ r \in J$, for all $u \in J$ and $r \in R$. A Jordan ideal $J$ which satisfies $J^* = J$ is called a $*$-Jordan ideal. If $J$ is a nonzero Jordan ideal of a ring $R$, then $2[R, R]J \subseteq J$ and $2J[R, R] \subseteq J$ ([12, Lemma 2.4]). Moreover, from [1] (see the proof of Lemma 3) we have $4j^2R \subseteq J$ and $4Rj^2 \subseteq J$ for all $j \in J$. Since $4jrj = 2\{j(jr+rj)+(jr+rj)j\}-\{2j^2r+r2j^2\}$, it follows that $4jRj \subseteq J$ for all $j \in J$ (see [1], proof of Theorem 3).

Now we discuss some basic properties of $*$-prime ring which shall be used frequently throughout the text. For the details, one can look into [8], [9] and [10].

($I$) Every $*$-prime ring is semiprime.

($II$) Let $J$ be a nonzero Jordan ideal of a 2-torsion free $*$-prime ring $R$ such that $aJb = a^*Jb = 0$. Then $a = 0$ or $b = 0$ (see [9, Lemma 2]).

($III$) Let $J$ be a nonzero Jordan ideal of a 2-torsion free $*$-prime ring $R$ such that $aJ = 0$ (resp. $Ja = 0$). Then $a = 0$. In fact, if $aJ = 0$ (resp. $Ja = 0$), then $aJa = 0 = aJa^*$ (resp. $aJa = 0 = a^*Ja$), and by ($II$) $a = 0$.

($IV$) Let $R$ be a 2-torsion free $*$-prime ring and $J$ a nonzero $*$-Jordan ideal of $R$. If $J \subseteq Z(R)$, then $R$ is commutative (see [10, Lemma 3]).

($V$) Let $R$ be a 2-torsion free $*$-prime ring and $J$ a nonzero $*$-Jordan ideal of $R$. If $aJa = 0$, then $a = 0$. In fact if $aJa = 0$, then $aJaJa^* = 0$ and, by ($II$), either $a = 0$ or $aJa^* = 0$ in which case, because of $aJa = 0$, we get $a = 0$.

Long ago Herstein [5] proved that if a prime ring $R$ of characteristic different from two admits a derivation $d$ such that $d(x)d(y) = d(y)d(x)$ holds for all $x, y \in R$, then $R$ is commutative. Motivated by this result Bell and Daif [2] obtained the same result by considering the identity $d[x, y] = 0$ for all $x, y$ in a nonzero ideal of $R$. Later, Daif and Bell [3] established commutativity of semiprime ring satisfying $d([x, y]) = [x, y]$ for all $x, y$ in a nonzero ideal of $R$, and $d$ a derivation of $R$. Further, in the year 1997 M. Hongan [6] established commutativity of 2-torsion free semiprime ring $R$ which admits a derivation $d$ satisfying $d([x, y]) + [x, y] \in Z(R)$ for all $x, y \in I$ or $d([x, y]) - [x, y] \in Z(R)$ for all $x, y \in I$, where $I$ is an ideal of $R$. In the present paper we generalize these results for $*$-prime ring $R$ satisfying any one of the properties: (i) $d[x, y] = 0$, (ii) $d([x, y]) - [x, y] \in Z(R)$, (iii) $d([x, y]) + [x, y] \in Z(R)$, (iv) $d(xoy) = 0$, (v) $d(xoy) - xoy \in Z(R)$, (vi) $d(xoy) + xoy \in Z(R)$ for all $x, y \in J$, a nonzero Jordan ideal of $R$.

## 2. Differential identities with commutator

The following two basic commutator identities $[x, yz] = y[x, z] + [x, y]z$ and $[xy, z] = x[y, z] + [x, z]y$ shall be used frequently, throughout the text, without any specific mention. We begin with the following lemmas which are essential for developing the proof of our main results. The proof of Lemma 1 can be seen in [6] while Lemma 2 and Lemma 3 are essentially proved in [8] and [10] respectively.

**Lemma 1** ([6, Corollary 1]). *Let $R$ be a 2-torsion free semiprime ring. If $R$ admits a derivation $d$ such that either $d([x, y]) + [x, y] \in Z(R)$ or $d([x, y]) - [x, y] \in Z(R)$ for all $x, y \in R$, then $R$ is commutative.*

**Lemma 2** ([8, Lemma 3]). *Let $R$ be a 2-torsion free $*$-prime ring and $J$ a nonzero $*$-Jordan ideal of $R$. If $R$ admits a derivation $d$ such that $d(x^2) = 0$, for all $x \in J$, then $d = 0$.*

**Lemma 3** ([10, Theorem 1]). *Let $R$ be a 2-torsion free $*$-prime ring and $J$ a $*$-Jordan ideal of $R$. If $R$ admits a derivation $d$ which is centralizing on $J$, i.e. $[d(x), x] \in Z(R)$ for all $x \in J$, then $R$ is commutative.*

Very recently, the results obtained by Bell and Daif [2] were further extended to $*$-prime ring by Oukhtite and Salhi (see [11, Theorem 1.3]) who proved that if $R$ is a 2-torsion free $*$-prime ring and $I$ a nonzero $*$-ideal of $R$ such that $*$ commutes with a derivation $d$ of $R$ and $d([x, y]) = 0$, for all $x, y \in I$, then $R$ is commutative. It is easy to see that in the hypothesis of the above result "$*$ *commutes with the derivation* $d$" can be avoided. Now we prove the following lemma, which improves Theorem 1.3 of [11] for the case when the underlying identity belongs to the center of a $*$-prime ring.

**Lemma 4.** *Let $R$ be a 2-torsion free $*$-prime ring. If $R$ admits a nonzero derivation $d$ such that $d([x, y]) \in Z(R)$ for all $x, y \in R$, then $R$ is commutative.*

Proof: Suppose that

$$d([x, y]) \in Z(R) \quad \text{for all} \quad x, y \in R. \tag{1}$$

We claim that $Z(R) \neq 0$. Indeed, otherwise equation (1) reduces to

$$d([x, y]) = 0 \quad \text{for all} \quad x, y \in R.$$

Hence $R$ is commutative by Theorem 1.3 of [11], and thus $R = 0$, a contradiction. Now replacing $x$ by $xz$ in (1), where $0 \neq z \in Z(R)$, we get $[x, y]d(z) \in Z(R)$. This yields that

$$[[x, y], r]Rd(z) = 0 \quad \text{for all} \quad x, y, r \in R. \tag{2}$$

In the light of $*$-primeness, either $[[x, y], r] = 0$ for all $x, y, r \in R$ and hence $R$ is commutative or $d(z) = 0$. Assume that $d(z) = 0$ for all $z \in Z(R)$. Then

$$d^2([x, y]) = 0 \quad \text{for all} \quad x, y \in R. \tag{3}$$

Replacing $y$ by $xy$ in (3) and employing (3) we obtain

$$d^2(x)[x, y] + 2d(x)d([x, y]) = 0 \quad \text{for all} \quad x, y \in R. \tag{4}$$

Substituting $yx$ for $y$ in (4) we get

$$d(x)[x, y]d(x) = 0 \quad \text{for all} \quad x, y \in R. \tag{5}$$

Writing $yd(x)r$ instead of $y$ in (5) and using (5) we obtain

$$d(x)y[x, d(x)]rd(x) = 0 \quad \text{for all} \quad r, x, y \in R. \tag{6}$$

Right multiplication of the last equation by $y[x, d(x)]$ yields

$$d(x)y[x, d(x)]Rd(x)y[x, d(x)] = 0 \quad \text{for all} \quad x, y \in R. \tag{7}$$

Since $R$ is semiprime, (7) forces that

$$d(x)y[x,d(x)] = 0 \text{ for all } x,y \in R. \tag{8}$$

Left multiplication of the last equation by $x$ implies that

$$xd(x)y[x,d(x)] = 0 \text{ for all } x,y \in R. \tag{9}$$

Replacing $y$ by $xy$ in (8) we obtain

$$d(x)xy[x,d(x)] = 0 \text{ for all } x,y \in R. \tag{10}$$

Subtracting (10) from (9) we find that

$$[d(x),x]R[d(x),x] = 0 \text{ for all } x \in R. \tag{11}$$

Once again using semiprimeness, we conclude that

$$[d(x),x] = 0 \text{ for all } x \in R. \tag{12}$$

Hence $R$ is commutative by Lemma 3. □

**Lemma 5.** *Let $R$ be a 2-torsion free $*$-prime ring and $J$ be a non-zero $*$-Jordan ideal of $R$ such that $[J,J] \subseteq Z(R)$. Then $R$ is commutative.*

Proof: If $[J,J] \subseteq Z(R)$, then $[[x,y],r] = 0$ for all $x,y \in J$ and $r \in R$. Now replacing $x$ by $2x^2$ and $y$ by $4yx^2$, we find that $[x^2,y][r,x^2] = 0$ for all $x,y \in J$ and $r \in R$. Further replacing $r$ by $ry$ we obtain that $[x^2,y]R[x^2,y] = 0$. But since every $*$-prime ring is semiprime the latter relation yields that $[x^2,y] = 0$, for all $x,y \in J$. Again replace $y$ by $2[r,s]y$ to get $[r,s][x^2,s] = 0$ for all $r,s \in R$ and $x \in J$. Now substituting $x^2r$ for $r$, we obtain $[x^2,s]R[x^2,s] = 0$. This forces that $[x^2,s] = 0$ for all $x \in J$. This yields that $[xy+yx,s] = 0$ for all $x,y \in J$ and $s \in R$. Hence replacing $x$ by $2x^2$ we arrive at $2[x^2y+yx^2,s] = 0$. Now since $R$ is 2-torsion free and $x^2 \in Z(R)$, we find that $x^2[y,s] = 0$. This implies that $x^2R[y,s] = 0 = x^2R[y,s]^*$ for all $x,y \in J$ and $s \in R$. Now using $*$-primeness we find that either $J \subseteq Z(R)$ or $x^2 = 0$. But if $x^2 = 0$ for all $x \in J$ then $xy+yx = 0$ for all $x,y \in J$. This yields that $x(rx+xr)+(rx+xr)x = 0$, for all $x \in J$ and $r \in R$ and hence $2xRx = 0$. But since $R$ is 2-torsion free and semiprime, we find that $x = 0$ for all $x \in J$, a contradiction. On the other hand if $J \subseteq Z(R)$, then $R$ is commutative by $(IV)$. □

**Theorem 1.** *Let $R$ be a 2-torsion free $*$-prime ring and $J$ be a nonzero $*$-Jordan ideal of $R$. If $R$ admits a nonzero derivation $d$ such that $d([x,y]) \in Z(R)$ for all $x,y \in J$, then $R$ is commutative.*

Proof: It can be easily seen that $d$ commutes with $*$. We have

$$d([x,y]) \in Z(R) \text{ for all } x,y \in J. \tag{13}$$

Let us consider $J_1 = \{x \in J \mid d(x) \in J\}$. Our claim is that $J_1 \neq 0$. Since $d(2x^2) = 2d(x^2) = 2(d(x) \circ x) \in J$, it follows that $2x^2 \in J_1$ for all $x \in J$. Hence if we assume that $J_1 = 0$, then 2-torsion freeness yields $x^2 = 0$ for all $x \in J$ and thus $x \circ y = 0$ for all $x, y \in J$. In particular, $x \circ (2[r,s]y) = 0$ so that $x \circ ([r,s]y) = 0$ for all $x, y \in J$, $r, s \in R$. Since $x \circ ([r,s]y) = [x,[r,s]]y + [r,s](x \circ y)$, the last equation reduces to $[x,[r,s]]y = 0$ and thus $[x,[r,s]]J = 0$ for all $x \in J$, $r, s \in R$. Therefore, $[x,[r,s]] = 0$ for all $x \in J$, $r, s \in R$, proving that $J \subseteq Z(R)$. Since $x^2 = 0$ for all $x \in J$, $xRx = 0$ for all $x \in J$ so that $J = 0$, a contradiction. It is also easy to see that $J_1$ is a $*$-Jordan ideal of $R$. If $J \cap Z(R) = 0$, then because of

$$d([x, 2y^2]) = [d(x), 2y^2] + [x, 2d(y) \circ y] \in Z(R) \cap J \quad \text{for all} \quad x, y \in J_1$$

we get

$$d([x, 2y^2]) = 0 \quad \text{for all} \quad x, y \in J_1. \tag{14}$$

Replacing $x$ by $4xy^2$ in (14) we arrive at

$$[x, y^2]d(y^2) = 0 \quad \text{for all} \quad x, y \in J_1. \tag{15}$$

Substituting $2[r,s]x$ for $x$ in (15) where $r, s \in R$ we obtain

$$[[r,s], y^2]J_1 d(y^2) = 0 \quad \text{for all} \quad y \in J_1,\ r, s \in R. \tag{16}$$

For $y \in Sa_*(R) \cap J_1$ we obtain $2y^2 \in Z(R) \cap J_1$ or $d(y^2) = 0$. But $2y^2 \in Z(R) \cap J_1$ forces $y^2 = 0$ so that in both the cases we arrive at $d(y^2) = 0$. Let $y \in J_1$. Since $y - y^*, y + y^* \in Sa_*(R) \cap J_1$, $d(y + y^*)^2 = d(y - y^*)^2 = 0$. Therefore, $d((y^*)^2) = -d(y^2)$. Replacing $y$ by $y^*$ in (16) we obtain

$$([[r,s], y^2])^* J_1 d(y^2) = 0 \quad \text{for all} \quad y \in J_1,\ r, s \in R. \tag{17}$$

Combining (16) with (17) and reasoning as above we deduce that

$$d(y^2) = 0 \quad \text{for all} \quad y \in J_1. \tag{18}$$

Hence by Lemma 2, $d = 0$, which is a contradiction. Consequently, $Z(R) \cap J \neq 0$.

Let $0 \neq z \in Z(R) \cap J$. Replacing $2xz$ instead of $x$ in (13) we obtain $[x,y]d(z) \in Z(R)$ so that $[[x,y],r]d(z) = 0$ for all $x, y \in J$, $r \in R$ and therefore

$$[[x,y],r]Rd(z) = 0 \quad \text{for all} \quad x, y \in J,\ r \in R. \tag{19}$$

As $[[x,y],r]^* Rd(z) = 0$ by (19), either $d(z) = 0$ or $[J,J] \subseteq Z(R)$. If $d(z) = 0$ for all $z \in Z(R) \cap J$ then replacing $x$ by $2rz$ in (13) and using (13) we obtain

$$d([r,y]) \in Z(R) \quad \text{for all} \quad y \in J,\ r \in R. \tag{20}$$

Substituting $2zs$ for $y$ in (20) we find that

$$d([r,s]) \in Z(R) \quad \text{for all} \quad r, s \in R \tag{21}$$

and hence $R$ is commutative by Lemma 4. If $[J,J]\subseteq Z(R)$, then $R$ is commutative by Lemma 5. $\square$

If $d([x,y])\in Z(R)$ then in that case we find the following theorem which improves the result of [2] for the case when $R$ is 2-torsion free.

**Theorem 2.** *Let $R$ be a prime ring of characteristic different from two and $J$ be a nonzero Jordan ideal of $R$. If $R$ admits a nonzero derivation $d$ such that $d([x,y])\in Z(R)$ for all $x,y\in J$, then $R$ is commutative.*

Proof: Assume that $d$ is a nonzero derivation of $R$ such that $d([x,y])\in Z(R)$ for all $x,y\in J$. Let $\mathcal{D}$ be the additive mapping defined on $\mathcal{R}=R\times R^0$ by $\mathcal{D}(x,y)=(d(x),0)$. Clearly, $\mathcal{D}$ is a nonzero derivation of $\mathcal{R}$. Moreover, if we set $\mathcal{J}=J\times J$, then $\mathcal{J}$ is a $*_{ex}$-Jordan ideal of $\mathcal{R}$ and $\mathcal{D}([x,y])\in Z(\mathcal{R})$ for all $x,y\in\mathcal{J}$. Since $\mathcal{R}$ is a $*_{ex}$-prime ring, in view of Theorem 1 we deduce that $\mathcal{R}$ is commutative and a fortiori $R$ is commutative. $\square$

**Corollary 1** ([7, Theorem 1])**.** *Let $R$ be a 2-torsion free $*$-prime ring and $J$ be a nonzero $*$-Jordan ideal of $R$. If $R$ admits a nonzero derivation $d$ such that $d([x,y])=0$ for all $x,y\in J$, then $R$ is commutative.*

**Theorem 3.** *Let $R$ be a 2-torsion free $*$-prime ring and $J$ a nonzero $*$-Jordan ideal of $R$. Then the following statements are equivalent:*

(i) *$R$ is commutative;*
(ii) *$R$ admits a derivation $d$ such that $d([x,y])-[x,y]\in Z(R)$ for all $x,y\in J$;*
(iii) *$R$ admits a derivation $d$ such that $d([x,y])+[x,y]\in Z(R)$ for all $x,y\in J$.*

Proof: Obviously (i) $\Longrightarrow$ (iii). Moreover, (iii) $\Longrightarrow$ (ii) indeed if $d([x,y])+[x,y]\in Z(R)$ then the derivation $(-d)$ satisfies $(-d)([x,y])-[x,y]\in Z(R)$.

(ii) $\Longrightarrow$ (i). Suppose that $R$ satisfies (ii). If $d=0$ then $[J,J]\subseteq Z(R)$ and $R$ is commutative by Lemma 5. Assume that $d$ is a nonzero derivation such that

$$d([x,y])-[x,y]\in Z(R)\quad\text{for all}\quad x,y\in J. \tag{22}$$

Define $J_1$ as in Theorem 1. If $J\cap Z(R)=0$, then because of

$$d([x,2y^2])-[x,2y^2]=[d(x),2y^2]+[x,2d(y)\circ y]-[x,2y^2]\in Z(R)\cap J$$

for all $x,y\in J_1$, we get

$$d([x,2y^2])-[x,2y^2]=0\quad\text{for all}\quad x,y\in J_1. \tag{23}$$

Replacing $x$ by $4xy^2$ in (23) we arrive at

$$[x,y^2]d(y^2)=0\quad\text{for all}\quad x,y\in J_1. \tag{24}$$

Since equation (24) is the same as equation (15), arguing as in the proof of Theorem 1 we are forced to $Z(R)\cap J\neq 0$.

Let $0 \neq z \in Z(R) \cap J$. Writing $2xz$ instead of $x$ in (22) and employing (22) we obtain $[x, y]d(z) \in Z(R)$ and thus

$$[[x, y], r]Rd(z) = 0 \text{ for all } x, y \in J \text{ and } r \in R. \tag{25}$$

Since equation (25) is the same as equation (19), arguing as in the proof of Theorem 1 we arrive at $R$ is commutative or $d(z) = 0$.

Assume that $d(z) = 0$. Replacing $x$ by $2zr = z \circ r$ in (22), where $r \in R$, we obtain

$$d([r, y]) - [r, y] \in Z(R) \text{ for all } y \in J,\ r \in R. \tag{26}$$

Replacing $y$ by $2zs$ in (26) where $s \in R$ we get

$$d([r, s]) - [r, s] \in Z(R) \text{ for all } r, s \in R, \tag{27}$$

and hence $R$ is commutative by Lemma 1. □

**Theorem 4.** *Let $R$ be a 2-torsion free $*$-prime ring and $J$ a nonzero $*$-Jordan ideal of $R$. Then $R$ is commutative if and only if $R$ admits a derivation $d$ such that for all $x, y \in J$, either $d([x, y]) - [x, y] \in Z(R)$ or $d([x, y]) + [x, y] \in Z(R)$.*

PROOF: Obviously, every commutative ring satisfies $d([x, y]) - [x, y] \in Z(R)$ and $d([x, y]) + [x, y] \in Z(R)$.

Conversely, for each fixed $x \in J$ we put $J_x = \{y \in J \mid d([x, y]) - [x, y] \in Z(R)\}$ and $J'_x = \{y \in J \mid d([x, y]) + [x, y] \in Z(R)\}$. Then obviously $J_x$ and $J_x^{'}$ are additive subgroups of $J$ whose union is $J$. But a group cannot be a set theoretic union of two of its proper subgroups. Hence either $J = J_x$ or $J = J_x^{'}$. Further using similar arguments as above, we find that $J = \{x \in J \mid J_x = J\}$ or $J = \{x \in J \mid J_x^{'} = J\}$. Therefore $R$ is commutative by Theorem 3. □

Using similar arguments as used in proving Theorem 2 and employing Theorem 3, we get the following theorem which improves Theorem 1 of [3].

**Theorem 5.** *Let $R$ be a prime ring of characteristic different from two and $J$ a nonzero Jordan ideal of $R$. If $R$ admits a derivation $d$ such that either $d([x, y]) - [x, y] \in Z(R)$ for all $x, y \in J$, or $d([x, y]) + [x, y] \in Z(R)$ for all $x, y \in J$, then $R$ is commutative.*

The following is an immediate consequence of Theorem 3.

**Corollary 2** ([7, Theorem 3])**.** *Let $R$ be a 2-torsion free $*$-prime ring and $J$ be a nonzero $*$-Jordan ideal of $R$. If $R$ admits a nonzero derivation $d$ such that $d([x, y]) = [x, y]$ for all $x, y \in J$, then $R$ is commutative.*

## 3. Differential identities with anti-commutator

This section is devoted to a question whether the commutativity of the ring still holds if the commutator in the statements of the preceding section is replaced by the anti-commutator. We have investigated this problem and obtained similar results.

**Theorem 6.** *Let $R$ be a 2-torsion free $*$-prime ring and $J$ be a nonzero $*$-Jordan ideal of $R$. If $R$ admits a nonzero derivation $d$ such that $d(x \circ y) \in Z(R)$ for all $x, y \in J$, then $R$ is commutative.*

PROOF: Suppose that

$$d(x \circ y) \in Z(R) \quad \text{for all} \quad x, y \in J. \tag{28}$$

If $J \cap Z(R) = 0$ then, because of $d(x \circ y) \in Z(R) \cap J$, we get

$$d(x \circ y) = 0 \quad \text{for all} \quad x, y \in J. \tag{29}$$

Hence $d = 0$ by Lemma 2, a contradiction. Accordingly, $J \cap Z(R) \neq 0$.

Let $0 \neq z \in J \cap Z(R)$. Replacing $x$ by $2zx = z \circ x$ in (28) we arrive at

$$d(x \circ y)z + (x \circ y)d(z) \in Z(R) \quad \text{for all} \quad x, y \in J. \tag{30}$$

Comparing (28) with (30) we obtain $(x \circ y)d(z) \in Z(R)$ so that

$$[(x \circ y), r]Rd(z) = 0 \quad \text{for all} \quad x, y \in J, \; r \in R. \tag{31}$$

In particular, we have

$$[x^2, r]Rd(z) = ([x^2, r])^* Rd(z) = 0 \quad \text{for all} \quad x \in J, \; r \in R. \tag{32}$$

Consequently, either $[x^2, r] = 0$ or $d(z) = 0$ for all $z \in J \cap Z(R)$. If $d(J \cap Z(R)) = 0$, then equation (28) assures that

$$d^2(x \circ y) = 0 \quad \text{for all} \quad x, y \in J. \tag{33}$$

Substituting $2x^2$ for $x$ and $2y^2$ for $y$ in (33) and using both (28) and (33) we arrive at

$$d(x^2)Rd(y^2) = 0 \quad \text{for all} \quad x, y \in J \tag{34}$$

and so that

$$d(x^2)Rd(x^2) = 0 \quad \text{for all} \quad x \in J \tag{35}$$

and the semiprimeness gives $d(x^2) = 0$ for all $x \in J$. Hence again by Lemma 2, $d = 0$, a contradiction. Thus $[x^2, r] = 0$ for all $x \in J$, $r \in R$. Now applying similar method as used in the proof of Lemma 5, we find that $R$ is commutative. □

Further using similar arguments as used to prove Theorem 2, with application of Theorem 6, one can prove the following.

**Theorem 7.** *Let $R$ be a prime ring of characteristic different from two and $J$ be a nonzero Jordan ideal of $R$. If $R$ admits a nonzero derivation $d$ such that $d(x \circ y) \in Z(R)$ for all $x, y \in J$, then $R$ is commutative.*

**Theorem 8.** *Let $R$ be a 2-torsion free $*$-prime ring and $J$ be a nonzero $*$-Jordan ideal of $R$. Then the following conditions are equivalent:*

(i) *$R$ is commutative;*
(ii) *$R$ admits a derivation $d$ such that $d(x \circ y) - x \circ y \in Z(R)$ for all $x, y \in J$;*
(iii) *$R$ admits a derivation $d$ such that $d(x \circ y) + x \circ y \in Z(R)$ for all $x, y \in J$.*

Proof: Clearly (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (i). Assume that $R$ satisfies (ii). If $d = 0$ then $x^2 \in Z(R)$ for all $x \in J$ and arguing as in the proof of Lemma 5 we arrive at $R$ is commutative. Suppose that $d$ is a nonzero derivation such that

$$d(x \circ y) - x \circ y \in Z(R) \text{ for all } x, y \in J. \tag{36}$$

If $Z(R) \cap J = 0$, then by (36) we obtain

$$d(x \circ y) = x \circ y \text{ for all } x, y \in J. \tag{37}$$

Writing $x^2$ instead of $x$ and $4yx^2$ instead of $y$ in (37) we get

$$(x^2 \circ y)d(x^2) = 0 \text{ for all } x, y \in J. \tag{38}$$

Substituting $2[r, s]y$ for $y$ in (38), where $r, s \in R$, we obtain

$$[x^2, [r, s]]yd(x^2) = 0$$

and therefore

$$[x^2, [r, s]]Jd(x^2) = 0 \text{ for all } x \in J, \ r, s \in R. \tag{39}$$

Since equation (39) is the same as equation (16), arguing as in the proof of Theorem 1 we find that $d = 0$, a contradiction. Consequently, $J \cap Z(R) \neq 0$.

Let $0 \neq z \in J \cap Z(R)$. Replacing $x$ by $2xz = x \circ z$ in (36) we find that $(x \circ y)d(z) \in Z(R)$ and hence

$$[x^2, r]Rd(z) = [x^2, r]^* Rd(z) = 0 \text{ for all } x \in J, \ r \in R. \tag{40}$$

$R$ being $*$-prime, from (40) it follows that either $d(Z(R) \cap J) = 0$ or $[x^2, r] = 0$ for all $x \in J$, $r \in R$. Assume that $d(Z(R) \cap J) = 0$. From (36) it follows that

$$d^2(x \circ y) - d(x \circ y) = 0 \text{ for all } x, y \in J. \tag{41}$$

In particular, for $0 \neq y \in J \cap Z(R)$ equation (41) leads to

$$(d^2(x) - d(x))y = 0 \text{ for all } x \in J, \tag{42}$$

and hence $(d^2(x) - d(x))Ry = (d^2(x) - d(x))Ry^* = 0$ for all $x \in J$. $R$ being $*$-prime implies that

$$d^2(x) - d(x) = 0 \text{ for all } x \in J. \tag{43}$$

Writing $4xu^2$ instead of $x$ in (43), where $u \in J$, we obtain

$$(44) \qquad d(x)d(u^2) = 0 \text{ for all } x, u \in J.$$

Replacing $x$ by $4u^2x$ in (44) we find that $d(u^2)Jd(u^2) = 0$ for all $u \in J$. Now application of $(V)$ assures that $d(u^2) = 0$ for all $u \in J$, and hence by Lemma 2, $d = 0$, a contradiction. Consequently, $[x^2, r] = 0$ for all $x \in J$, $r \in R$ and hence using similar arguments as given in the proof of Lemma 5, we get the required result. □

Using similar procedure as used to prove Theorem 4, we get the following.

**Theorem 9.** *Let $R$ be a 2-torsion free $*$-prime ring and $J$ a nonzero $*$-Jordan ideal of $R$. Then $R$ is commutative if and only if $R$ admits a derivation $d$ such that for all $x, y \in J$, either $d(x \circ y) - x \circ y \in Z(R)$ or $d(x \circ y) + x \circ y \in Z(R)$.*

If $R$ is a prime ring of characteristic different from two, then one can prove the following.

**Theorem 10.** *Let $R$ be a prime ring of characteristic different from two and $J$ be a nonzero Jordan ideal of $R$. If $R$ admits a derivation $d$ such that either $d(x \circ y) - x \circ y \in Z(R)$ for all $x, y \in J$, or $d(x \circ y) + x \circ y \in Z(R)$ for all $x, y \in J$, then $R$ is commutative.*

**Corollary 3.** *Let $R$ be a 2-torsion free $*$-prime ring and $J$ be a nonzero $*$-Jordan ideal of $R$. Suppose that $R$ admits a derivation $d$. Then the following conditions are equivalent:*

(i) *$d(xy) - xy \in Z(R)$ for all $x, y \in J$;*
(ii) *$d(xy) + xy \in Z(R)$ for all $x, y \in J$;*
(iii) *$d(xy) - yx \in Z(R)$ for all $x, y \in J$;*
(iv) *$d(xy) + yx \in Z(R)$ for all $x, y \in J$;*
(v) *$R$ is commutative.*

The following example demonstrates that Theorems 1, 3, 4, 6 and 8 cannot be extended to semiprime rings.

**Example 1.** Let $(R, \sigma)$ be a noncommutative prime ring with involution. Let us consider $\mathcal{R} = R \times Q[X]$. It is obvious to see that $\mathcal{R}$ is semiprime. Moreover, if we define $(r, P(X))^* = (\sigma(r), P(X))$, then $*$ is an involution of $\mathcal{R}$ for which $J = \{0\} \times Q[X]$ is a nonzero $*$-Jordan ideal. Furthermore, $D(r, P(X)) = (0, P'(X))$ is a nonzero derivation of $\mathcal{R}$ such that $D([u, v]) \in Z(\mathcal{R})$, $D([u, v]) - [u, v] \in Z(\mathcal{R})$, $D(u \circ v) \in Z(\mathcal{R})$, $D(u \circ v) - u \circ v \in Z(\mathcal{R})$, for all $u, v \in J$ but $\mathcal{R}$ is not commutative.

The following example shows that in Theorems 1 and 6 the hypothesis that $J$ is a $*$-Jordan ideal is crucial.

**Example 2.** Let $R$ be a noncommutative prime ring which admits a nonzero derivation $d$ and let $\mathcal{R} = R \times R^0$. If we set $J = R \times 0$, then $J$ is a nonzero Jordan ideal of the $*_{ex}$-prime ring $\mathcal{R}$. Furthermore, if we define $D(x, y) = (0, d(y))$, then

$D$ is a derivation of $\mathcal{R}$ which satisfies $D[u,v] \in Z(\mathcal{R})$, $D(u \circ v) \in Z(\mathcal{R})$ for all $u, v \in J$. However $\mathcal{R}$ is noncommutative.

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## References


[1] Awtar R., *Lie and Jordan structure in prime rings with derivations*, Proc. Amer. Math. Soc. **41** (1973), 67–74.

[2] Bell H.E., Daif M.N., *On derivations and commutativity in prime rings*, Acta Math. Hungar. **66** (1995), 337–343.

[3] Daif M.N., Bell H.E., *Remarks on derivations on semiprime rings*, Internat. J. Math. Math. Sci. **15** (1992), 205–206.

[4] Herstein I.N., *Rings with Involution*, Univ. Chicago Press, Chicago, 1976.

[5] Herstein I.N., *A note on derivations*, Canad. Math. Bull. **21** (1978), 369–370.

[6] Hongan M., *A note on semiprime rings with derivation*, Internat. J. Math. Math. Sci. **20** (1997), no. 2, 413–415.

[7] Mamouni A., Oukhtite L., *Derivations satisfying certain algebraic identities on Jordan ideals*, Arab. J. Math. **1** (2012), no. 3, 341–346.

[8] Mamouni A., Oukhtite L., *Generalized derivations centralizing on Jordan ideals of rings with involution*, submitted.

[9] Oukhtite L., *On Jordan ideals and derivations in rings with involution*, Comment. Math. Univ. Carolin. **51** (2010), no. 3, 389–395.

[10] Oukhtite L., *Posner's second theorem for Jordan ideals in rings with involution*, Expo. Math. **29** (2011), 415–419.

[11] Oukhtite L., Salhi S., *On derivations in $\sigma$-prime rings*, Int. J. Algebra **1** (2007), no. 5, 241–246.

[12] Zaidi S.M.A., Ashraf A., Ali S., *On Jordan ideals and left $(\theta,\theta)$-derivations in prime rings*, Int. J. Math. Math. Sci. 2004, no. 37–40, 1957–1964.


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