On generalized f-harmonic morphisms

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Abstract. In this paper, we study the characterization of generalized f-harmonic morphisms between Riemannian manifolds. We prove that a map between Riemannian manifolds is an f-harmonic morphism if and only if it is a horizontally weakly conformal map satisfying some further conditions. We present new properties generalizing Fuglede-Ishihara characterization for harmonic morphisms ([Fuglede B., *Harmonic morphisms between Riemannian manifolds*, Ann. Inst. Fourier (Grenoble) **28** (1978), 107–144], [Ishihara T., *A mapping of Riemannian manifolds which preserves harmonic functions*, J. Math. Kyoto Univ. **19** (1979), no. 2, 215–229]).

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1. Introduction

Consider a smooth map $\varphi : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds and let $f : M \times N \longrightarrow (0, +\infty)$ be a smooth positive function. The map φ is said to be f-harmonic (in a generalized sense) if it is a critical point of the f-energy functional

$$E_f(\varphi) = \frac{1}{2}\int_M f(x, \varphi(x))\,|d\varphi|^2\, v_g, \tag{1.1}$$

The Euler-Lagrange equation associated to the f-energy functional is

$$\tau_f(\varphi) \equiv f_\varphi \tau(\varphi) + d\varphi(\operatorname{grad}^M f_\varphi) - e(\varphi)(\operatorname{grad}^N f) \circ \varphi = 0, \tag{1.2}$$

where $f_\varphi : M \longrightarrow (0, +\infty)$ is a smooth positive function defined by

$$f_\varphi(x) = f(x, \varphi(x)), \quad \forall x \in M, \tag{1.3}$$

$\tau(\varphi) = \operatorname{trace}_g \nabla d\varphi$ is the tension field of φ, and $e(\varphi) = \frac{1}{2}|d\varphi|^2$ is the energy density of φ. $\tau_f(\varphi)$ is called the f-tension field of φ ([4]).

In particular, if $\varphi : M \longrightarrow N$ has no critical points, i.e. $|d_x\varphi| \neq 0$, then harmonic maps, p-harmonic maps and F-harmonic maps ([1]) are f-harmonic maps with $f = 1$, $f = |d\varphi|^{p-2}$ and $f = F'(\frac{|d\varphi|^2}{2})$ respectively.

Let $f_1 : M \longrightarrow (0, \infty)$ be a smooth function. If $f(x, y) = f_1(x)$ for all $(x, y) \in M \times N$, then $\tau_f(\varphi) = \tau_{f_1}(\varphi) = f_1\tau(\varphi) + d\varphi(\operatorname{grad}^M f_1)$. Moreover, $\varphi : M \longrightarrow N$

is f-harmonic if and only if it is f_1-harmonic in the sense of A. Lichnerowicz [9] and N. Course [3].

The identity map $Id : (\mathbb{R}^m, \langle\cdot,\cdot\rangle_{\mathbb{R}^m}) \longrightarrow (\mathbb{R}^m, \langle\cdot,\cdot\rangle_{\mathbb{R}^m})$ is f-harmonic if it satisfies the system of differential equation

$$\frac{\partial f}{\partial x^i} + \frac{2-m}{2}\frac{\partial f}{\partial y^i} = 0, \tag{1.4}$$

for all $i = 1, \ldots, m$, where $f \in C^\infty(\mathbb{R}^m \times \mathbb{R}^m)$ be a smooth positive function. Let $F \in C^\infty(\mathbb{R}^m)$ be a smooth positive function, then the function of type $f(x^1, \ldots, x^m, y^1, \ldots, y^m) = F(y^1 - \frac{2-m}{2}x^1, \ldots, y^m - \frac{2-m}{2}x^m)$ satisfies the system of differential equation (1.4).

For more details and examples of f-harmonic maps (in a generalized sense), we can refer to [4] and [5].

2. f-harmonic morphisms

Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth mapping between Riemannian manifolds. The critical set of φ is the set $C_\varphi = \{x \in M \mid d_x\varphi = 0\}$. The map φ is said to be horizontally weakly conformal or semi-conformal if for each $x \in M\backslash C_\varphi$, the restriction of $d_x\varphi$ to $\mathcal{H}_x$ is surjective and conformal, where the horizontal space $\mathcal{H}_x$ is the orthogonal complement of $\mathcal{V}_x = Ker\, d_x\varphi$. The horizontal conformality of φ implies that there exists a function $\lambda : M\backslash C_\varphi \longrightarrow \mathbb{R}_+$ such that for all $x \in M\backslash C_\varphi$ and $X, Y \in \mathcal{H}_x$

$$h(d_x\varphi(X), d_x\varphi(Y)) = \lambda(x)^2 g(X, Y). \tag{2.1}$$

The map φ is horizontally weakly conformal at x with dilation $\lambda(x)$ if and only if in any local coordinates (y^α) on a neighbourhood of $\varphi(x)$,

$$g(\operatorname{grad}^M \varphi^\alpha, \operatorname{grad}^M \varphi^\beta) = \lambda^2(h^{\alpha\beta} \circ \varphi) \qquad (\alpha, \beta = 1, \ldots, n). \tag{2.2}$$

Let $f : M \times \mathbb{R} \longrightarrow (0, +\infty)$, $(x, t) \longmapsto f(x, t)$ be a smooth function.

Definition 2.1. A C^2-function $u : U \longrightarrow \mathbb{R}$ defined on an open subset U of M is called f-harmonic if

$$\Delta_f^M u \equiv f_u\, \Delta^M u + du(\operatorname{grad}^M f_u) - e(u)\,(f')_u = 0, \tag{2.3}$$

where $f_u : M \longrightarrow (0, +\infty)$ is a smooth function defined by

$$f_u(x) = f(x, u(x)), \quad x \in U, \tag{2.4}$$

$(f')_u : M \longrightarrow (0, +\infty)$ is a smooth function defined by

$$(f')_u(x) = \frac{\partial f}{\partial t}(x, u(x)), \quad x \in U. \tag{2.5}$$

Definition 2.2. The map $\varphi : (M, g) \longrightarrow (N, h)$ is called a f-harmonic morphism if, for every harmonic function $v : V \longrightarrow \mathbb{R}$ defined on an open subset V of N with $\varphi^{-1}(V)$ non-empty, the composition $v \circ \varphi$ is f-harmonic on $\varphi^{-1}(V)$.

Theorem 2.1. *Let $\varphi : (M^m, g) \longrightarrow (N^n, h)$ be a smooth map. Let $f : M \times \mathbb{R} \longrightarrow (0, +\infty)$ be a smooth function. Then, the following are equivalent:*

(1) *φ is an f-harmonic morphism;*

(2) *φ is a horizontally weakly conformal with dilation λ satisfying*

$$f_{\varphi^\alpha}\, \tau(\varphi)^\alpha + g(\mathrm{grad}^M f_{\varphi^\alpha}, \mathrm{grad}^M \varphi^\alpha) - \frac{1}{2}\lambda^2 (f')_{\varphi^\alpha} (h^{\alpha\alpha} \circ \varphi) = 0, \tag{2.6}$$

for all $\alpha = 1, \dots, n$ and in any local coordinates (y^α) on N;

(3) *there exists a smooth positive function λ on M such that*

$$\Delta_f^M (v \circ \varphi) = f_{v\circ\varphi}\, \lambda^2\, (\Delta^N v) \circ \varphi,$$

for every smooth function $v : V \longrightarrow \mathbb{R}$ defined on an open subset V of N.

We will need the following lemma to prove the theorem.

Lemma 2.1 ([8])**.** *Let y_0 be a point in N^n, let (y^γ) be normal coordinates on N centered at y_0 and let $\{c_\gamma, c_{\alpha\beta}\}_{\alpha,\beta,\gamma=1}^n$ be constants with $c_{\alpha\beta} = c_{\beta\alpha}$ and $\sum_\alpha c_{\alpha\alpha} = 0$. Then there exists a neighborhood V of y_0 in N and a harmonic function $v : V \longrightarrow \mathbb{R}$ such that*

$$\frac{\partial v}{\partial y^\alpha}(y_0) = c_\alpha, \quad \frac{\partial^2 v}{\partial y^\alpha \partial y^\beta}(y_0) = c_{\alpha\beta}, \tag{2.7}$$

for all $\alpha, \beta, \gamma = 1, \dots, n$.

Proof of Theorem 2.1: Suppose $\varphi : (M^m, g) \longrightarrow (N^n, h)$ is a f-harmonic morphism. If $x_0 \in M$, consider systems of local coordinates (x^i) and (y^α) around x_0, $y_0 = \varphi(x_0)$, respectively, where we assume that (y^α) are normal, centered at y_0. To prove the horizontal conformality of φ, we apply Lemma 2.1, that is, we may for every sequence $(c_{\alpha\beta})_{\alpha,\beta=1}^n$ with $c_{\alpha\beta} = c_{\beta\alpha}$ and $\sum_\alpha c_{\alpha\alpha} = 0$ choose a harmonic function v such that

$$\frac{\partial v}{\partial y^\alpha}(y_0) = 0, \quad \frac{\partial^2 v}{\partial y^\alpha \partial y^\beta}(y_0) = c_{\alpha\beta}, \tag{2.8}$$

for all $\alpha, \beta = 1, \dots, n$. By assumption, the function $v \circ \varphi$ is f-harmonic in a neighbourhood of x_0, so by Definition 2.1

$$\begin{aligned} 0 =& \Delta_f^M (v \circ \varphi) \\ =& f_{v\circ\varphi}\, \Delta^M (v \circ \varphi) + dv(d\varphi(\mathrm{grad}^M f_{v\circ\varphi})) - e(v \circ \varphi)\, (f')_{v\circ\varphi}. \end{aligned} \tag{2.9}$$

In particular, since at x_0 we have

$$dv(d\varphi(\mathrm{grad}^M f_{v\circ\varphi})) = 0, \tag{2.10}$$

we get

$$e(v \circ \varphi) = 0. \tag{2.11}$$

By (2.9), (2.10) and (2.11) we have

$$\begin{aligned} 0 &= \Delta^M (v \circ \varphi) \\ &= dv(\tau(\varphi)) + \operatorname{trace}_g \nabla dv(d\varphi, d\varphi) \\ &= \operatorname{trace}_g \nabla dv(d\varphi, d\varphi). \end{aligned} \tag{2.12}$$

Since at x_0 we have

$$\nabla dv = \sum_{\alpha,\beta} \frac{\partial^2 v}{\partial y^\alpha \partial y^\beta} dy^\alpha \otimes dy^\beta = \sum_{\alpha,\beta} c_{\alpha\beta} dy^\alpha \otimes dy^\beta, \tag{2.13}$$

by (2.8), (2.12) and (2.13), we obtain

$$\begin{aligned} 0 &= \sum_{\alpha,\beta} g(\operatorname{grad}^M \varphi^\alpha, \operatorname{grad}^M \varphi^\beta) c_{\alpha\beta} \\ &= \sum_{\alpha} g(\operatorname{grad}^M \varphi^\alpha, \operatorname{grad}^M \varphi^\alpha) c_{\alpha\alpha} + \sum_{\alpha \neq \beta} g(\operatorname{grad}^M \varphi^\alpha, \operatorname{grad}^M \varphi^\beta) c_{\alpha\beta}. \end{aligned} \tag{2.14}$$

We subtract

$$0 = \sum_{\alpha} g(\operatorname{grad}^M \varphi^1, \operatorname{grad}^M \varphi^1) c_{\alpha\alpha}. \tag{2.15}$$

By (2.14) and (2.15), we obtain

$$\begin{aligned} 0 = &\sum_{\alpha} \Big[g(\operatorname{grad}^M \varphi^\alpha, \operatorname{grad}^M \varphi^\alpha) - g(\operatorname{grad}^M \varphi^1, \operatorname{grad}^M \varphi^1) \Big] c_{\alpha\alpha} \\ &+ \sum_{\alpha \neq \beta} g(\operatorname{grad}^M \varphi^\alpha, \operatorname{grad}^M \varphi^\beta) c_{\alpha\beta}. \end{aligned} \tag{2.16}$$

Let $\alpha_0 \neq 1$ and let

$$c_{\alpha\beta} = \begin{cases} 1, & \text{if } \alpha = \beta = 1; \\ -1, & \text{if } \alpha = \beta = \alpha_0; \\ 0, & \text{if } \alpha = \beta \neq 1, \alpha_0; \\ 0, & \text{if } \alpha \neq \beta. \end{cases}$$

Then by (2.16), we have

$$g(\operatorname{grad}^M \varphi^{\alpha_0}, \operatorname{grad}^M \varphi^{\alpha_0}) = g(\operatorname{grad}^M \varphi^1, \operatorname{grad}^M \varphi^1). \tag{2.17}$$

Then

$$g(\operatorname{grad}^M \varphi^{\alpha}, \operatorname{grad}^M \varphi^{\alpha}) = g(\operatorname{grad}^M \varphi^1, \operatorname{grad}^M \varphi^1), \tag{2.18}$$

for all $\alpha = 1, \dots, n$. Let $\alpha_0 \neq \beta_0$ and let

$$c_{\alpha\beta} = \begin{cases} 1, & \text{if } \alpha = \alpha_0 \text{ and } \beta = \beta_0; \\ 0, & \text{if } \alpha \neq \alpha_0 \text{ or } \beta \neq \beta_0; \\ 0, & \text{if } \alpha = \beta. \end{cases}$$

Then by (2.16), we have

$$g(\operatorname{grad}^M \varphi^{\alpha_0}, \operatorname{grad}^M \varphi^{\beta_0}) = 0. \tag{2.19}$$

So we have

$$g(\operatorname{grad}^M \varphi^{\alpha}, \operatorname{grad}^M \varphi^{\beta}) = 0, \tag{2.20}$$

for all $\alpha \neq \beta = 1, \dots, n$. It follows from (2.18) and (2.20) that the f-harmonic morphism φ is horizontally weakly conformal map

$$g(\operatorname{grad}^M \varphi^{\alpha}, \operatorname{grad}^M \varphi^{\beta}) = \lambda^2\, \delta_{\alpha\beta}, \tag{2.21}$$

for all $\alpha, \beta = 1, \dots, n$. For every C^2-function $v : V \longrightarrow \mathbb{R}$ defined on an open subset V of N, we have

$$\begin{aligned} \Delta_f^M(v \circ \varphi) &= f_{v\circ\varphi}\, \Delta^M(v \circ \varphi) + dv(d\varphi(\operatorname{grad}^M f_{v\circ\varphi})) - e(v \circ \varphi)(f')_{v\circ\varphi} \\ &= f_{v\circ\varphi} dv(\tau(\varphi)) + f_{v\circ\varphi} \operatorname{trace}_g \nabla dv(d\varphi, d\varphi) \\ &\quad + dv(d\varphi(\operatorname{grad}^M f_{v\circ\varphi})) - e(v \circ \varphi)(f')_{v\circ\varphi}. \end{aligned} \tag{2.22}$$

Since φ is horizontally weakly conformal map, we obtain

$$\begin{aligned} \Delta_f^M(v \circ \varphi) = f_{v\circ\varphi} dv(\tau(\varphi)) &+ f_{v\circ\varphi}\lambda^2(\Delta^N v) \circ \varphi \\ &+ dv(d\varphi(\operatorname{grad}^M f_{v\circ\varphi})) - e(v \circ \varphi)(f')_{v\circ\varphi}. \end{aligned} \tag{2.23}$$

By choosing v to be a harmonic function and since φ is an f-harmonic morphism, we conclude that

$$f_{v\circ\varphi} dv(\tau(\varphi)) + dv(d\varphi(\operatorname{grad}^M f_{v\circ\varphi})) - e(v \circ \varphi)(f')_{v\circ\varphi} = 0,$$

i.e. in any local coordinates (y^α) on N, we have

$$f_{\varphi^\alpha}\, \tau(\varphi)^\alpha + g(\operatorname{grad}^M f_{\varphi^\alpha}, \operatorname{grad}^M \varphi^\alpha) - \frac{1}{2}\lambda^2 (f')_{\varphi^\alpha}(h^{\alpha\alpha} \circ \varphi) = 0,$$

for all $\alpha = 1, \dots, n$.

Thus, we obtain the implication (1) $\Longrightarrow$ (2). Furthermore, the implication (2) $\Longrightarrow$ (3) follows from the formula (2.23). The implication (3) $\Longrightarrow$ (1) is trivial. $\square$

Example 2.1. The identity map $Id : (\mathbb{R}^m, \langle\cdot,\cdot\rangle_{\mathbb{R}^m}) \longrightarrow (\mathbb{R}^m, \langle\cdot,\cdot\rangle_{\mathbb{R}^m})$ is f-harmonic morphism if f satisfies the system of differential equation

$$\frac{\partial f}{\partial x^i} + \frac{1}{2}\frac{\partial f}{\partial t} = 0, \tag{2.24}$$

for all $i = 1, \dots, m$, where $f \in C^\infty(\mathbb{R}^m \times \mathbb{R})$ is a smooth positive function. Let $F \in C^\infty(\mathbb{R}^m)$ be a smooth positive function, then the function of the type $f(x^1, \dots, x^m, t) = F(t - \frac{1}{2}x^1, \dots, t - \frac{1}{2}x^m)$, satisfies the system of differential equation (2.24).

If $f(x,t) = 1$ for all $(x,t) \in M \times \mathbb{R}$, the condition (2.6) is equivalent to the condition $\tau(\varphi) = 0$ i.e. φ is harmonic. We arrive at the following corollary.

Corollary 2.1 ([6], [8])**.** *A smooth map $\varphi : M \longrightarrow N$ between Riemannian manifolds is a harmonic morphism if and only if $\varphi : M \longrightarrow N$ is both harmonic and horizontally weakly conformal.*

If $f(x,t) = f_1(x)$ for all $(x,t) \in M \times \mathbb{R}$, where $f_1 \in C^\infty(M)$ is a smooth positive function, the condition (2.6) is equivalent to the condition $f_1\, \tau(\varphi) + d\varphi(\operatorname{grad}^M f_1) = 0$ i.e. φ is f_1-harmonic. We arrive at the following corollary.

Corollary 2.2 ([10])**.** *A smooth map $\varphi : M \longrightarrow N$ between Riemannian manifolds is a f_1-harmonic morphism if and only if $\varphi : M \longrightarrow N$ is both f_1-harmonic and horizontally weakly conformal with $f_1 \in C^\infty(M)$ being a smooth positive function.*

Let $f : M \times \mathbb{R} \longrightarrow (0, +\infty)$, $(x,t) \longmapsto f(x,t)$ be a smooth function.

Corollary 2.3. *Let $\varphi : M \longrightarrow N$ be an f-harmonic morphism between Riemannian manifolds with dilation λ_1 and $\psi : N \longrightarrow P$ a harmonic morphism between Riemannian manifolds with dilation λ_2. Then the composition $\psi \circ \varphi : M \longrightarrow P$ is an f-harmonic morphism with dilation $\lambda_1(\lambda_2 \circ \varphi)$.*

Proof: This follows from the fact that

$$\Delta_f^M(v \circ \varphi) = f_{v\circ\varphi}\, \lambda_1^2\, (\Delta^N v) \circ \varphi,$$

for every smooth function $v : V \longrightarrow \mathbb{R}$ defined on an open subset V of N, and

$$\Delta^N(u \circ \psi) = \lambda_2^2\, (\Delta^P u) \circ \psi,$$

for every smooth function $u : U \longrightarrow \mathbb{R}$ defined on an open subset U of P. So that

$$\begin{aligned} \Delta_f^M(u \circ \psi \circ \varphi) &= f_{u\circ\psi\circ\varphi}\lambda_1^2\, (\Delta^N(u \circ \psi)) \circ \varphi \\ &= f_{u\circ\psi\circ\varphi}\lambda_1^2\, (\lambda_2 \circ \varphi)^2 (\Delta^P u) \circ \psi \circ \varphi. \end{aligned}$$

□

Corollary 2.4. *Let $\varphi : (M, g) \longrightarrow (N, h)$ be a smooth map of two Riemannian manifolds. If $f(x,t) = f_1(x)\, f_2(t)$ for all $(x,t) \in M \times \mathbb{R}$, where $f_1 \in C^\infty(M)$ is a smooth positive function and $f_2 \in C^\infty(\mathbb{R})$ is a smooth positive function. Then, the following are equivalent:*

(1) *φ is an f-harmonic morphism;*
(2) *φ is a horizontally weakly conformal with dilation λ satisfying*

$$(f_2 \circ \varphi^\alpha)\, \tau_{f_1}(\varphi)^\alpha + \frac{1}{2}\lambda^2 f_1 (f_2' \circ \varphi^\alpha)(h^{\alpha\alpha} \circ \varphi) = 0, \tag{2.25}$$

for all $\alpha = 1, \dots, n$ and in any local coordinates (y^α) on N.

Proof: By Theorem 2.1, the map $\varphi : (M, g) \longrightarrow (N, h)$ is f-harmonic morphism if and only if $\varphi : (M, g) \longrightarrow (N, h)$ is a horizontally weakly conformal with dilation λ satisfying the condition

$$f_{\varphi^\alpha}\, \tau(\varphi)^\alpha + g(\operatorname{grad}^M f_{\varphi^\alpha}, \operatorname{grad}^M \varphi^\alpha) - \frac{1}{2}\lambda^2 (f')_{\varphi^\alpha}(h^{\alpha\alpha} \circ \varphi) = 0,$$

for all $\alpha = 1, \dots, n$, and in any local coordinates (y^α) on N, i.e.

$$\begin{aligned} &f_1 (f_2 \circ \varphi^\alpha)\, \tau(\varphi)^\alpha + f_1 g(\operatorname{grad}^M (f_2 \circ \varphi^\alpha), \operatorname{grad}^M \varphi^\alpha) \\ &\quad + (f_2 \circ \varphi^\alpha) g(\operatorname{grad}^M f_1, \operatorname{grad}^M \varphi^\alpha) - \frac{1}{2}\lambda^2 f_1 (f_2' \circ \varphi^\alpha)(h^{\alpha\alpha} \circ \varphi) = 0, \end{aligned} \tag{2.26}$$

because $f_{\varphi^\alpha} = f_1 (f_2 \circ \varphi^\alpha)$.

Let $\tau_{f_1}(\varphi) = f_1\, \tau(\varphi) + d\varphi(\operatorname{grad}^M f_1)$ be the f_1-tension field of φ, then one has

$$\tau_{f_1}(\varphi)^\alpha = f_1 \tau(\varphi)^\alpha + g(\operatorname{grad}^M f_1, \operatorname{grad}^M \varphi^\alpha). \tag{2.27}$$

By (2.26) and (2.27), we obtain

$$\begin{aligned} &(f_2 \circ \varphi^\alpha)\, \tau_{f_1}(\varphi)^\alpha + f_1 g(\operatorname{grad}^M (f_2 \circ \varphi^\alpha), \operatorname{grad}^M \varphi^\alpha) \\ &- \frac{1}{2}\lambda^2 f_1 (f_2' \circ \varphi^\alpha)(h^{\alpha\alpha} \circ \varphi) = 0, \end{aligned} \tag{2.28}$$

the second term on the left-hand side of (2.28) is

$$\begin{aligned} f_1 g(\operatorname{grad}^M (f_2 \circ \varphi^\alpha), \operatorname{grad}^M \varphi^\alpha) &= f_1 (f_2' \circ \varphi^\alpha) g(\operatorname{grad}^M \varphi^\alpha, \operatorname{grad}^M \varphi^\alpha) \\ &= \lambda^2 f_1 (f_2' \circ \varphi^\alpha)(h^{\alpha\alpha} \circ \varphi). \end{aligned}$$

□

In the case where $f_2 = 1$, we recover the result obtained by Y.L. Ou [10] of f_1-harmonic morphisms (in the sense of A. Lichnerowicz [9] and N. Course [3]).

Proposition 2.1. *Let (M, g) be a Riemannian manifold. A smooth map*

$$\varphi : (M, g) \longrightarrow (\mathbb{R}^n, \langle \cdot, \cdot \rangle_{\mathbb{R}^n}), \quad x \longmapsto (\varphi^1(x), \dots, \varphi^n(x))$$

is an f-harmonic morphism if and only if its components φ^α are f-harmonic functions whose gradients are orthogonal and of the same norm at each point.

Proof: Let us notice that the condition (2.6) of Theorem 2.1 becomes

$$f_{\varphi^\alpha}\,\Delta^M\varphi^\alpha + g(\operatorname{grad}^M f_{\varphi^\alpha}, \operatorname{grad}^M \varphi^\alpha) - e(\varphi^\alpha)(f')_{\varphi^\alpha} = 0,$$

for all $\alpha = 1,\dots,n$, i.e. the functions φ^α are f-harmonic. □

Proposition 2.2. *Let $\varphi : (M,g) \longrightarrow (\mathbb{R}^n, \langle\cdot,\cdot\rangle_{\mathbb{R}^n})$ be a harmonic morphism of two Riemannian manifolds. Then $\varphi : (M,g) \longrightarrow (\mathbb{R}^n, \langle\cdot,\cdot\rangle_{\mathbb{R}^n})$ is f-harmonic morphism with $f(x,t) = f_1(x)\,e^{t+c}$ for all $(x,t) \in M\times\mathbb{R}$ and $f_1 \in C^\infty(M)$ being a smooth positive function defined by the components of φ as follows*

$$f_1 = e^{-\frac{1}{2}(\varphi^1+\cdots+\varphi^n)},$$

where $c \in \mathbb{R}_+$.

Proof: The map $\varphi : (M,g) \longrightarrow (\mathbb{R}^n, \langle\cdot,\cdot\rangle_{\mathbb{R}^n})$ where $\varphi = (\varphi^1,\dots,\varphi^n)$ is harmonic morphism if and only if it is harmonic horizontally and weakly conformal with dilation λ. Let $f_1 = e^{-\frac{1}{2}(\varphi^1+\cdots+\varphi^n)}$, so that

$$\tau_{f_1}(\varphi)^\alpha = f_1\tau(\varphi)^\alpha + g(\operatorname{grad}^M f_1, \operatorname{grad}^M \varphi^\alpha) = g(\operatorname{grad}^M f_1, \operatorname{grad}^M \varphi^\alpha),$$

because φ is harmonic. One has

$$\begin{aligned}\operatorname{grad}^M f_1 &= -\frac{1}{2}e^{-\frac{1}{2}(\varphi^1+\cdots+\varphi^n)}(\operatorname{grad}^M\varphi^1 + \cdots + \operatorname{grad}^M\varphi^n)\\ &= -\frac{1}{2}f_1(\operatorname{grad}^M\varphi^1 + \cdots + \operatorname{grad}^M\varphi^n).\end{aligned}$$

So we get

$$\tau_{f_1}(\varphi)^\alpha = -\frac{1}{2}f_1\Big(g(\operatorname{grad}^M\varphi^1, \operatorname{grad}^M\varphi^\alpha) + \cdots + g(\operatorname{grad}^M\varphi^n, \operatorname{grad}^M\varphi^\alpha)\Big).$$

Since φ is horizontally and weakly conformal with dilation λ, we obtain

$$\tau_{f_1}(\varphi)^\alpha = -\frac{1}{2}\lambda^2 f_1(\langle\cdot,\cdot\rangle_{\mathbb{R}^n})^{\alpha\alpha}\circ\varphi = -\frac{1}{2}\lambda^2 f_1. \tag{2.29}$$

Let $f(x,t) = f_1(x)\,e^{t+c}$ for all $(x,t)\in M\times\mathbb{R}$, where $c\in\mathbb{R}_+$. Then the condition (2.25) is equivalent to (2.29). Finally, by Corollary 2.4 the map φ is f-harmonic morphism. □

Example 2.2. Let (M,g) be a Riemannian manifold, $\gamma : M \longrightarrow (0,\infty)$ be a smooth function and let $M\times_{\gamma^2}\mathbb{R}^n$ be the warped product equipped with the Riemannian metric $G_\gamma = g + \gamma^2\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$. The natural projection

$$\pi_2 : (M\times_{\gamma^2}\mathbb{R}^n, G_\gamma) \longrightarrow (\mathbb{R}^n, \langle\cdot,\cdot\rangle_{\mathbb{R}^n}),$$

is harmonic morphism ([2]). According to Proposition 2.2 the natural projection π_2 is f-harmonic morphism with

$$f(x,y_1,\dots,y_n,t)=e^{-\frac{1}{2}(y^1+\dots+y^n)+t+c},\quad c\in\mathbb{R}_+$$

for all $(x,y_1,\dots,y_n,t)\in M\times\mathbb{R}^n\times\mathbb{R}$.

Example 2.3. Let $H^m=(\mathbb{R}^{m-1}\times\mathbb{R}^*_+,\frac{1}{x_m^2}\langle\cdot,\cdot\rangle_{\mathbb{R}^m})$. The projection

$$\pi_1:H^m\longrightarrow(\mathbb{R}^{m-1},\langle\cdot,\cdot\rangle_{\mathbb{R}^{m-1}}),\quad (x_1,\dots,x_{m-1},x_m)\longmapsto a\,(x_1,\dots,x_{m-1}),$$

where $a\in\mathbb{R}\backslash\{0\}$ is harmonic morphism ([2]). According to Proposition 2.2 the projection π_1 is f-harmonic morphism with

$$f(x_1,\dots,x_{m-1},x_m,t)=e^{-\frac{a}{2}(x_1+\dots+x_{m-1})+t+c},\quad c\in\mathbb{R}_+$$

for all $(x_1,\dots,x_{m-1},x_m,t)\in H^m\times\mathbb{R}$.

Example 2.4. (1) Let $\varphi:(\mathbb{R}^2\backslash\{0\},\langle\cdot,\cdot\rangle_{\mathbb{R}^2})\longrightarrow(\mathbb{R}^2\backslash\{0\},\langle\cdot,\cdot\rangle_{\mathbb{R}^2})$ be defined by

$$\varphi(x,y)=\big(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2}\big).$$

Then φ is a horizontally and weakly conformal map with dilation $\lambda(x,y)=\frac{1}{x^2+y^2}$, and φ is f-harmonic morphism with

$$f(x,y,t)=F\Big(2\,t-\frac{x+y}{x^2+y^2}\Big),$$

where $F:\mathbb{R}\longrightarrow(0,\infty)$ is a smooth function. Indeed, we have

$$\varphi^1(x,y)=\frac{x}{x^2+y^2}\,,\quad \varphi^2(x,y)=\frac{y}{x^2+y^2}\,,\quad f_{\varphi^1}(x,y)=F\Big(\frac{x-y}{x^2+y^2}\Big),$$

$$f_{\varphi^2}(x,y)=F\Big(\frac{y-x}{x^2+y^2}\Big),\quad \Delta^{\mathbb{R}^2}\varphi^1=\Delta^{\mathbb{R}^2}\varphi^2=0,$$

$$\operatorname{grad}^{\mathbb{R}^2}\varphi^1=\Big(\frac{y^2-x^2}{(x^2+y^2)^2}\,,-\frac{2xy}{(x^2+y^2)^2}\Big),$$

$$\operatorname{grad}^{\mathbb{R}^2}\varphi^2=\Big(-\frac{2xy}{(x^2+y^2)^2}\,,\frac{x^2-y^2}{(x^2+y^2)^2}\Big),$$

$$\operatorname{grad}^{\mathbb{R}^2}f_{\varphi^1}=F'\Big(\frac{x-y}{x^2+y^2}\Big)\Big(\frac{-x^2+y^2+2xy}{(x^2+y^2)^2},-\frac{x^2-y^2+2xy}{(x^2+y^2)^2}\Big),$$

$$\operatorname{grad}^{\mathbb{R}^2}f_{\varphi^2}=F'\Big(\frac{y-x}{x^2+y^2}\Big)\Big(\frac{x^2-y^2-2xy}{(x^2+y^2)^2},\frac{x^2-y^2+2xy}{(x^2+y^2)^2}\Big),$$

$$\langle \operatorname{grad}^{\mathbb{R}^2} \varphi^1, \operatorname{grad}^{\mathbb{R}^2} f_{\varphi^1} \rangle_{\mathbb{R}^2} = \frac{F'\Big(\frac{x-y}{x^2+y^2}\Big)}{(x^2+y^2)^2},$$

$$\langle \operatorname{grad}^{\mathbb{R}^2} \varphi^2, \operatorname{grad}^{\mathbb{R}^2} f_{\varphi^2} \rangle_{\mathbb{R}^2} = \frac{F'\Big(\frac{y-x}{x^2+y^2}\Big)}{(x^2+y^2)^2},$$

$$e(\varphi^1) = e(\varphi^2) = \frac{1}{2(x^2+y^2)^2}, \ (f')_{\varphi^1} = 2F'\Big(\frac{x-y}{x^2+y^2}\Big), \ (f')_{\varphi^2} = 2F'\Big(\frac{y-x}{x^2+y^2}\Big).$$

By (2.3) the functions φ^1 and φ^2 are f-harmonic and by Proposition 2.1 the map φ is f-harmonic morphism. With the same method we find that:

(2) Let $\psi : (\mathbb{R}^3\backslash\{0\}, \langle \cdot, \cdot \rangle_{\mathbb{R}^3}) \longrightarrow (\mathbb{R}^3\backslash\{0\}, \langle \cdot, \cdot \rangle_{\mathbb{R}^3})$ be defined by

$$\psi(x,y,z) = \Big(\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2}\Big).$$

Then ψ is f-harmonic morphism with

$$f(x,y,z,t) = \frac{F\Big(2\,t - \frac{x+y+z}{x^2+y^2+z^2}\Big)}{x^2+y^2+z^2},$$

where $F : \mathbb{R} \longrightarrow (0, \infty)$ is a smooth function. Here ψ is a horizontally and weakly conformal map with dilation $\lambda(x,y,z) = \frac{1}{x^2+y^2+z^2}$.

Remark 2.1. Using Proposition 2.1, we can construct many examples for f-harmonic morphisms (in a generalized sense).

Proposition 2.2 remains true for the map $\varphi : (M,g) \longrightarrow (N,h)$, where N is an open subsets of $\mathbb{R}^n$ and $h = e^{\alpha(y)}\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ is a metric conformally equivalent to the standard inner product on $\mathbb{R}^n$.

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