On characterized subgroups of Abelian topological groups X and the group of all X-valued null sequences

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Abstract. Let X be an Abelian topological group. A subgroup H of X is characterized if there is a sequence $\mathbf{u} = \{u_n\}$ in the dual group of X such that $H = \{x \in X : (u_n, x) \to 1\}$. We reduce the study of characterized subgroups of X to the study of characterized subgroups of compact metrizable Abelian groups. Let $c_0(X)$ be the group of all X-valued null sequences and $\mathfrak{u}_0$ be the uniform topology on $c_0(X)$. If X is compact we prove that $c_0(X)$ is a characterized subgroup of $X^{\mathbb{N}}$ if and only if $X \cong \mathbb{T}^n \times F$, where $n \geq 0$ and F is a finite Abelian group. For every compact Abelian group X, the group $c_0(X)$ is a $\mathfrak{g}$-closed subgroup of $X^{\mathbb{N}}$. Some general properties of $(c_0(X), \mathfrak{u}_0)$ and its dual group are given. In particular, we describe compact subsets of $(c_0(X), \mathfrak{u}_0)$.

Keywords: group of null sequences; T-sequence; characterized subgroup; T-characterized subgroup; $\mathfrak{g}$-closed subgroup

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1. Introduction

Notation and preliminaries. In the article all topological groups are assumed to be Hausdorff. Let X be an Abelian topological group. The filter of all open neighborhoods at zero we denote by $\mathcal{N}(X)$. As usual, $X^{\mathbb{N}}$ denotes the group of all sequences $\mathbf{x} = (x_n)_{n\in\mathbb{N}}$ and $X^{(\mathbb{N})}$ denotes the subgroup of $X^{\mathbb{N}}$ of all sequences eventually equal to zero. Unless otherwise stated we consider $X^{\mathbb{N}}$ with the product topology $\mathfrak{p}_X$ and write simply $X^{\mathbb{N}} = (X^{\mathbb{N}}, \mathfrak{p}_X)$.

It is easy to check that the collection $\{V^{\mathbb{N}} : V \in \mathcal{N}(X)\}$ forms a basis at 0 for a group topology in $X^{\mathbb{N}}$. This topology is called *the uniform topology* and is denoted by $\mathfrak{u}_X$ [21]. Following [21], denote by $c_0(X)$ the following subgroup of $X^{\mathbb{N}}$

$$c_0(X) := \left\{ (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} : \lim_n x_n = 0 \right\}.$$

The uniform group topology on $c_0(X)$ induced from $(X^{\mathbb{N}}, \mathfrak{u}_X)$ we denote by $\mathfrak{u}_0$. If $X = \mathbb{R}$, then $(c_0(\mathbb{R}), \mathfrak{u}_0)$ coincides with the classical Banach space c_0. The groups of the form $(c_0(X), \mathfrak{u}_0)$ were thoroughly studied in [21].

For an Abelian topological group X we denoted by $\widehat{X}$ the group of all continuous characters on X. The group $\widehat{X}$ endowed with the compact-open topology is denoted by $X^\wedge$. Recall that X is called *minimally almost periodic* (*MinAP*) if

$\widehat{X} = \{0\}$, and X is called *maximally almost periodic* (*MAP*) if $\widehat{X}$ separates the points of X.

Let $\mathbf{u} = \{u_n\}_{n\in\omega}$ be a sequence in an Abelian group G. We say that a sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$ in G is *trivial* if there is $g \in G$ such that $u_n = g$ for almost all indices n. In the case $g = 0$ we say that $\mathbf{u}$ is *eventually equal to zero*. For a nontrivial $\mathbf{u}$, in general, no Hausdorff group topology may exist in which $\mathbf{u}$ converges to zero. A very important question of whether there exists a Hausdorff group topology τ on G such that $u_n \to 0$ in (G, τ), especially for the integers, has been studied by many authors — Graev [32], Nienhuys [39], and others. A complete answer to this question was given by Protasov and Zelenyuk in [41]. Following [41], we say that a sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$ in an Abelian group G is a *T-sequence* if there is a Hausdorff group topology on G in which u_n converges to zero. The finest group topology with this property we denote by $\tau_{\mathbf{u}}$.

The counterpart of the above question for *precompact* group topologies on $\mathbb{Z}$ is studied by Raczkowski [42]. Recall that an Abelian topological group X is called *precompact* if it is Hausdorff and for every open neighborhood U of zero there exists a finite subset F of X such that $U + F = X$. Following [5], [6] and motivated by [42], we say that a sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$ is a *TB-sequence* in an Abelian group G if there is a precompact Hausdorff group topology on G in which u_n converges to zero. Clearly, every TB-sequence is a T-sequence, but the converse assertion, in general, does not hold. Note that the sequence $\mathbf{e} = \{e_n\}_{n\in\mathbb{N}}$, where $e_1 = (1, 0, 0, \dots)$, $e_2 = (0, 1, 0, \dots), \dots$, is a TB-sequence in $\mathbb{Z}^{(\mathbb{N})}$. The topology $\tau_{\mathbf{e}}$ and the dual group of $(\mathbb{Z}^{(\mathbb{N})}, \tau_{\mathbf{e}})$ are described in [26] explicitly.

Main results. Let $\mathbf{u} = \{u_n\}_{n\in\omega}$ be a sequence in the dual group $X^\wedge$ of an Abelian topological group X. Following [22], set

$$s_{\mathbf{u}}(X) = \{x \in X : (u_n, x) \to 1\}. \tag{1}$$

In [5] the following simple criterion for $\mathbf{u}$ to be a TB-sequence was obtained:

Fact 1 ([5]). *A sequence $\mathbf{u}$ in a (discrete) Abelian group G is a TB-sequence if and only if the subgroup $s_{\mathbf{u}}(X)$ of the (compact) dual $X = G^\wedge$ is dense.*

The subgroups of the form $s_{\mathbf{u}}(X)$ of a compact metrizable abelian group X have been studied by many authors, especially in the case $X = \mathbb{T}$, see [3], [7], [9], [10], [11], [13], [24], [37], [38]. It was proved by Borel [13] that every countable subgroup of $\mathbb{T}$ has the form (1) for an appropriate $\mathbf{u}$. Larcher [38] proved (see also [5]) that for the celebrated Fibonacci's sequence $\mathbf{f} = \{f_n\}$, defined by $f_0 = f_1 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for all $n \in \mathbb{N}$, one has $s_{\mathbf{f}}(\mathbb{T}) = \langle\alpha\rangle$, where α is the positive solution of the equation $x^2 - x - 1 = 0$ (namely, the Golden Ratio) taken modulo 1. Note that the interest in the study of the groups of the form $s_{\mathbf{u}}(X)$ is motivated also by some applications to Diophantine approximation, dynamical systems and ergodic theory (see [12], [40], [45]).

Motivated by the above circumstances, the following notion was proposed in [22]:

Definition 2 ([22])**.** Let H be a subgroup of a topological Abelian group X. If there exists a sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$ of characters of X such that $H = s_{\mathbf{u}}(X)$ we say that H is *characterized* (by $\mathbf{u}$) and that $\mathbf{u}$ *characterizes* H.

Note that for the torus $\mathbb{T}$ this notion was already defined in [11]. Characterized subgroups has been studied by many authors, see, for example, [8], [11], [19], [20], [22], [27]. In particular, the main theorem of [20] and [8] asserts that every countable subgroup of a compact metrizable Abelian group is characterized.

It is important to emphasize that there is no restriction on a sequence $\mathbf{u}$ in Definition 2. Let X be a compact Abelian group. If a characterized subgroup H of X is dense, then a characterizing sequence is also a TB-sequence by Fact 1. On the other hand, if X is not connected, then every open proper subgroup H of X is characterized [19], but any characterizing sequence for H is not a T-sequence [30]. The question for which characterized subgroups of compact Abelian groups one can find characterizing sequences which are also T-sequences is considered in [30]. This question is of independent interest because every T-sequence $\mathbf{u}$ naturally defines the Protasov-Zelenyuk topology $\tau_{\mathbf{u}}$ satisfying the following dual property:

Fact 3 ([27], [28])**.** *Let H be a T-characterized subgroup of an infinite compact Abelian group X by a T-sequence $\mathbf{u}$. Then $\widehat{(\widehat{X}, \tau_{\mathbf{u}})} = H (= s_{\mathbf{u}}(X))$. Moreover, if X is metrizable, then $(\widehat{X}, \tau_{\mathbf{u}})^\wedge$ is Polish.*

This motivates us to introduce the following notion:

Definition 4 ([30])**.** Let H be a subgroup of a compact Abelian group X. We say that H is a *T-characterized subgroup* of X if there exists a T-sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$ in $\widehat{X}$ such that $H = s_{\mathbf{u}}(X)$.

Note that every Abelian topological group X is a T-characterized subgroup of X by the zero sequence. Clearly, every T-characterized subgroup of X is also a characterized subgroup of X. Nevertheless, as it was noticed above, there are compact groups which have characterized non-T-characterized subgroups.

The previous discussion emphasizes the importance of the following general problem:

Problem 5. *Describe the characterized and T-characterized subgroups of an Abelian topological group.*

As it was noticed in [16], [18], every characterized subgroup H of an Abelian topological group X is an $F_{\sigma\delta}$-subgroup of X, and hence H is a Borel subset of X. In [19], [30], the Borel hierarchy of characterized and T-characterized subgroups of compact Abelian groups is studied in detail, where the authors reduced the study of characterized and T-characterized subgroups of *compact* Abelian groups to the study of characterized and T-characterized ones of compact *metrizable* Abelian groups. It is natural to ask whether such a reduction can be given in the general case. In the next theorem we give a positive answer to this question:

Theorem 6. *For a subgroup H of an Abelian topological group X, the following statements are equivalent.*

(i) *H is a characterized (respectively, T-characterized) subgroup of X.*

(ii) *H has a subgroup H_0 such that*

(1) *H_0 is a closed G_δ-subgroup of X;*

(2) *there exists a continuous monomorphism i from X/H_0 into a compact metrizable Abelian group K with dense image such that $i(H/H_0) = i(X/H_0) \cap s_{\widetilde{\mathbf{u}}}(K)$ for some sequence (respectively, T-sequence) $\widetilde{\mathbf{u}}$ in $\widehat{K}$.*

(iii) *There exists a continuous homomorphism p of X into a compact metrizable Abelian group K with dense image and a characterized (respectively, T-characterized) subgroup H' of K such that $H = p^{-1}(H')$.*

Note that in the proofs of Lemma 11 and Theorem 6 we describe explicitly the groups H_0 and K and homomorphisms i and p.

Let X be a nontrivial Abelian topological group. To check that a given subgroup H of X is characterized is one of the most interesting constituent of Problem 5. The subgroup $c_0(X)$ is a very natural subgroup of the direct product $X^{\mathbb{N}}$ (note that, as a dense subgroup, $c_0(X)$ is never a direct summand of $X^{\mathbb{N}}$). So one can ask:

Problem 7. *For which Abelian topological groups X the group $c_0(X)$ is a characterized (respectively, T-characterized) subgroup of $X^{\mathbb{N}}$?*

An answer to this question is of independent interest since it determines also one of the properties of the group $(c_0(X), \mathfrak{u}_0)$.

Our interest in the study of groups of the form $(c_0(X), \mathfrak{u}_0)$ is explained by the following arguments. Firstly, the assignment [21]

$$X \to \mathfrak{F}_0(X) := (c_0(X), \mathfrak{u}_0)$$

defines a functor $\mathfrak{F}_0 : \mathbf{TopAb} \to \mathbf{TopAb}$, where **TopAb** is the category of Hausdorff Abelian topological groups and continuous homomorphisms. By Proposition 3.4 and Remark 3.5 in [21], this functor preserves many important topological properties as (sequential) completeness, metrizability, separability, maximal almost periodicity, local quasi-convexity and connectedness. Secondly, if X is metrizable and connected, then the Bohr modification of $(c_0(X), \mathfrak{u}_0)$ is a connected precompact metrizable non-Mackey group by Theorem 1.8 in [21] (the definition of Mackey groups and their basic properties see [15], [21]). Thirdly, Rolewicz in [43] observed that a complete metrizable group $(c_0(\mathbb{T}), \mathfrak{u}_0)$ is monothetic (see a proof of this fact in [23, pp. 20–21]). So monothetic Polish groups need not be compact nor discrete. Moreover, the dual group of $(c_0(\mathbb{T}), \mathfrak{u}_0)$ is countable [26]. In particular, the last two arguments show that the groups of the form $(c_0(X), \mathfrak{u}_0)$ represent a nice source of (counter)examples in different areas of topological algebra. Fourthly, the group $(c_0(\mathbb{T}), \mathfrak{u}_0)$ plays an essential role in the theory of characterized subgroups of compact Abelian groups due to Theorem 1.28 of [28].

Taking into account a role of compact groups in **TopAb**, it is natural to consider Problem 7 for compact Abelian groups. By Theorem 1 of [26], we have $(\mathbb{Z}^{(\mathbb{N})}, \tau_{\mathbf{e}})^{\wedge} = (c_0(\mathbb{T}), \mathfrak{u}_0)$. Hence $c_0(\mathbb{T}) = s_{\mathbf{e}}(\mathbb{T}^{\mathbb{N}})$ is a T-characterized subgroup of $\mathbb{T}^{\mathbb{N}}$ by Fact 3. The next theorem gives a complete answer to Problem 7 for compact X.

Theorem 8. *For a compact Abelian group X the following are equivalent:*

(i) *$c_0(X)$ is a characterized subgroup of $X^{\mathbb{N}}$;*
(ii) *$c_0(X)$ is a T-characterized subgroup of $X^{\mathbb{N}}$;*
(iii) *$X \cong \mathbb{T}^n \times F$, where $n \geq 0$ and F is a finite Abelian group (maybe trivial).*

Theorem 8 shows that to be a characterized subgroup is a very strong property for $c_0(X)$. Let us consider a weaker one.

Let H be a subgroup of an Abelian topological group X. Following [22] we say that H is $\mathfrak{g}$-*closed* if

$$H = \bigcap_{\mathbf{u} \in \widehat{X}^{\mathbb{N}}} \{s_{\mathbf{u}}(X) : \ H \leq s_{\mathbf{u}}(X)\} .$$

The set of all T-sequences in the dual group $\widehat{X}$ of a compact Abelian group X we denote by $\mathcal{T}_s(\widehat{X})$. In analogy to the closure operator $\mathfrak{g}$, the operator $\mathfrak{g}_T$ is defined as follows [30]

$$\mathfrak{g}_T(H) := \bigcap_{\mathbf{u} \in \mathcal{T}_s(\widehat{X})} \{s_{\mathbf{u}}(X) : \ H \leq s_{\mathbf{u}}(X)\} ,$$

and we say that H is $\mathfrak{g}_T$-*closed* if $H = \mathfrak{g}_T(H)$.

Clearly, if H is a characterized (respectively, T-characterized) subgroup of an Abelian topological group X, then it is also $\mathfrak{g}$-closed in X. Thus the property to be $\mathfrak{g}$-closed is weaker than the property to be characterized. So it is natural to ask:

Problem 9. *For which Abelian topological groups X the group $c_0(X)$ is a $\mathfrak{g}$-closed (respectively, $\mathfrak{g}_T$-closed) subgroup of $X^{\mathbb{N}}$?*

It is easy to show (see Proposition 45 below) that, if $c_0(X)$ is $\mathfrak{g}$-closed in $X^{\mathbb{N}}$, then X is MAP.

In the next theorem we give a complete answer to Problem 9 for compact groups (see Theorem 48 for a more general statement):

Theorem 10. *For every compact Abelian group X the group $c_0(X)$ is $\mathfrak{g}_T$-closed in $X^{\mathbb{N}}$.*

The article is organized as follows. Theorem 6 is proved in Section 2. Also in this section we characterize Abelian topological groups which have no proper characterized or proper T-characterized subgroups (see Proposition 12). The case when X is itself a characterized subgroup of X by a *nontrivial* sequence is of independent interest. We call such groups *self-characterized*. A complete description of self-characterized Abelian topological groups is given in Theorem 19. In Theorem 21 we show that a locally compact Abelian group X is self-characterized if and only if X is not compact.

Some general properties of the groups of the form $(c_0(X), \mathfrak{u}_0)$ are obtained in Section 3. In particular, we give a complete description of compact subsets of $(c_0(X), \mathfrak{u}_0)$ (see Proposition 31). In Section 4 we prove some general assertions concerning the dual group of $(c_0(X), \mathfrak{u}_0)$. Making use of these results we prove Theorems 8 and 10 in Sections 5 and 6 respectively.

2. On characterized subgroups of Abelian topological groups

Let G be an Abelian group. The subgroup of G generated by a subset A we denote by $\langle A\rangle$. The group G endowed with the discrete topology we denote by G_d.

Let $p : X \to Y$ be a continuous homomorphism of Abelian topological groups. The mapping $p^\wedge : Y^\wedge \to X^\wedge$, defined by $p^\wedge(\chi) = \chi \circ p$ for $\chi \in Y^\wedge$, is called *the adjoint homomorphism* of p.

For a subset A of an Abelian topological group X, *the annihilator* of A in $\widehat{X}$ is $A^\perp := \{u \in \widehat{X} : (u, x) = 1, \forall x \in A\}$, while for a subset B of $\widehat{X}$, *the annihilator of B in X* is $B^\top := \{x \in X : (u, x) = 1, \forall u \in B\}$.

Let X be a MAP Abelian topological group and G be an arbitrary subgroup of $X^\wedge$ which separates the points of X. Denote by $T_G := \sigma(X, G)$ the weakest (Hausdorff) topology on X such that all characters of G are continuous with respect to T_G. Then T_G is a precompact group topology on X. The weak* group topology on $\widehat{X}$ we denote by $\sigma(\widehat{X}, X)$.

For a sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$ in the dual group of X, set [19]

$$K_{\mathbf{u}} := \bigcap_{n\in\omega} \ker(u_n).$$

In what follows we use the next generalization of Lemma 2.2 in [19] and Lemma 2.4 of [30].

Lemma 11. *For an Abelian topological group X, let $\mathbf{u} = \{u_n\}_{n\in\omega}$ be a sequence (respectively, a T-sequence) in $\widehat{X}$. Then the following holds.*

(i) *The subgroup $K_{\mathbf{u}}$ is a closed G_δ-subgroup of X such that $K_{\mathbf{u}} \subseteq s_{\mathbf{u}}(X)$ and $K_{\mathbf{u}} = \langle \mathbf{u}\rangle^\top$.*

(ii) *There is a sequence (respectively, a T-sequence) $\widetilde{\mathbf{u}}$ in $(X/K_{\mathbf{u}})^\wedge$ such that:*
  - (a) *$\mathbf{u} = q^\wedge(\widetilde{\mathbf{u}})$, where $q^\wedge$ is the adjoint homomorphism of the quotient map $q : X \to X/K_{\mathbf{u}}$;*
  - (b) *$s_{\mathbf{u}}(X)/K_{\mathbf{u}} = s_{\widetilde{\mathbf{u}}}(X/K_{\mathbf{u}})$, i.e., $\widetilde{\mathbf{u}}$ characterizes $s_{\mathbf{u}}(X)/K_{\mathbf{u}}$ in $X/K_{\mathbf{u}}$;*
  - (c) *there exists a continuous monomorphism i from $X/K_{\mathbf{u}}$ into the compact metrizable Abelian group K, where $K = \langle\widetilde{\mathbf{u}}\rangle_d^\wedge$, with dense image such that $i(s_{\mathbf{u}}(X)/K_{\mathbf{u}}) = i(X/K_{\mathbf{u}}) \cap s_{\widetilde{\mathbf{u}}}(K)$.*

Proof: (i) It is trivial that $K_{\mathbf{u}}$ is a closed subgroup of X contained in $s_{\mathbf{u}}(X)$. Let us prove the equality $K_{\mathbf{u}} = \langle \mathbf{u}\rangle^{\top}$. Clearly, $\langle \mathbf{u}\rangle^{\top} \subseteq K_{\mathbf{u}}$. Conversely, if $h \in K_{\mathbf{u}}$, then $(u_n, h) = 1$ for every $n \in \omega$. Hence $(g,h) = 1$ for every $g \in \langle \mathbf{u}\rangle$. Thus $K_{\mathbf{u}} \subseteq \langle \mathbf{u}\rangle^{\top}$. We will prove that $K_{\mathbf{u}}$ is a G_δ-subgroup of X at the end of the proof of item (ii).

(ii) For every $n \in \omega$, define the character $\widetilde{u}_n$ of $X/K_{\mathbf{u}}$ as follows: $(\widetilde{u}_n, q(x)) = (u_n, x)$ ($\widetilde{u}_n$ is well-defined since $K_{\mathbf{u}} \subseteq \ker u_n$). Then $\widetilde{\mathbf{u}} = \{\widetilde{u}_n\}_{n\in\omega}$ is a sequence of continuous characters of $X/K_{\mathbf{u}}$ such that $q^{\wedge}(\widetilde{u}_n) = u_n$. Assume now that $\mathbf{u}$ is a T-sequence. Then, taking into account that $q^{\wedge}$ is injective, $\widetilde{\mathbf{u}}$ is a T-sequence in $(X/K_{\mathbf{u}})^{\wedge}$ by Lemma 2.2 of [30]. This proves item (a).

Let us show that $s_{\mathbf{u}}(X)/K_{\mathbf{u}} = s_{\widetilde{\mathbf{u}}}(X/K_{\mathbf{u}})$. Indeed, for every $h + K_{\mathbf{u}} \in s_{\mathbf{u}}(X)/K_{\mathbf{u}}$, by definition, we have $(\widetilde{u}_n, h+K_{\mathbf{u}}) = (u_n, h) \to 1$. Thus $s_{\mathbf{u}}(X)/K_{\mathbf{u}} \subseteq s_{\widetilde{\mathbf{u}}}(X/K_{\mathbf{u}})$. Conversely, if $x + K_{\mathbf{u}} \in s_{\widetilde{\mathbf{u}}}(X/K_{\mathbf{u}})$, then $(\widetilde{u}_n, x + K_{\mathbf{u}}) = (u_n, x) \to 1$. This yields $x \in s_{\mathbf{u}}(X)$. Therefore $x+K_{\mathbf{u}} \in s_{\mathbf{u}}(X)/K_{\mathbf{u}}$, and the item (b) is proved.

Let us prove item (c). For the sake of simplicity we set $H := X/K_{\mathbf{u}}$ and $G := \langle \widetilde{\mathbf{u}}\rangle$.

We claim that H is MAP and G separates the points of H. Indeed, if $x+K_{\mathbf{u}} \neq 0$, then, by the definition of $K_{\mathbf{u}}$, there exists $n \in \omega$ such that $(\widetilde{u}_n, x + K_{\mathbf{u}}) = (u_n, x) \neq 1$.

Since G separates the points of H, T_G is a precompact group topology on H. Clearly, the identity inclusion r from H onto (H, T_G) is continuous. Denote by K' the completion of (H, T_G). Since $\widehat{K'} = \widehat{(H, T_G)} = G$ [17], by Pontryagin's duality theorem we obtain that $K' = G_d^{\wedge} = K$. Since G is countable, K is a metrizable compact Abelian group. Let $j : (H, T_G) \to K$ be the natural embedding.

Set $i := j \circ r$. Then i is an injective continuous homomorphism from H into K with dense image. Naturally identifying $\widehat{K}$ with $\widehat{H}$, we obtain that $(\widetilde{u}_n, i(h)) = (\widetilde{u}_n, h)$ for every $h \in H$ and every $n \in \mathbb{N}$. So that $\widetilde{\mathbf{u}}$ is a T-sequence in $\widehat{K}$ if it is a T-sequence in $\widehat{H}$.

Let us show that $i(s_{\mathbf{u}}(X)/K_{\mathbf{u}}) = i(X/K_{\mathbf{u}}) \cap s_{\widetilde{\mathbf{u}}}(K)$. Let $x + K_{\mathbf{u}} \in s_{\mathbf{u}}(X)/K_{\mathbf{u}}$. By item (b), $x + K_{\mathbf{u}} \in s_{\widetilde{\mathbf{u}}}(X/K_{\mathbf{u}})$, and hence

$$(\widetilde{u}_n, i(x + K_{\mathbf{u}})) = (\widetilde{u}_n, x + K_{\mathbf{u}}) \to 1.$$

So $i(x + K_{\mathbf{u}}) \in s_{\widetilde{\mathbf{u}}}(K)$. Thus $i(s_{\mathbf{u}}(X)/K_{\mathbf{u}}) \subseteq i(X/K_{\mathbf{u}}) \cap s_{\widetilde{\mathbf{u}}}(K)$.

Conversely, let $i(x + K_{\mathbf{u}}) \in i(X/K_{\mathbf{u}}) \cap s_{\widetilde{\mathbf{u}}}(K)$ for an $x \in X$. Then

$$(\widetilde{u}_n, x + K_{\mathbf{u}}) = (\widetilde{u}_n, i(x + K_{\mathbf{u}})) \to 1,$$

and hence $x + K_{\mathbf{u}} \in s_{\widetilde{\mathbf{u}}}(X/K_{\mathbf{u}})$. So $x + K_{\mathbf{u}} \in s_{\mathbf{u}}(X)/K_{\mathbf{u}}$ by item (b). Thus $i(X/K_{\mathbf{u}}) \cap s_{\widetilde{\mathbf{u}}}(K) \subseteq i(s_{\mathbf{u}}(X)/K_{\mathbf{u}})$. The item (c) is proved.

Since K is a metrizable compact group, the trivial subgroup $Z = \{0\}$ is a G_δ-subgroup of K. Hence $K_{\mathbf{u}} = (i \circ q)^{-1}(Z)$ is a G_δ-subgroup of X. $\square$

Proof of Theorem 6: (i)$\Rightarrow$(ii) Let $H = s_{\mathbf{u}}(X)$ for a sequence (respectively, a T-sequence) $\mathbf{u}$ in $\widehat{X}$. Set $H_0 := K_{\mathbf{u}}$. Then the assertion follows from Lemma 11.

(ii)$\Rightarrow$(i) Denote by q the quotient map from X onto X/H_0. Let $\widetilde{\mathbf{u}} = \{\widetilde{u}_n\}_{n\in\omega}$ be a sequence (respectively, T-sequence) in $\widehat{K}$ such that $i(H/H_0) = i(X/H_0) \cap s_{\widetilde{\mathbf{u}}}(K)$.

Set $p := i \circ q$. Since i has dense image, the adjoint map $p^\wedge$ of p is injective. Set $\mathbf{u} := p^\wedge(\widetilde{\mathbf{u}})$. Then $\mathbf{u}$ is a sequence (respectively, a T-sequence by Lemma 2.2 of [30]) in $\widehat{X}$. Let us show that $H = s_{\mathbf{u}}(X)$.

If $x \in H$, then

$$(u_n, x) = (p^\wedge(\widetilde{u}_n), x) = (\widetilde{u}_n, p(x)) \to 1,$$

because $p(x) \in s_{\widetilde{\mathbf{u}}}(K)$. Thus $H \subseteq s_{\mathbf{u}}(X)$.

Conversely, if $x \in s_{\mathbf{u}}(X)$, then

$$(u_n, x) = (\widetilde{u}_n, p(x)) = (\widetilde{u}_n, i(x + H_0)) \to 1,$$

and hence $i(x+H_0) \in i(X/H_0) \cap s_{\widetilde{\mathbf{u}}}(K)$. By hypothesis, we obtain that $i(x+H_0) \in i(H/H_0)$. Since i is injective, we have $x + H_0 \in H/H_0$. So $x \in H$ because H_0 is a subgroup of H. Thus $s_{\mathbf{u}}(X) \subseteq H$.

(ii)$\Rightarrow$(iii) Let $p := i \circ q$, where $q : X \to X/H_0$ is the quotient map, and $H' := s_{\widetilde{\mathbf{u}}}(K)$. Since $\ker(p) = H_0 \subseteq H$ and $p(H) = i(H/H_0) = s_{\widetilde{\mathbf{u}}}(K) \cap p(X) = H' \cap p(X)$, we obtain that $H = p^{-1}(H')$.

(iii)$\Rightarrow$(ii) Set $H_0 := \ker(p)$ and $i : X/H_0 \to K, i(x + H_0) := p(x)$. Then $H_0 = p^{-1}(\{0\})$ is a closed G_δ-subgroup of X (since $\{0\}$ is a G_δ-subgroup of K). This proves (1). Clearly, i is a continuous monomorphism with dense image and

$$i(H/H_0) = p(H) = p(X) \cap H' = i(X/H_0) \cap H'.$$

Thus (2) is fulfilled. The theorem is proved. $\square$

In the next proposition we describe all Abelian topological groups X which have only one characterized (respectively, T-characterized) subgroup, namely, X itself (cf. the paragraph before Theorem 1.10 in [30]). Note that, by the definition of T-sequences, every T-sequence either is eventually equal to zero or has infinitely many elements.

Proposition 12. *Let X be an Abelian topological group.*

(i) *X is the unique characterized subgroup of X if and only if X is MinAP.*
(ii) *X is the unique T-characterized subgroup of X if and only if $\widehat{X}$ is finite.*

Proof: (i) Assume that X is the unique characterized subgroup of X. Suppose for a contradiction that X has a non-zero continuous character g. Set $\mathbf{u} = \{u_n\}_{n\in\omega}$, where $u_n = g$ for every $n \in \omega$. Clearly, $s_{\mathbf{u}}(X) \neq X$. This contradiction shows that X is MinAP.

Conversely, if X is MinAP, it is clear that X is the unique characterized subgroup of X.

(ii) Let $\widehat{X}$ be finite. Then every T-sequence $\mathbf{u}$ in $\widehat{X}$ is eventually equal to zero. Hence $s_{\mathbf{u}}(X) = X$. Thus X is the unique T-characterized subgroup of X.

Assume that X is the unique T-characterized subgroup of X. We have to show that $\widehat{X}$ is finite. Suppose for a contradiction that $\widehat{X}$ is infinite. It is well-known that $\widehat{X}$ contains a countably infinite subgroup G which either has infinite exponent or is isomorphic to the direct sum of the form $F^{(\mathbb{N})}$ for some non-zero finite group F. By Corollary 4.7 of [29], G admits a MinAP group topology generated by a T-sequence $\widetilde{\mathbf{u}} = \{\widetilde{u}_n\}_{n\in\omega}$ in G. Since $s_{\widetilde{\mathbf{u}}}(G_d^\wedge) = \widehat{(G,\tau_{\widetilde{\mathbf{u}}})}$ by Fact 3, we obtain that $s_{\widetilde{\mathbf{u}}}(G_d^\wedge) = \{0\}$. Corollary 2 in [26] implies that $G = \langle\widetilde{\mathbf{u}}\rangle$. Consider the sequence $\widetilde{\mathbf{u}}$ as a sequence $\mathbf{u}$ in $\widehat{X}$. By Lemma 2.2 of [30], $\mathbf{u}$ is also a T-sequence in $\widehat{X}$. Now Lemma 11(i) yields that $K_{\mathbf{u}}$ is a proper subgroup of X. To obtain a contradiction with our supposition it is enough to show that $K_{\mathbf{u}} = s_{\mathbf{u}}(X)$.

We have to prove only that $s_{\mathbf{u}}(X) \subseteq K_{\mathbf{u}}$. Note that $\mathbf{u} = q^\wedge(\widetilde{\mathbf{u}})$, where $q^\wedge$ is the adjoint homomorphism of the quotient map $q : X \to X/K_{\mathbf{u}}$. Since $s_{\widetilde{\mathbf{u}}}(K) = \{0\}$, where $K = \langle\widetilde{\mathbf{u}}\rangle_d^\wedge = G_d^\wedge$, Lemma 11(ii) implies $s_{\mathbf{u}}(X)/K_{\mathbf{u}} = \{0\}$. Hence $K_{\mathbf{u}} = s_{\mathbf{u}}(X)$ is a proper T-characterized subgroup of X. This contradiction shows that $\widehat{X}$ must be finite. □

In Corollary 3 of [27] it is proved that for every metrizable compact Abelian group X, the equality $s_{\mathbf{u}}(X) = X$ implies that the sequence $\mathbf{u}$ is eventually equal to zero. An analogous result is known for every compact Abelian group (although we do not know any exact reference to this). In Theorem 21 below we show that this property characterizes compact groups in the class of all locally compact Abelian groups. The last fact justifies the next definition.

Definition 13. An Abelian topological group X is called *self-characterized* if there exists a nontrivial sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$ in $\widehat{X}$ such that

$$s_{\mathbf{u}}(X) = X.$$

The next proposition follows easily from definitions.

Proposition 14. *Let X be an Abelian topological group.*

(1) *X is self-characterized if and only if the precompact group $(\widehat{X}, \sigma(\widehat{X}, X))$ contains a nontrivial convergent sequence.*
(2) *X is self-characterized by a sequence $\mathbf{u}$ in $\widehat{X}$ if and only if $\mathbf{u}$ is a nontrivial sequence converging to zero in the weak* topology $\sigma(\widehat{X}, X)$. In this case, $\mathbf{u}$ is a TB-sequence in $\widehat{X}$ and it contains infinitely many different elements.*

Proof: Let X be self-characterized by a nontrivial sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$ in $\widehat{X}$. So that $(u_n, x) \to 1$ for every $x \in X$. This is equivalent to the fact that $\mathbf{u}$ is nontrivial and $u_n \to 0$ in the weak* group topology $\sigma(\widehat{X}, X)$ on $\widehat{X}$, which proves (1) and (2). Since the topology $\sigma(\widehat{X}, X)$ is precompact, $\mathbf{u}$ is a TB-sequence in $\widehat{X}$. The sequence $\mathbf{u}$ contains infinitely many different elements because it is nontrivial and the topology $\sigma(\widehat{X}, X)$ is Hausdorff. □

Since weak* topology is weaker than the compact-open one, Proposition 14 immediately implies the next useful sufficient condition for X to be self-characterized:

Corollary 15. *Let X be an Abelian topological group such that $X^\wedge$ contains a nontrivial convergent sequence. Then X is self-characterized.*

We use the next example to prove Theorem 21.

Example 16. Let X be an Abelian topological group.

(1) If $X = \mathbb{R}$, then X is self-characterized.
(2) If X is a locally compact non-compact Abelian group containing an open compact subgroup K, then X is self-characterized.
(3) If X is MinAP, then X is not self-characterized.

PROOF: (1) follows from Corollary 15.

(2) Since X is not compact, $X^\wedge$ contains an infinite open compact subgroup which is topologically isomorphic to $(X/K)^\wedge$ by [35, 23.25]. It is well-known that every infinite compact Abelian group contains a nontrivial convergent sequence. Now the assertion follows from Corollary 15.

(3) is trivial. □

Remark 17. Note that there is a self-characterized precompact Abelian group H such that $H^\wedge$ does not contain nontrivial convergent sequences. In other words, there exists a precompact Abelian group H such that $\widehat{H}$ *has no* nontrivial convergent sequences in the compact-open topology, but $\widehat{H}$ *has* nontrivial convergent sequences in the weak* topology. Indeed, let H be an arbitrary dense proper characterized subgroup of a compact metrizable Abelian group X (such subgroups exist by Lemma 3.6 in [19]). So $H = s_{\mathbf{u}}(X)$ for some sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$ in $X^\wedge$. Since H is proper, $\mathbf{u}$ is nontrivial by Corollary 3 in [27]. Let τ be the induced topology on H from X. Clearly, $\mathbf{u} \subset (H,\tau)^\wedge$. Since $(u_n, h) \to 1$ for every $h \in H$ by definition, (H,τ) is self-characterized. On the other hand, $(H,\tau)^\wedge$ is discrete by [4], [14], and hence $(H,\tau)^\wedge$ does not contain any nontrivial convergent sequence.

Proposition 18. *If an Abelian topological group X is self-characterized, then the direct product $X \times Y$ is self-characterized for every Abelian topological group Y.*

PROOF: Let $s_{\mathbf{u}}(X) = X$ for some nontrivial sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$ in $\widehat{X}$. Set $\mathbf{v} = \{v_n\}_{n\in\omega} \in \widehat{X \times Y}$, where $v_n = (u_n, 0_{\widehat{Y}})$ for every $n \in \omega$. Then $\mathbf{v}$ is nontrivial. Clearly, $s_{\mathbf{v}}(X \times Y) = X \times Y$. □

In the next theorem we describe all self-characterized Abelian groups.

Theorem 19. *For an Abelian topological group X, the following statements are equivalent.*

(1) *X is self-characterized.*
(2) *There exists an infinite compact metrizable Abelian group K and a continuous homomorphism p from X into K with dense image such that $p(X)$ is contained in a proper characterized subgroup of K.*

Proof: (1)$\Rightarrow$(2) Let $s_{\mathbf{u}}(X) = X$ for some nontrivial sequence $\mathbf{u}$ in $\widehat{X}$. Then $\mathbf{u}$ is a TB-sequence in $\widehat{X}$ containing infinitely many different elements by Proposition 14. Hence $\langle \mathbf{u} \rangle$ is a countably infinite Abelian group.

By Lemma 11(ii), there exists a T-sequence $\widetilde{\mathbf{u}}$ in $(X/K_{\mathbf{u}})^\wedge$ such that $s_{\mathbf{u}}(X)/K_{\mathbf{u}} = s_{\widetilde{\mathbf{u}}}(X/K_{\mathbf{u}})$ and $\mathbf{u} = q^\wedge(\widetilde{\mathbf{u}})$, where $q^\wedge$ is the adjoint homomorphism of the quotient map $q : X \to X/K_{\mathbf{u}}$.

Set $K := \langle \widetilde{\mathbf{u}} \rangle_d^\wedge$ and $p := i \circ q$, where the injective homomorphism i is defined in Lemma 11(ii). Since $q^\wedge$ is injective the group $\langle \widetilde{\mathbf{u}} \rangle$ is countably infinite, and hence K is an infinite compact metrizable Abelian group. Now Lemma 11(ii) implies that p is a continuous homomorphism from X into K with dense image such that $p(X) \subseteq s_{\widetilde{\mathbf{u}}}(K)$. Since $\widetilde{\mathbf{u}}$ is nontrivial, Corollary 3 of [27] implies that $s_{\widetilde{\mathbf{u}}}(K)$ is a proper subgroup of K.

(2)$\Rightarrow$(1) Let $p(X)$ be contained in a proper characterized subgroup $s_{\mathbf{v}}(K)$ of K, where $\mathbf{v} = \{v_n\}_{n\in\omega}$ is a sequence in $K^\wedge$. The dual homomorphism $p^\wedge$ of p is injective because $p(X)$ is dense in K. Since $s_{\mathbf{v}}(K)$ is proper, $\mathbf{v}$ is nontrivial by Corollary 3 in [27]. Set $\mathbf{u} = \{u_n\}_{n\in\omega}$, where $u_n = p^\wedge(v_n)$ for every $n \in \omega$. Then $\mathbf{u}$ is nontrivial because $p^\wedge$ is injective and $\mathbf{v}$ is nontrivial.

Let us show that $s_{\mathbf{u}}(X) = X$. Indeed, for every $x \in X$, we have $p(x) \in s_{\mathbf{v}}(K)$, and hence

$$(u_n, x) = (v_n, p(x)) \to 1.$$

Thus $s_{\mathbf{u}}(X) = X$. Since $\mathbf{u}$ is nontrivial the group X is self-characterized. $\square$

Remark 20. Let X be an Abelian topological group. Then X is not self-characterized if and only if from the equality $s_{\mathbf{u}}(X) = X$ for a sequence $\mathbf{u}$ in $X^\wedge$ it follows that $\mathbf{u}$ is eventually equal to zero. Indeed, if X is not self-characterized and $s_{\mathbf{u}}(X) = X$ for a sequence $\mathbf{u}$ in $X^\wedge$, then $\mathbf{u}$ is trivial by definition. So $X = \ker(u_n)$ for all sufficiently large n. Hence $u_n = 0$ for all sufficiently large n. Thus $\mathbf{u}$ is eventually equal to zero. The converse is trivial.

Theorem 21. *For a locally compact Abelian group X, the following statement are equivalent.*

- (i) *X is not self-characterized, i.e., if $s_{\mathbf{u}}(X) = X$ for a sequence $\mathbf{u}$ in $X^\wedge$, then $\mathbf{u}$ is eventually equal to zero.*
- (ii) *X is compact.*

Proof: (i)$\Rightarrow$(ii) Assume that X is not self-characterized. It is well-known [35, 24.30] that $X \cong \mathbb{R}^n \times H$, where $n \in \omega$ and H is a locally compact Abelian group containing an open compact subgroup. If X is not compact, then either $n > 0$ or $n = 0$ and H is not compact. Now Example 16 and Proposition 18 imply that X is self-characterized. This contradiction shows that X is compact.

(ii)$\Rightarrow$(i) Assume that X is compact. For every continuous homomorphism p from X into a compact metrizable Abelian group K with dense image, we have $p(X) = K$. So $p(X)$ is not a proper subgroup of K. Thus X is not self-characterized by Theorem 19. $\square$

Remark 22. There exist also precompact non-compact Abelian groups which are not self-characterized. Indeed, Hart and Kunen [33], [34] showed that every infinite compact Abelian group X has a dense Borel subgroup H such that H is not contained in any proper characterized subgroup of X (subgroups with this property are called $\mathfrak{g}$-dense [22]). Let us show that the precompact non-compact group $H = (H, T_{\widehat{X}})$ is not self-characterized.

Let a sequence $\mathbf{u}$ in $(H, T_{\widehat{X}})^\wedge$ be such that $s_{\mathbf{u}}(H) = H$. Since $\widehat{(H, T_{\widehat{X}})} = \widehat{X}$ by [17], we obtain that $\mathbf{u} \subset \widehat{X}$. Hence H considered as a subgroup of X is contained in $s_{\mathbf{u}}(X)$. By the choice of H, we obtain that $s_{\mathbf{u}}(X) = X$. Now Theorem 21 implies that $\mathbf{u}$ is eventually equal to zero. Thus H is not self-characterized.

3. Some general properties of the groups of the form $(c_0(X), \mathfrak{u}_0)$

In this section, for an Abelian topological group X, we obtain some general properties of the groups of the form $c_0(X)$ endowed with the uniform topology $\mathfrak{u}_0$. In particular, we describe all compact subsets of the group $(c_0(X), \mathfrak{u}_0)$.

Proposition 23. *For every Abelian topological groups X and Y, the groups $(c_0(X) \times c_0(Y), \mathfrak{u}_0^X \times \mathfrak{u}_0^Y)$ and $(c_0(X \times Y), \mathfrak{u}_0)$ are topologically isomorphic.*

Proof: For every $(x_n)_{n\in\mathbb{N}} \in c_0(X)$ and every $(y_n)_{n\in\mathbb{N}} \in c_0(Y)$, set

$$i\left((x_n), (y_n)\right) := \left((x_n, y_n)\right)_{n\in\mathbb{N}}.$$

Clearly, i is an algebraic isomorphism from $c_0(X) \times c_0(Y)$ onto $c_0(X \times Y)$. It is easy to check that, for every $U \in \mathcal{N}(X)$ and each $V \in \mathcal{N}(Y)$, we have

$$i\left((U^{\mathbb{N}} \cap c_0(X)) \times (V^{\mathbb{N}} \cap c_0(Y))\right) = (U \times V)^{\mathbb{N}} \cap c_0(X \times Y).$$

Thus i is a topological isomorphism. $\square$

Proposition 24. *Let H be a* (*respectively, closed or open*) *subgroup of an Abelian topological group X. Then the identity map*

$$i : (c_0(H), \mathfrak{u}_0) \to (c_0(X), \mathfrak{u}_0), \text{ where } i(h_n) = (h_n) \text{ for every } (h_n)_{n\in\mathbb{N}} \in c_0(H),$$

is an embedding (*respectively, onto a closed subgroup or an open one*).

Proof: Clearly, i is well-defined. For every $U \in \mathcal{N}(H)$ choose $V \in \mathcal{N}(X)$ such that $U = V \cap H$. Then

$$\begin{aligned} i\left(U^{\mathbb{N}} \cap c_0(H)\right) &= \{(h_n) \in c_0(X) : \ h_n \in U, \forall n \in \mathbb{N}, \text{ and } h_n \to 0\} \\ &= V^{\mathbb{N}} \cap i(c_0(H)). \end{aligned}$$

Thus i is an embedding.

Assume that H is a closed subgroup of X. For every $(x_n) \in c_0(X) \setminus c_0(H)$ there exists an index m such that $x_m \notin H$. Choose $V \in \mathcal{N}(X)$ such that $(x_m + V) \cap H = \emptyset$. Then $((x_n) + V^{\mathbb{N}}) \cap c_0(H) = \emptyset$. Thus $i(c_0(H))$ is a closed subgroup of $(c_0(X), \mathfrak{u}_0)$.

Suppose now that H is an open subgroup of X. Then $c_0(H) = H^{\mathbb{N}} \cap c_0(X)$ is open in $(c_0(X), \mathfrak{u}_0)$. □

In what follows we identify $(c_0(H), \mathfrak{u}_0)$ with its image $i(c_0(H))$ and say that $(c_0(H), \mathfrak{u}_0)$ is a subgroup of $(c_0(X), \mathfrak{u}_0)$.

Let A be an arbitrary Abelian group. For every $n \in \mathbb{N}$, define an injective homomorphism $\nu_n : A \to A^{(\mathbb{N})}$ by

$$\nu_n(a) = (0, \dots, 0, a, 0 \dots),$$

where $a \in A$ is placed in the position n. Define projections $p_n : A^{\mathbb{N}} \to A^{\mathbb{N}}$ and $\pi_n : A^{\mathbb{N}} \to A$ by

$$p_n(\mathbf{a}) := (0, \dots, 0_n, a_{n+1}, a_{n+2}, \dots) \text{ and } \pi_n(\mathbf{a}) := a_n,$$

and set

$$\mathbf{a}_n := (a_1, \dots, a_n, 0, \dots) = \sum_{i=1}^{n} \nu_i(a_i) = \mathbf{a} - p_n(\mathbf{a}) \in A^{(\mathbb{N})},$$

where $\mathbf{a} = (a_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}}$. Clearly, if A is an Abelian topological group, then the restrictions of p_n and π_n onto $c_0(X)$ are continuous in the uniform topology $\mathfrak{u}_0$.

For the convenience of the reader we prove the next lemma from [21]:

Lemma 25 ([21, Lemma 3.3(b)])**.** *Let $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in (c_0(X), \mathfrak{u}_0)$ for an Abelian topological group X. Then $\mathbf{x}_n \to \mathbf{x}$ in $(c_0(X), \mathfrak{u}_0)$. In particular, $X^{(\mathbb{N})}$ is dense in $(c_0(X), \mathfrak{u}_0)$.*

Proof: Let $V \in \mathcal{N}(X)$. Choose $n_0 \in \mathbb{N}$ such that $x_n \in V$ for every $n > n_0$. Then

$$\mathbf{x} - \mathbf{x}_n = p_n(\mathbf{x}) \in V^{\mathbb{N}}, \quad \forall n > n_0.$$

Thus $\mathbf{x}_n \to \mathbf{x}$ in $\mathfrak{u}_0$. □

In the next proposition we consider the completion of the groups of the form $(c_0(X), \mathfrak{u}_0)$.

Proposition 26. *Let $\bar{X}$ be a completion of an Abelian topological group X. Then $(c_0(\bar{X}), \mathfrak{u}_0)$ is a completion of $(c_0(X), \mathfrak{u}_0)$. In particular, algebraically,*

$$\widehat{(c_0(X), \mathfrak{u}_0)} = \widehat{(c_0(\bar{X}), \mathfrak{u}_0)} \quad (\textit{and } \widehat{X} = \widehat{\bar{X}}).$$

Proof: By Proposition 3.4 of [21], the group $(c_0(\bar{X}), \mathfrak{u}_0)$ is complete. Since $(c_0(X), \mathfrak{u}_0)$ is a dense subgroup of $(c_0(\bar{X}), \mathfrak{u}_0)$ by Proposition 24 and Lemma 25, the group $(c_0(\bar{X}), \mathfrak{u}_0)$ is a completion of $(c_0(X), \mathfrak{u}_0)$ by Theorem 3.6.14 of [2]. The last assertion follows from Corollary 3.6.17 in [2]. □

Let X and Y be topological groups. Following Siwiec [44], a continuous homomorphism $p : X \to Y$ is called *sequence-covering* if it is surjective and for every sequence $\{y_n\}$ converging to the unit e_Y there is a sequence $\{x_n\}$ converging to e_X such that $p(x_n) = y_n$.

Proposition 27. *Let H be a closed subgroup of an Abelian topological group X and $q : X \to X/H$ be the quotient map. Then the quotient group $(c_0(X), \mathfrak{u}_0)/c_0(H)$ embeds into $(c_0(X/H), \mathfrak{u}_0)$ as a dense subgroup under the natural map j defined by the formula*

$$j(\mathbf{x} + c_0(H)) := (q(x_n))_{n\in\mathbb{N}} \in c_0(X/H),\ \forall \mathbf{x} = (x_n)_{n\in\mathbb{N}} \in c_0(X).$$

If, in addition, the quotient map q is sequence-covering, then j is a topological isomorphism.

Proof: For every $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in c_0(X)$, we have $q(x_n) \to 0$ in X/H. Thus j is well-defined. Clearly, j is injective. Since $(X/H)^{(\mathbb{N})} = j(X^{(\mathbb{N})})$ Lemma 25 implies that j has dense image. Denote by G the image of j.

To prove that j is an embedding, it is enough to show that for every $V \in \mathcal{N}(X)$,

$$j\left(\left[c_0(X) \cap V^{\mathbb{N}}\right] + c_0(H)\right) = U^{\mathbb{N}} \cap G,\ \text{ where } U := q(V). \tag{2}$$

The inclusion "$\subseteq$" is clear. Let us prove the converse one. Let $\widetilde{\mathbf{x}} = (x_n + H)_{n\in\mathbb{N}} \in (U^{\mathbb{N}} \cap G)$. Since $\widetilde{\mathbf{x}} \in G$ we can find $\mathbf{x}' = (x'_n)_{n\in\mathbb{N}}$ in $c_0(X)$ such that $q(x'_n) = x_n + H$ for every $n \in \mathbb{N}$. Since $x'_n \to 0_X$ there exists a natural number m such that $x'_n \in V$ for every $n > m$. And since $x_n + H \in q(V)$, we can choose $y_1, \dots, y_m \in V$ such that $q(y_n) = x_n + H$ for every $1 \le n \le m$. Set

$$\mathbf{z} := (z_n)_{n\in\mathbb{N}},\ \text{ where } z_n = \begin{cases} y_n, & \text{if } \ 1 \le n \le m, \\ x'_n, & \text{if } \ m < n. \end{cases}$$

By construction, $\mathbf{z} \in c_0(X) \cap V^{\mathbb{N}}$ and $j(\mathbf{z} + c_0(H)) = \widetilde{\mathbf{x}}$. This proves the equality (2). Thus j is an embedding.

Assume now that the quotient map q is sequence-covering. To prove that j is a topological isomorphism it is enough to show that j is surjective. Let $\widetilde{\mathbf{x}} = (x_n + H)_{n\in\mathbb{N}} \in c_0(X/H)$. Since $x_n + H \to 0$ in X/H and q is sequence covering, there exists a sequence $\{x'_n\}_{n\in\mathbb{N}}$ in X such that $x'_n \to 0_X$ and $q(x'_n) = x_n + H$ for every $n \in \mathbb{N}$. So $\mathbf{x}' := (x'_n)_{n\in\mathbb{N}}$ belongs to $c_0(X)$ and $j(\mathbf{x}' + c_0(H)) = \widetilde{\mathbf{x}}$. Thus j is surjective. $\square$

Corollary 28. *Let H be a closed subgroup of an Abelian Čech-complete topological group X. Then the quotient group $(c_0(X), \mathfrak{u}_0)/c_0(H)$ is topologically isomorphic to $(c_0(X/H), \mathfrak{u}_0)$.*

Proof: Every quotient homomorphism of a Čech-complete topological group is sequence-covering by Theorem 1.2 of [1]. So the assertion follows from Proposition 27. $\square$

For example, if $X = \mathbb{R}$ and $H = \mathbb{Z}$, we obtain that $c_0/\mathbb{Z}^{(\mathbb{N})} \cong (c_0(\mathbb{T}), \mathfrak{u}_0)$ (see [26]).

To describe compact subsets in $(c_0(X), \mathfrak{u}_0)$ we need the next notion which is used repeatedly in the article.

Definition 29. Let $\{K_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of closed subsets of a topological group X. We say that $\{K_n\}$ is a *null-sequence* of closed subsets in X, if for every open neighborhood U of the unit, $K_n \subseteq U$ for all sufficiently large $n \in \mathbb{N}$.

Lemma 30. *Let $\{K_n\}_{n\in\mathbb{N}}$ be a null-sequence of compact subsets of an Abelian topological group X. Set $K := \prod_{n\in\mathbb{N}} K_n$. Then K is a compact subset of $(c_0(X), \mathfrak{u}_0)$.*

PROOF: Let us show first that $K \subset c_0(X)$. Let $\mathbf{x} = (x_n) \in K$. For every $U \in \mathcal{N}(X)$, choose $m \in \mathbb{N}$ such that $K_n \subseteq U$ for every $n > m$. Then $x_n \in U$ for every $n > m$. Thus $\mathbf{x} \in c_0(X)$.

To show that K is compact we have to prove that every net $S = \{\mathbf{x}_i\}_{i\in I}$, where I is a directed set, has a cluster point [25, 3.1.23]. By the Tychonoff theorem, S has a cluster point $\mathbf{x}$ in the product topology $\mathfrak{p}$ on K. For our purpose it is enough to show that $\mathbf{x}$ is a cluster point of S also in the uniform topology $\mathfrak{u}_0$. Fix arbitrarily $U \in \mathcal{N}(X)$ and $i' \in I$. Choose a natural number m such that $K_n \subseteq U$ for every $n > m$. Since $\mathbf{x}$ is a cluster point of S in $\mathfrak{p}$, there exists an index $i_0 \in I$, $i_0 > i'$, such that $\mathbf{x}_{i_0} \in \mathbf{x} + (U^m \times \prod_{n>m} K_n)$. By the choice of m we have also $\mathbf{x}_{i_0} \in \mathbf{x} + (U^{\mathbb{N}} \cap c_0(X))$. Thus $\mathbf{x}$ is a cluster point of S in the topology $\mathfrak{u}_0$. Therefore K is compact in $\mathfrak{u}_0$ (and $\mathfrak{u}_0|_K = \mathfrak{p}$). $\square$

In the next proposition we describe compact subsets in $(c_0(X), \mathfrak{u}_0)$.

Proposition 31. *Let X be an Abelian topological group. For a closed subset K of $(c_0(X), \mathfrak{u}_0)$, the following statements are equivalent.*

(1) *K is compact.*
(2) *the sequence $\{\pi_n(K)\}_{n\in\mathbb{N}}$ is a null-sequence of compact subsets of X.*

PROOF: (1)⇒(2) Since π_n is continuous, $\pi_n(K)$ is a compact subset of X for every $n \in \mathbb{N}$. Suppose for a contradiction that $\{\pi_n(K)\}_{n\in\mathbb{N}}$ is not a null-sequence. So that there exists a $U \in \mathcal{N}(X)$ and an increasing sequence of indices $\{n_l\}_{l\in\mathbb{N}}$ such that $\pi_{n_l}(K) \not\subseteq U$. For every $l \in \mathbb{N}$ take an element $\mathbf{x}_l \in K$ such that

$$\pi_{n_l}(\mathbf{x}_l) \notin U. \tag{3}$$

Since K is compact, the sequence $\{\mathbf{x}_l\}_{l\in\mathbb{N}}$ has a cluster point $\mathbf{x} = (x_n)$ in K. Choose arbitrarily $V \in \mathcal{N}(X)$ such that $V + V \subseteq U$. Since $\mathbf{x} \in c_0(X)$ we can choose $m \in \mathbb{N}$ such that $\pi_n(\mathbf{x}) \in V$ for every $n \geq m$. Since the set $\{\mathbf{x}_l\}_{l\in\mathbb{N}} \cap (\mathbf{x} + V^{\mathbb{N}})$ is infinite, we can find $l_0 \in \mathbb{N}$ such that $n_{l_0} > m$ and $\mathbf{x}_{l_0} \in \mathbf{x} + V^{\mathbb{N}}$. In particular, $\pi_{n_{l_0}}(\mathbf{x}_{l_0}) \in x_{n_{l_0}} + V \subseteq V + V \subseteq U$, which contradicts (3).

(2)⇒(1) The set $K' := \prod_{n\in\mathbb{N}} \pi_n(K)$ is a compact subset of $(c_0(X), \mathfrak{u}_0)$ by Lemma 30. Hence K is compact in $\mathfrak{u}_0$ as a closed subset of K'. $\square$

4. On the dual group of $(c_0(X), \mathfrak{u}_0)$

In this section we consider some general properties of the dual group of $(c_0(X), \mathfrak{u}_0)$ which we use in the sequel.

Let X be an Abelian topological group. We denote by $\mathfrak{b}_X$ the box topology on $X^{\mathbb{N}}$. Clearly, $\mathfrak{u}_X \leq \mathfrak{b}_X$. Set

$$\mathfrak{p}' := \mathfrak{p}_X|_{X^{(\mathbb{N})}}, \ \mathfrak{u}' := \mathfrak{u}_X|_{X^{(\mathbb{N})}} = \mathfrak{u}_0|_{X^{(\mathbb{N})}} \text{ and } \mathfrak{b}' := \mathfrak{b}_X|_{X^{(\mathbb{N})}}.$$

Then $\mathfrak{p}' \leq \mathfrak{u}' \leq \mathfrak{b}'$. In what follows we shall omit the subscript X. Recall that we defined an injective homomorphism $\nu_n : X \to X^{(\mathbb{N})}$ by

$$\nu_n(x) = (0, \dots, 0, x, 0, \dots),$$

where $x \in X$ is placed in the position n. Clearly, ν_n is an embedding for every group topology $\mathfrak{p}', \mathfrak{u}'$ or $\mathfrak{b}'$ on $X^{(\mathbb{N})}$. Also we consider ν_n as an embedding of X into $(X^{\mathbb{N}}, \mathfrak{p})$, $(X^{\mathbb{N}}, \mathfrak{u})$ or $(X^{\mathbb{N}}, \mathfrak{b})$.

Let us note that by Kaplan's theorem [36], the map $\eta \mapsto (\eta \circ \nu_n)_{n\in\mathbb{N}}$ is a topological isomorphism of $(X^{(\mathbb{N})}, \mathfrak{b}')^\wedge$ onto $((X^\wedge)^{\mathbb{N}}, \mathfrak{p})$ such that

$$(4) \qquad (\eta, \mathbf{x}) = \prod_{n\in\mathbb{N}} (\eta, \nu_n(x_n)), \ \forall \mathbf{x} = (x_n) \in X^{(\mathbb{N})}, \ \forall \eta \in (X^{(\mathbb{N})}, \mathfrak{b}')^\wedge.$$

Also Kaplan [36] proved that $(X^{\mathbb{N}}, \mathfrak{p})^\wedge$ and $((X^\wedge)^{(\mathbb{N})}, \mathfrak{b}')$ are topologically isomorphic by the topological isomorphism $\chi \mapsto (\chi \circ \nu_n)_{n\in\mathbb{N}}$ for which

$$(5) \qquad (\chi, \mathbf{x}) = \prod_{n\in\mathbb{N}} (\chi, \nu_n(x_n)), \ \forall \mathbf{x} = (x_n) \in X^{\mathbb{N}}, \ \forall \chi \in (X^{\mathbb{N}}, \mathfrak{p})^\wedge.$$

So we will identify $(X^{(\mathbb{N})}, \mathfrak{b}')^\wedge$ with $((X^\wedge)^{\mathbb{N}}, \mathfrak{p})$ and $(X^{\mathbb{N}}, \mathfrak{p})^\wedge$ with $((X^\wedge)^{(\mathbb{N})}, \mathfrak{b}')$.

Note that the projection $\pi_n : (X^\wedge)^{\mathbb{N}} \to X^\wedge$ onto nth coordinate is the adjoint homomorphism of $\nu_n : X \to (X^{(\mathbb{N})}, \mathfrak{b}')$. Indeed, for every $\mathbf{g} = (g_n) \in (X^\wedge)^{\mathbb{N}}$ and every $x \in X$, by (4) and (5), we have

$$(\pi_n(\mathbf{g}), x) = (g_n, x) = (\mathbf{g}, \nu_n(x)).$$

Set

$$t : \left(X^{(\mathbb{N})}, \mathfrak{b}'\right) \to (c_0(X), \mathfrak{u}_0), \ t(\mathbf{x}) = \mathbf{x}, \ \forall \mathbf{x} \in X^{(\mathbb{N})},$$

and

$$j : (c_0(X), \mathfrak{u}_0) \to \left(X^{\mathbb{N}}, \mathfrak{p}\right), \ j(\mathbf{x}) = \mathbf{x}, \ \forall \mathbf{x} \in c_0(X).$$

Then $j \circ t$ is a continuous inclusion of $(X^{(\mathbb{N})}, \mathfrak{b}')$ into $(X^{\mathbb{N}}, \mathfrak{p})$ such that

$$(6) \qquad (j \circ t)^\wedge(\mathbf{g}) = \mathbf{g}, \ \forall \mathbf{g} \in \left((X^\wedge)^{(\mathbb{N})}, \mathfrak{b}'\right),$$

since

$$((j \circ t)^\wedge(\mathbf{g}), \mathbf{x}) = (\mathbf{g}, j \circ t(\mathbf{x})) = (\mathbf{g}, \mathbf{x}), \ \forall \mathbf{x} \in \left(X^{(\mathbb{N})}, \mathfrak{b}'\right).$$

Proposition 32. *Let X be an Abelian topological group. Then the following holds.*

(1) *The identity inclusion $t : (X^{(\mathbb{N})}, \mathfrak{b}') \to (c_0(X), \mathfrak{u}_0)$ is a continuous monomorphism with dense image.*

(2) *The adjoint homomorphism $t^\wedge$ of t is a continuous monomorphism with dense image from $(c_0(X), \mathfrak{u}_0)^\wedge$ to $(X^\wedge)^{\mathbb{N}}$. Moreover, if $\chi \in (c_0(X), \mathfrak{u}_0)^\wedge$, then $t^\wedge(\chi) = (g_n)_{n\in\mathbb{N}} \in (X^\wedge)^{\mathbb{N}}$, where $g_n := \chi \circ \nu_n$ for every $n \in \mathbb{N}$.*

Proof: (1) follows from the inequality $\mathfrak{u}' \leq \mathfrak{b}'$ and Lemma 25.

(2) Since t has dense image, $t^\wedge$ is a continuous monomorphism. The homomorphism $t^\wedge$ has dense image because, by (6),

$$\widehat{X}^{(\mathbb{N})} = t^\wedge\left(j^\wedge(\widehat{X}^{(\mathbb{N})})\right) \subseteq t^\wedge\left((c_0(X), \mathfrak{u}_0)^\wedge\right),$$

and $\widehat{X}^{(\mathbb{N})}$ is dense in $(X^\wedge)^{\mathbb{N}}$. Further, for every natural number n and every $x \in X$, we have

$$\begin{aligned}(\pi_n(t^\wedge(\chi)), x) &= (t^\wedge(\chi), \nu_n(x)) = (\chi, t \circ \nu_n(x)) \\ &= (\chi, \nu_n(x)) = (\chi \circ \nu_n, x) = (g_n, x).\end{aligned} \qquad \Box$$

Moreover, in Proposition 39 below we show that also $j^\wedge$ has dense image.

Taking into account Proposition 32, in what follows we identify (algebraically) the dual group $(c_0(X), \mathfrak{u}_0)^\wedge$ with its image $t^\wedge((c_0(X), \mathfrak{u}_0)^\wedge) \subseteq \widehat{X}^{\mathbb{N}}$. We also say that an element $\mathbf{g} = (g_n)_{n\in\mathbb{N}} \in \widehat{X}^{\mathbb{N}}$ *determines* a character of $(c_0(X), \mathfrak{u}_0)$ if $\mathbf{g}$ belongs to $(c_0(X), \mathfrak{u}_0)^\wedge$. It is natural to ask:

Question 33. *Let X be an Abelian topological group. Which elements of $\widehat{X}^{\mathbb{N}}$ determine continuous characters of $(c_0(X), \mathfrak{u}_0)$?*

Below we give a necessary condition and a sufficient one for $\mathbf{g} \in \widehat{X}^{\mathbb{N}}$ to determine a continuous character of $(c_0(X), \mathfrak{u}_0)$. We start from the necessary condition. The next proposition is proved in [21] (see item (2) before Notation 4.5 and Proposition 5.1(b)). We give its proof to make the paper more self-contained.

Proposition 34 ([21])**.** *Let X be an Abelian topological group.*

(1) *If an element $\mathbf{g} = (g_n)_{n\in\mathbb{N}} \in \widehat{X}^{\mathbb{N}}$ determines a continuous character χ of $(c_0(X), \mathfrak{u}_0)$, then $\mathbf{g}$ satisfies the condition:*

(α) *for every $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in c_0(X)$ there exists a limit*

$$\lim_{n\to\infty} \prod_{i=1}^{n} (g_i, x_i).$$

(2) *If an element* $\mathbf{g} = (g_n)_{n\in\mathbb{N}} \in \widehat{X}^{\mathbb{N}}$ *satisfies the condition* (α), *then* $\mathbf{g}$ *determines a character* χ *of* $(c_0(X), \mathfrak{u}_0)$ *by the formula*

$$(\chi, \mathbf{x}) := \lim_{n\to\infty} \prod_{i=1}^{n} (g_i, x_i), \ \forall \mathbf{x} = (x_n)_{n\in\mathbb{N}} \in c_0(X).$$

Proof: (1) Fix arbitrarily $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in (c_0(X), \mathfrak{u}_0)$. By Lemma 25, we have $\mathbf{x}_n = \sum_{i=1}^{n} \nu_i(x_i) \to \mathbf{x}$ in $(c_0(X), \mathfrak{u}_0)$. Let $\mathbf{g} = t^{\wedge}(\chi)$ for some $\chi \in (c_0(X), \mathfrak{u}_0)^{\wedge}$. Then Proposition 32(2) implies

$$(\chi, \mathbf{x}) = \lim_{n\to\infty} (\chi, \mathbf{x}_n) = \lim_{n\to\infty} \prod_{i=1}^{n} (\chi, \nu_i(x_i)) = \lim_{n\to\infty} \prod_{i=1}^{n} (g_i, x_i).$$

Thus $\mathbf{g}$ satisfies condition (α).

(2) is trivial. □

Now we give a simple sufficient condition.

Proposition 35. *Let* X *be an Abelian topological group and let* $\mathbf{g} = (g_n)_{n\in\mathbb{N}} \in \widehat{X}^{\mathbb{N}}$ *satisfy the following condition*

$(*)$ *there exists an open subgroup* H *of* X *and a natural number* m *such that* $g_n \in H^{\perp}$ *for every* $n \geq m$.

Then $\mathbf{g}$ *determines a continuous character* χ *of* $(c_0(X), \mathfrak{u}_0)$ *by the formula*

$$(\chi, \mathbf{x}) := \lim_{n\to\infty} \prod_{i=1}^{n} (g_i, x_i), \ \forall \mathbf{x} = (x_n)_{n\in\mathbb{N}} \in c_0(X).$$

Proof: Let us show first that $\mathbf{g}$ determines a character χ of $(c_0(X), \mathfrak{u}_0)$. Indeed, for every $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in c_0(X)$ choose a natural number n_0 such that $x_n \in H$ for every $n > n_0$. Then

$$\lim_{n\to\infty} \prod_{i=1}^{n} (g_i, x_i) = \prod_{i=1}^{n_0} (g_i, x_i).$$

Thus $\mathbf{g}$ defines a character χ of $(c_0(X), \mathfrak{u}_0)$ by Proposition 34(2).

We claim that χ is continuous. Indeed, fix arbitrarily $\varepsilon > 0$. Choose $U \in \mathcal{N}(X)$ such that $U \subseteq H$ and

$$\left| 1 - \prod_{i=1}^{m} (g_i, x_i) \right| < \varepsilon, \text{ for every } x_1, \dots, x_m \in U,$$

where m is taken from condition $(*)$. Then for every $\mathbf{x} = (x_n) \in U^{\mathbb{N}} \cap c_0(X)$ we obtain

$$|1 - (\chi, \mathbf{x})| = \left| 1 - \prod_{i=1}^{m} (g_i, x_i) \right| < \varepsilon.$$

So χ is continuous on $(c_0(X), \mathfrak{u}_0)$. □

Let X be an Abelian topological group. Following [21], define

$$c_0(X)^\beta := \left\{ \mathbf{g} = (g_n)_{n\in\mathbb{N}} \in \widehat{X}^{\mathbb{N}} : \exists \lim_{m\to\infty} \prod_{n=1}^{m} (g_n, x_n), \ \forall \mathbf{x} = (x_n)_{n\in\mathbb{N}} \in c_0(X) \right\}.$$

Taking into account Proposition 34 we set

$$c_0(X)^{\mathcal{H}} := \left\{ \mathbf{g} = (g_n)_{n\in\mathbb{N}} \in \widehat{X}^{\mathbb{N}} : \ \mathbf{g} \text{ satisfies condition } (*) \right\}.$$

Clearly, every $\mathbf{g} \in \widehat{X}^{(\mathbb{N})}$ satisfies condition $(*)$. Now Propositions 34 and 35 imply

$$\widehat{X}^{(\mathbb{N})} \subseteq c_0(X)^{\mathcal{H}} \subseteq (\widehat{c_0(X), \mathfrak{u}_0}) \subseteq c_0(X)^\beta. \tag{7}$$

In what follows the next notion will be useful. Let $\mathbf{g} = (g_n)_{n\in\mathbb{N}} \in \widehat{X}^{\mathbb{N}}$ for an Abelian topological group X. We say that $\mathbf{g}$ is *equicontinuous* if the sequence $\{g_n\}_{n\in\mathbb{N}}$ in $X^\wedge$ is equicontinuous, i.e., for every $\varepsilon > 0$ there is $U \in \mathcal{N}(X)$ such that

$$|1 - (g_n, x)| < \varepsilon, \quad \forall x \in U, \ \forall n \in \mathbb{N}.$$

Note that, if X is discrete (in particular, finite), then every $\mathbf{g} \in \widehat{X}^{\mathbb{N}}$ is trivially equicontinuous.

Lemma 36. *Let $\mathbf{g} = (g_n)_{n\in\mathbb{N}} \in \widehat{X}^{\mathbb{N}}$ for an Abelian topological group X. If $\mathbf{g} \in (\widehat{c_0(X), \mathfrak{u}_0})$, then $\mathbf{g}$ is equicontinuous.*

Proof: Take arbitrarily $\varepsilon > 0$. Since $\mathbf{g} \in (\widehat{c_0(X), \mathfrak{u}_0})$ there exists $U \in \mathcal{N}(X)$ such that

$$|1 - (\mathbf{g}, \mathbf{x})| < \varepsilon, \quad \forall \mathbf{x} \in U^{\mathbb{N}} \cap c_0(X).$$

In particular, for every $x \in U$, we have $\nu_n(x) \in U^{\mathbb{N}} \cap c_0(X)$ and

$$|1 - (g_n, x)| = |1 - (\mathbf{g}, \nu_n(x))| < \varepsilon.$$

Thus $\mathbf{g}$ is equicontinuous. $\square$

It is natural to ask (cf. Conjecture 8.2 in [21]):

Question 37. *For an Abelian topological group X, let $\mathbf{g} = (g_n)$ belong to $c_0(X)^\beta$ and be equicontinuous. Does $\mathbf{g}$ determine a continuous character of $(c_0(X), \mathfrak{u}_0)$?*

Let us recall (see [2]) that a topological group X is called a *P-group* if every G_δ-subset in X is open.

Proposition 38. *Let X be an Abelian P-group (in particular, discrete). Then*

(1) $c_0(X) = X^{(\mathbb{N})}$;

(2) $c_0(X)^{\mathcal{H}} = (\widehat{c_0(X), \mathfrak{u}_0}) = \widehat{X}^{\mathbb{N}}$.

Proof: (1) It is well-known that every compact subset of X is finite [2, §4.4]. So X does not contain nontrivial convergent sequences. Thus $c_0(X) = X^{(\mathbb{N})}$.

(2) By Proposition 35 it is enough to show that $\widehat{X}^{\mathbb{N}} \subseteq c_0(X)^{\mathcal{H}}$. Let $\mathbf{g} = (g_n)_{n\in\mathbb{N}} \in \widehat{X}^{\mathbb{N}}$. Since X has a linear topology [2, Lemma 4.4.1], for every $n \in \mathbb{N}$ we can choose an open subgroup U_n of X such that $g_n|_{U_n} = 0$. Set $U = \cap_{n\in\mathbb{N}} U_n$. Then U is an open subgroup of X by the definition of P-groups. Since $g_n \in U^{\perp}$ for every $n \in \mathbb{N}$, we have $\mathbf{g} \in c_0(X)^{\mathcal{H}}$. □

In the next proposition we prove that $\widehat{X}^{(\mathbb{N})}$ is dense in $(c_0(X), \mathfrak{u}_0)^\wedge$ for every Abelian topological group X.

Proposition 39. *Let $\mathbf{g} = (g_n)_{n\in\mathbb{N}} \in (c_0(X), \mathfrak{u}_0)^\wedge$ for an Abelian topological group X. Then $\mathbf{g}_n \to \mathbf{g}$ in $(c_0(X), \mathfrak{u}_0)^\wedge$. Therefore, $\widehat{X}^{(\mathbb{N})}$ is dense in $(c_0(X), \mathfrak{u}_0)^\wedge$.*

Proof: We have to show that for every compact subset K of $(c_0(X), \mathfrak{u}_0)$ and every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that, for every $n > n_0$,

$$\mathbf{g} - \mathbf{g}_n \in P(K, \varepsilon) := \{\mathbf{h} \in (c_0(X), \mathfrak{u}_0)^\wedge : |1 - (\mathbf{h}, \mathbf{x})| < \varepsilon, \forall \mathbf{x} \in K\}.$$

Choose a $U \in \mathcal{N}(X)$ such that

$$|1 - (\mathbf{g}, \mathbf{x})| < \varepsilon, \quad \forall \mathbf{x} = (x_n)_{n\in\mathbb{N}} \in U^{\mathbb{N}} \cap c_0(X).$$

By Proposition 31 we can choose $n_0 \in \mathbb{N}$ such that $\pi_n(K) \subseteq U$ for every $n > n_0$. In particular, $p_n(\mathbf{x}) \in U^{\mathbb{N}} \cap c_0(X)$ for every $\mathbf{x} \in K$ and every $n > n_0$. Then, for every $\mathbf{x} \in K$ and every $n > n_0$, we have

$$|1 - (\mathbf{g} - \mathbf{g}_n, \mathbf{x})| = |1 - (\mathbf{g}, p_n(\mathbf{x}))| < \varepsilon.$$

So $\mathbf{g}_n \to \mathbf{g}$ in $(c_0(X), \mathfrak{u}_0)^\wedge$. □

5. The proof of Theorem 8

In what follows we use the next lemma.

Lemma 40. *Let X and Y be Abelian topological groups. If $c_0(X)$ and $c_0(Y)$ are characterized in $X^{\mathbb{N}}$ and $Y^{\mathbb{N}}$ respectively, then $c_0(X \times Y)$ is a characterized subgroup of $(X \times Y)^{\mathbb{N}}$.*

Proof: Let $c_0(X) = s_{\mathbf{u}}(X^{\mathbb{N}})$ and $c_0(Y) = s_{\mathbf{u}}(Y^{\mathbb{N}})$ for some sequences $\mathbf{u} = \{u_n\}_{n\in\omega}$ in $\widehat{X^{\mathbb{N}}}$ and $\mathbf{v} = \{v_n\}_{n\in\omega}$ in $\widehat{Y^{\mathbb{N}}}$, where $u_n = (g_k^n)_{k\in\mathbb{N}}$ and $v_n = (h_k^n)_{k\in\mathbb{N}}$. Define the sequence $\mathbf{d} = \{d_n\}_{n\in\omega}$ in $(\widehat{X} \times \widehat{Y})^{(\mathbb{N})}$ as follows ($n \in \omega$)

$$d_{2n+1} = \left((g_1^n, 0_{\widehat{Y}}), (g_2^n, 0_{\widehat{Y}}), \dots\right) \text{ and } d_{2n} = \left((0_{\widehat{X}}, h_1^n), (0_{\widehat{X}}, h_2^n), \dots\right).$$

Then

$$\begin{aligned}((x_k, y_k))_{k\in\mathbb{N}} &\in s_{\mathbf{d}}\left((X \times Y)^{\mathbb{N}}\right) \\ &\Leftrightarrow \left(d_{2n+1}, ((x_k, y_k))_{k\in\mathbb{N}}\right) = (u_n, (x_k)) \to 1\end{aligned}$$

$$\begin{aligned} &\text{and } \left(d_{2n}, ((x_k, y_k))_{k\in\mathbb{N}}\right) = (v_n, (y_k)) \to 1 \\ \Leftrightarrow\ & (x_k) \in c_0(X) \text{ and } (y_k) \in c_0(Y) \\ \Leftrightarrow\ & ((x_k, y_k))_{k\in\mathbb{N}} \in c_0(X\times Y). \end{aligned}$$

Thus $c_0(X\times Y)$ is a characterized subgroup of $(X\times Y)^{\mathbb{N}}$ by the sequence $\mathbf{d}$. $\square$

Let X be an Abelian topological group and let $u = (g_m)_{m\in\mathbb{N}} \in \widehat{X}^{(\mathbb{N})}$. If $u \neq 0$, we set

$$N(u) := \{m\in\mathbb{N} : g_m \neq 0\}, \quad q(u) := \min\{N(u)\}, \quad w(u) := \max\{N(u)\},$$

and if $u = 0$, we set $N(u) := \emptyset$ and $q(u) = w(u) := \infty$. Put

$$C_u := \bigcap_{m\in\mathbb{N}} \ker(g_m).$$

Let $\mathbf{u} = \{u_n\}_{n\in\omega}$ be a sequence in $\widehat{X}^{(\mathbb{N})}$. For every $r\in\omega$, set

$$C_r(\mathbf{u}) := \bigcap_{n\geq r} C_{u_n}.$$

Then $\{C_r(\mathbf{u})\}_{r\in\omega}$ is an increasing sequence of closed G_δ-subgroups of X by Lemma 11.

To prove Theorem 8 we need some lemmas.

Lemma 41. *For every Abelian topological group X and each sequence $\mathbf{u} = \{u_n\}_{n\in\omega}$ in $\widehat{X^{\mathbb{N}}}$ we have*

$$\bigcup_{r\in\omega} (C_r(\mathbf{u}))^{\mathbb{N}} \subseteq s_{\mathbf{u}}(X^{\mathbb{N}}).$$

Proof: Let $u_n = (g_m^n)_{m\in\mathbb{N}}$ for every $n\in\omega$. Fix $r\in\omega$ and let $\mathbf{x} = (x_m) \in X^{\mathbb{N}}$, where $x_m \in C_r(\mathbf{u})$ for all $m\in\mathbb{N}$. Then, for every $n\geq r$, we have

$$(u_n, \mathbf{x}) = \prod_{m\in\mathbb{N}} (g_m^n, x_m) = 1.$$

Hence $\mathbf{x}\in s_{\mathbf{u}}(X^{\mathbb{N}})$. Thus $(C_r(\mathbf{u}))^{\mathbb{N}} \subseteq s_{\mathbf{u}}(X^{\mathbb{N}})$ for every $r\in\omega$. $\square$

Lemma 42. *Let an Abelian topological group X be not self-characterized and $\mathbf{u} = \{u_n\}_{n\in\omega}$ be a sequence in $\widehat{X^{\mathbb{N}}}$. If $c_0(X)\subseteq s_{\mathbf{u}}(X^{\mathbb{N}})$, then $q(u_n)\to\infty$.*

Proof: Let $u_n = (g_m^n)_{m\in\mathbb{N}}$ for every $n\in\omega$. Suppose for a contradiction that $q(u_n)\not\to\infty$. This means that there exists a natural number k and an increasing sequence of natural numbers $\{n_s\}$ such that the sequence $\mathbf{v} := \{g_k^{n_s}\}_{s\in\mathbb{N}} \in X^\wedge$ contains only non-zero elements. Since X is not self-characterized we have $s_{\mathbf{v}}(X)\neq X$ (see Remark 20). Take an arbitrary $x\in X\setminus s_{\mathbf{v}}(X)$ and put

$$\mathbf{x} := (x_n), \text{ where } x_k = x \text{ and } x_n = 0 \text{ if } n\neq k.$$

Clearly, $\mathbf{x} \in c_0(X)$. Since $(u_{n_s}, \mathbf{x}) = (g_k^{n_s}, x) \not\to 1$ by the choice of x, we obtain that $c_0(X) \not\subseteq s_{\mathbf{u}}(X^{\mathbb{N}})$, a contradiction. Thus $q(u_n) \to \infty$. $\square$

Recall that a decreasing sequence of closed subgroups $\{K_n\}_{n\in\mathbb{N}}$ of an Abelian topological group X is a null-sequence if, for every $U \in \mathcal{N}(X)$, $K_n \subseteq U$ for all sufficiently large $n \in \mathbb{N}$.

Lemma 43. *Let an Abelian topological group X be not self-characterized and $\mathbf{u} = \{u_n\}_{n\in\omega}$ be a sequence in $\widehat{X^{\mathbb{N}}}$. If $c_0(X) \subseteq s_{\mathbf{u}}(X^{\mathbb{N}})$, then for every null-sequence $\{K_n\}_{n\in\mathbb{N}}$ of closed subgroups of X there exist natural numbers r and n such that*

$$K_n \subseteq C_r(\mathbf{u}).$$

Proof: If $\mathbf{u}$ is eventually equal to zero the lemma is trivial. Let $\mathbf{u}$ contain infinitely many non-zero elements. Passing to the subsequence of $\mathbf{u}$ containing all non-zero elements of $\mathbf{u}$, without loss of generality we can suppose that $u_n \neq 0$ for every $n \in \omega$. Let $u_n = (g_m^n)_{m\in\mathbb{N}}$ for every $n \in \omega$. Since $u_n \neq 0$ we have $N(u_n) \neq \emptyset$ for every $n \in \omega$.

Suppose for a contradiction that there exists a null-sequence $\{K_n\}_{n\in\mathbb{N}}$ of closed subgroups of X such that for every natural numbers r and n

$$K_n \setminus C_r(\mathbf{u}) \neq \emptyset. \tag{8}$$

Step 1. Let us show that there exists an increasing sequence of natural numbers $1 \le r_1 < r_2 < \dots$ such that, for every $k \in \mathbb{N}$,

$$w(u_{r_k}) < q(u_{r_{k+1}}), \tag{9}$$

and

$$K_k \setminus C_{u_{r_k}} = K_k \setminus \left(\bigcap_{m\in N(u_{r_k})} \ker\left(g_m^{r_k}\right) \right) \neq \emptyset. \tag{10}$$

We build such a sequence by induction. Let $n = 1$. Then (8) implies that $K_1 \setminus C_r(\mathbf{u}) \neq \emptyset$ for every $r \in \mathbb{N}$. So there exists r_1, $1 \le r_1$, such that

$$K_1 \setminus C_{u_{r_1}} = K_1 \setminus \left(\bigcap_{m\in N(u_{r_1})} \ker\left(g_m^{r_1}\right) \right) \neq \emptyset.$$

Assume now that we found $r_1 < \dots < r_n$ satisfying (9), for $1 \le k \le n-1$, and (10), for $1 \le k \le n$. Lemma 42 implies that there is r_n', $r_n' > r_n$, such that $w(u_{r_n}) < q(u_s)$ for every natural number $s \ge r_n'$. By (8), there is r_n'', $r_n'' \ge r_n'$, such that $K_{n+1} \setminus C_r(\mathbf{u}) \neq \emptyset$ for every natural number $r \ge r_n''$. This means that there exists r_{n+1}, $r_{n+1} > r''_n > r'_n > r_n$, such that

$$K_{n+1} \setminus C_{u_{r_{n+1}}} = K_{n+1} \setminus \left(\bigcap_{m \in N(u_{r_{n+1}})} \ker\left(g_m^{r_{n+1}}\right) \right) \neq \emptyset.$$

By the choice of r_{n+1} we have also $w(u_{r_n}) < q(u_{r_{n+1}})$. So (9) and (10) are fulfilled.

Step 2. By (10), for every natural number n we can find $m_n \in N(u_{r_n})$ such that

(11) $$K_n \not\subset \ker\left(g_{m_n}^{r_n}\right).$$

Since K_n is a subgroup of X, (11) implies that there exists $x_{m_n} \in K_n$ such that

(12) $$\left|1 - \left(g_{m_n}^{r_n}, x_{m_n}\right)\right| \geq 0.25.$$

It follows from (9) that $m_1 < m_2 < \dots$. So we can define

$$\mathbf{x} := (x_k), \text{ where } x_k = \begin{cases} x_{m_n}, & \text{if } k = m_n \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\{K_n\}$ is a null-sequence of closed subgroups of X, clearly, $\mathbf{x} \in c_0(X)$. Now the choice of $\mathbf{x}$, (9) and (12) imply

$$|1 - (u_{r_n}, \mathbf{x})| = \left|1 - \left(g_{m_n}^{r_n}, x_{m_n}\right)\right| \geq 0.25.$$

Hence $\mathbf{x} \notin s_{\mathbf{u}}(X^{\mathbb{N}})$ and $c_0(X) \not\subset s_{\mathbf{u}}(X^{\mathbb{N}})$. This contradiction proves the lemma. □

Proposition 44. *Let an Abelian topological group X be not self-characterized. If $c_0(X)$ is a characterized subgroup of $X^{\mathbb{N}}$, then X has no nontrivial null-sequences of closed subgroups.*

Proof: Let $\mathbf{u}$ be a characterizing sequence for $c_0(X)$. Suppose for a contradiction that X has a nontrivial null-sequence of closed subgroups. By Lemma 43, $C_r(\mathbf{u}) \neq \{0\}$ for some natural number r. So $(C_r(\mathbf{u}))^{\mathbb{N}} \subseteq s_{\mathbf{u}}(X^{\mathbb{N}})$ by Lemma 41. Since $(C_r(\mathbf{u}))^{\mathbb{N}} \not\subset c_0(X)$ trivially, we obtain that $\mathbf{u}$ does not characterize $c_0(X)$. This contradiction shows that X has no nontrivial null-sequences of closed subgroups. □

Now we are in position to prove Theorem 8.

Proof of Theorem 8: (i)⇔(ii) immediately follows from Fact 1 since $c_0(X)$ is dense in $X^{\mathbb{N}}$.

(i)⇒(iii) Let $\mathbf{u} = \{u_n\}_{n\in\omega}$ characterize $c_0(X)$, where $u_n = (g_m^n)_{m\in\mathbb{N}} \in \widehat{X^{\mathbb{N}}}$. Denote by S the countable subgroup of $\widehat{X}$ generated by all the elements g_m^n, where $n \in \omega$ and $m \in \mathbb{N}$.

Let us show first that X is metrizable. Indeed, suppose for a contradiction that X is not metrizable. By Lemma 11, we obtain that $C_0(\mathbf{u}) = S^{\top}$ is a closed

G_δ-subgroup of X. Since X is compact, $(\widehat{X/C_0(\mathbf{u})}) = S$. Hence the quotient compact group $X/C_0(\mathbf{u})$ is metrizable. Since X is not metrizable we obtain that $C_0(\mathbf{u})$ is not trivial. Now Lemma 41 implies that $(C_0(\mathbf{u}))^{\mathbb{N}} \subseteq s_{\mathbf{u}}(X^{\mathbb{N}})$. Hence $c_0(X) \neq s_{\mathbf{u}}(X^{\mathbb{N}})$. This contradiction shows that X is metrizable.

Since X is compact, Theorem 21 and Proposition 44 imply that X has no nontrivial null-sequences of closed subgroups. Since X is metrizable, this means that X has no small subgroups. Thus X is a commutative compact Lie group. It is well-known that $X \cong \mathbb{T}^n \times F$, where $n \geq 0$ and F is a finite group (maybe trivial).

(iii)$\Rightarrow$(i) Let $X \cong \mathbb{T}^n \times F$ for some integer $n \geq 0$ and a finite group F.

As it was noticed in the introduction, $c_0(\mathbb{T})$ is a characterized subgroup of $\mathbb{T}^{\mathbb{N}}$. Since $c_0(F^{\mathbb{N}}) = F^{(\mathbb{N})}$ is a countable subgroup of the compact metrizable group $F^{\mathbb{N}}$, $c_0(F^{\mathbb{N}})$ is a characterized subgroup of $F^{\mathbb{N}}$ by [8], [20]. Now Lemma 40 implies that $c_0(X)$ is a characterized subgroup of $X^{\mathbb{N}}$. □

6. The proof of Theorem 10

We start from the next proposition:

Proposition 45. *Let X be an Abelian topological group. If $c_0(X)$ is $\mathfrak{g}$-closed in $X^{\mathbb{N}}$, then X is MAP.*

Proof: Suppose for a contradiction that X is not MAP. Denote by H the von Neumann radical of X, i.e., $H = \cap_{\chi\in\widehat{X}} \ker(\chi)$. So that $H \neq 0$. Clearly, $H^{\mathbb{N}} \not\subseteq c_0(X)$. On the other hand, for every sequence $\mathbf{u} \in \widehat{X^{\mathbb{N}}}$ we have $H^{\mathbb{N}} \subseteq s_{\mathbf{u}}(X^{\mathbb{N}})$ by equality (5). Hence $H^{\mathbb{N}} \subseteq \mathfrak{g}(c_0(X))$. Thus $c_0(X) \subsetneqq \mathfrak{g}(c_0(X))$, a contradiction. □

Let X be an Abelian topological group. For every $g \in \widehat{X}$ and every natural number n, set

$$u_n^g := \nu_n(g) = (0, \dots, 0, g, 0, \dots) \in \widehat{X}^{(\mathbb{N})},$$

where g is placed in the position n. For every $g \in \widehat{X}$, set $\mathbf{u}_g = \{u_n^g\}_{n\in\mathbb{N}} \subset \widehat{X}^{(\mathbb{N})}$.

Lemma 46. *Let X be an Abelian topological group. For every $g \in \widehat{X}$, the sequence $\mathbf{u}_g$ converges to zero in $(c_0(X), \mathfrak{u}_0)^\wedge$.*

Proof: Let K be a compact subset of $(c_0(X), \mathfrak{u}_0)$ and $\varepsilon > 0$. We have to show that there exists $m \in \mathbb{N}$ such that, for every $n > m$,

$$u_n^g \in P(K, \varepsilon) := \{\mathbf{g} \in (c_0(X), \mathfrak{u}_0)^\wedge : |1 - (\mathbf{g}, \mathbf{x})| < \varepsilon, \forall \mathbf{x} \in K\}.$$

Since $g \in \widehat{X}$, we can choose $U \in \mathcal{N}(X)$ such that

$$|1 - (g, x)| < \varepsilon, \quad \forall x \in U. \tag{13}$$

By Proposition 31, choose $m \in \mathbb{N}$ such that $\pi_n(K) \subseteq U$ for every $n > m$. Then, by (13), for every $n > m$ and every $\mathbf{x} \in K$, we have

$$|1 - (u_n^g, \mathbf{x})| = |1 - (g, \pi_n(\mathbf{x}))| < \varepsilon.$$

Thus $u_n^g \to 0$ in $(c_0(X), \mathfrak{u}_0)^\wedge$. □

Lemma 47. *Let X be an Abelian topological group. Then*

(i) $c_0(X) \subseteq \bigcap_{g\in\widehat{X}} s_{\mathbf{u}_g}(X^{\mathbb{N}})$;

(ii) $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \bigcap_{g\in\widehat{X}} s_{\mathbf{u}_g}(X^{\mathbb{N}})$ *if and only if* $x_n \to 0$ *in* $\sigma(X, \widehat{X})$.

Proof: (i) Let $\mathbf{x} = (x_k) \in c_0(X)$. Since $x_n \to 0$ in X, for every $g \in \widehat{X}$ we have

$$(u_n^g, \mathbf{x}) = (g, x_n) \to 1.$$

Hence $\mathbf{x} \in s_{\mathbf{u}_g}(X^{\mathbb{N}})$ for every $g \in \widehat{X}$. Thus $c_0(X) \subseteq \bigcap_{g\in\widehat{X}} s_{\mathbf{u}_g}(X^{\mathbb{N}})$.

(ii) By definition, $\mathbf{x} = (x_n)_{n\in\mathbb{N}} \in \bigcap_{g\in\widehat{X}} s_{\mathbf{u}_g}(X^{\mathbb{N}})$ if and only if $(u_n^g, \mathbf{x}) = (g, x_n) \to 1$ for every $g \in \widehat{X}$. But this exactly means that $x_n \to 0$ in $\sigma(X, \widehat{X})$. □

Let us recall that a MAP Abelian group (X, τ) *respects compactness* if the compact subsets of (X, τ) and $(X, \sigma(X, \widehat{X}))$ coincide. In analogy to this property we say that a MAP Abelian group (X, τ) *respects sequentiality* if (X, τ) and $(X, \sigma(X, \widehat{X}))$ have the same set of convergent sequences. Clearly, if X respects compactness, then it respects sequentiality. Glicksberg [31] proved that every locally compact Abelian group X respects compactness, and hence X respects sequentiality.

Theorem 48. *Let (X, τ) be a MAP Abelian group which respects sequentiality. Then $c_0(X)$ is a $\mathfrak{g}$-closed subgroup of $X^{\mathbb{N}}$. More precisely,*

$$c_0(X) = \bigcap_{g\in\widehat{X}} s_{\mathbf{u}_g}(X^{\mathbb{N}}). \tag{14}$$

Proof: The inclusion $c_0(X) \subseteq \bigcap_{g\in\widehat{X}} s_{\mathbf{u}_g}(X^{\mathbb{N}})$ follows from Lemma 47(i). Let us prove the converse inclusion in (14).

Let $\mathbf{x} = (x_k) \in \bigcap_{g\in\widehat{X}} s_{\mathbf{u}_g}(X^{\mathbb{N}})$. Lemma 47(ii) implies that $x_n \to 0$ in $\sigma(X, \widehat{X})$. Since X respects sequentiality, $x_n \to 0$ in τ as well. Thus $\mathbf{x} \in c_0(X)$ and (14) is proved. □

Now we are in position to prove Theorem 10.

Proof of Theorem 10: Since $c_0(X)$ is dense in $X^{\mathbb{N}}$, every sequence $\mathbf{u}$ such that $c_0(X) \leq s_{\mathbf{u}}(X^{\mathbb{N}})$ is a T-sequence by Fact 1. Now the theorem immediately follows from Theorem 48 and the above-mentioned Glicksberg's theorem [31]. □

We end with the following problems:

Problem 49. *Let X be an Abelian topological group such that $c_0(X)$ is $\mathfrak{g}$-closed in $X^{\mathbb{N}}$. Does X respect sequentiality?*

Problem 50. *Characterize MAP Abelian groups which respect sequentiality.*

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