Strong pseudocompact properties

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Abstract. For a free ultrafilter p on $\mathbb{N}$, the concepts of strong pseudocompactness, strong p-pseudocompactness and pseudo-ω-boundedness were introduced in [Angoa J., Ortiz-Castillo Y.F., Tamariz-Mascarúa A., *Ultrafilters and properties related to compactness*, Topology Proc. **43** (2014), 183–200] and [García-Ferreira S., Ortiz-Castillo Y.F., *Strong pseudocompact properties of certain subspaces of* $\mathbb{N}^*$, submitted]. These properties in a space X characterize the pseudocompactness of the hyperspace $\mathcal{K}(X)$ of compact subsets of X with the Vietoris topology. In this paper, we study the strong pseudocompactness and strong p-pseudocompactness of certain spaces. Besides, we established a relationship between these kind of properties and a result involving topological groups of I. Protasov [*Discrete subsets of topological groups*, Math. Notes **55** (1994), no. 1–2, 101–102].

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Preliminaries and introduction

In this article, every space will be Tychonoff and infinite. The Greek letter ω represents the first infinite cardinal number. For a space X, we let $\mathcal{D}(X)$ be the set of all discrete subsets of X. If X is a topological space and $A \subseteq X$, then we denote by $cl_X(A)$ (or simply $cl(A)$) the closure of A in X. The *Stone-Čech* compactification $\beta\mathbb{N}$ of the discrete space of natural numbers $\mathbb{N}$ will be identified with the set of all ultrafilters on $\mathbb{N}$ and its remainder $\mathbb{N}^*$ will be identified with the set of all free ultrafilters on $\mathbb{N}$. If $A \subseteq \mathbb{N}$, then $\hat{A} = cl_{\beta(\mathbb{N})}A = \{p \in \beta(\mathbb{N}) : A \in p\}$ is a basic clopen subset of $\beta(\mathbb{N})$, and $A^* = \hat{A} \setminus A = \{p \in \mathbb{N}^* : A \in p\}$ is a basic clopen subset of $\mathbb{N}^*$. Given two ultrafilters $p, q \in \beta\mathbb{N}$, we say that $p \leq_{RK} q$ if there exists a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that $\bar{f}(q) = p$, where $\bar{f}$ is the Stone extension of f to $\beta\mathbb{N}$. This relation is known as the *Rudin-Keisler* pre-order on $\beta\mathbb{N}$. We say that two ultrafilters p and q are $\leq_{RK}$*-equivalent* if $p \leq_{RK} q$ and $q \leq_{RK} p$ (in symbols, $p \sim q$); they are $\leq_{RK}$*-comparable* if either $p \leq_{RK} q$ or $q \leq_{RK} p$; and they are $\leq_{RK}$*-incomparable* if they are not $\leq_{RK}$-comparable. For $p, q \in \mathbb{N}^*$, it is known that p and q are RK-equivalent iff there is a bijection $f : \mathbb{N} \to \mathbb{N}$ such that

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$\bar{f}(p) = q$ (see [6]). The *type* of $p \in \mathbb{N}^*$ is the set $T(p) = \{q \in \mathbb{N}^* : p \sim q\}$. For an infinite set X, we write $[X]^\omega := \{A \subseteq X : |A| = \omega\}$. Those notions used and not defined in this article have the meaning given to them in [7].

Following the paper [11], given a space X, $p \in \mathbb{N}^*$ and a sequence $(S_n)_{n\in\mathbb{N}}$ of nonempty subsets of X, we say that $x \in X$ is a *p-limit* of $(S_n)_{n\in\mathbb{N}}$, in symbols $x = p - \lim_{n\to\infty} S_n$, if $\{n \in \mathbb{N} : S_n \cap W \neq \emptyset\} \in p$ for each neighborhood W of x. In particular, if $(x_n)_{n\in\mathbb{N}}$ is a sequence of points of X and x is a p-limit point of this sequence, then the point x is unique and we simply write $x = p - \lim_{n\to\infty} x_n$ rather than $x = p - \lim_{n\to\infty}\{x_n\}$. In some cases, the set of p-limit points, denoted by $L(p, (S_n)_{n\in\mathbb{N}})$, of a sequence $(S_n)_{n\in\mathbb{N}}$ of nonempty subsets of X can have more than one point. The concept of p-limit point of a sequence of points has been around in the mathematical literature for a long time and has been considered by several mathematicians (for instance, R.A. Bernstein [4], H. Furstenberg [8, p. 179] and E. Akin [1, p. 5, 61]). For $p \in \mathbb{N}^*$, a topological space X is named *p-pseudocompact* if every sequence of nonempty open subsets of X has a p-limit point. This concept of p-pseudocompactness was introduced by Ginsburg and Saks [11]. In the literature, a space X is called *pseudo-ω-bounded* if for each sequence $(U_n)_{n\in\mathbb{N}}$ of nonempty open subsets of X there is a compact subset K of X such that $K \cap U_n \neq \emptyset$ for each $n \in \mathbb{N}$, a space X is called *ultrapseudocompact* if X is p-pseudocompact for all $p \in \mathbb{N}^*$. As in the paper [3], a space X is called *strongly p-pseudocompact*, for $p \in \mathbb{N}^*$, if for each sequence $(U_n)_{n\in\mathbb{N}}$ of nonempty open subsets of X there are a sequence $(x_n)_{n\in\mathbb{N}}$ of points in X and $x \in X$ such that $x = p-\lim_{n\to\infty} x_n$ and $x_n \in U_n$ for all $n \in \mathbb{N}$. Base on this notion, a space X is called *strongly pseudocompact* if for each sequence $(U_n)_{n\in\mathbb{N}}$ of nonempty open subsets of X there are $p \in \mathbb{N}^*$, a sequence $(x_n)_{n\in\mathbb{N}}$ of points in X and $x \in X$ such that $x = p - \lim_{n\to\infty} x_n$ and $x_n \in U_n$ for all $n \in \mathbb{N}$ (see [9]). All this notions arose from the following result concerning hyperspaces. Given a space X, $\mathcal{K}(X)$ denotes the hyperspace of compact subsets of X with the Vietoris topology.

Theorem 0.1 ([2, Theorem 2.2]). *Let X be a topological space. Then the following statements are equivalent:*

(1) *X is pseudo-ω-bounded,*
(2) *$\mathcal{K}(X)$ is pseudocompact,*
(3) *$\mathcal{K}(X)$ is p-pseudocompact for some $p \in \mathbb{N}^*$, and*
(4) *$\mathcal{K}(X)$ is strongly p-pseudocompact for some $p \in \mathbb{N}^*$.*

It is easy to check that the strong pseudocompactness of $\mathcal{K}(X)$ is equivalent to any statement of the previous theorem. Also, the properties of p-pseudocompactness and strong p-pseudocompactness are equivalent under locally compactness, and for every space X satisfying $\mathbb{N} \subseteq X \subseteq \beta\mathbb{N}$ (see [3]). The first examples of p-pseudocompact spaces non-strongly p-pseudocompact, was given in [9] (actually, ultrapseudocompact non-strongly pseudocompact spaces). An example of a strongly pseudocompact non-ultrapseudocompact space is a countable compact space whose square is not pseudocompact. In Section 1, we give an example of a strongly pseudocompact, ultrapseudocompact non-strongly p-pseudocompact space for several $p \in \mathbb{N}^*$. In the second section, we use some ideas of I. Protasov [12] to give an approach to the answer to the next question:

Question 0.2. *Is it true that every pseudocompact group G is strongly pseudocompact?*

Unfortunately, we could not answer this question.

1. Strong pseudocompactness and p-pseudocompactness

Clearly, strong pseudocompactness is stronger than pseudocompactness and weaker than countable compactness. It is proved in [3] that the set of RK-predecessors $P_{RK}(p)$ of p is a strongly p-pseudocompact, non-p-compact space, for any $p \in \mathbb{N}^*$. Also it is showed in [9] that the subspace of weak P-points of $\mathbb{N}^*$ is an ultrapseudocompact, non-strongly pseudocompact space. The only missing implication between these properties is from the strong pseudocompactness to the p-pseudocompactness. In the present section, we will show that this implication is false. To construct our example we shall need the following two lemmas. The first one is a well-known result due to Z. Frolík (for a proof see [5, Lemma 8.2]).

Lemma 1.1. *Let $S, T \in [\mathbb{N}^*]^\omega$. If $cl(S) \cap cl(T) \neq \emptyset$, then either $cl(S) \cap T \neq \emptyset$ or $S \cap cl(T) \neq \emptyset$.*

To the rest of this paper, we simplify our notation introducing the following definition.

Definition 1.2. Let X be a space. Given $S \in [\mathcal{P}(X) \setminus \{\emptyset\}]^\omega$ and $p \in \mathbb{N}^*$, the *p-boundary* of S in X is the set

$$\mathcal{B}_p(S) = \{x \in X : x = p - \lim_{n\to\infty} x_n, \{x_n : n \in \mathbb{N}\}^1 \subseteq \bigcup S \text{ and}$$

$$\forall A \in S(|\{n : x_n \in A\}| < \omega)\}.$$

Lemma 1.3. *Let $\mathcal{U}$ be an infinite family of nonempty pairwise disjoint clopen sets of $\mathbb{N}^*$ and let p, $q \in \mathbb{N}^*$. If there exist two countable sets $S_1, S_2 \subseteq \mathcal{U}$ such that $\mathcal{B}_p(S_1) \cap \mathcal{B}_q(S_2) \neq \emptyset$, then p and q are RK-equivalent.*

Proof: Fix $z \in \mathcal{B}_p(S_1) \cap \mathcal{B}_q(S_2)$. Choose $\{x_n : n \in \mathbb{N}\} \subseteq \bigcup S_1$ and $\{y_m : m \in \mathbb{N}\} \subseteq \bigcup S_2$ satisfying the conditions of Definition 1.2. In particular, $z = p - \lim_{n\to\infty} x_n = q - \lim\ y_m$. Consider the sets

$$A = \{n \in \mathbb{N} : \exists m \in \mathbb{N}(x_n = y_m)\} \text{ and } B = \{m \in \mathbb{N} : \exists n \in \mathbb{N}(y_m = x_n)\}.$$

Claim: We have that $A \in p$, $B \in q$, $\{m \in \mathbb{N} : \exists n \in A \cap C(y_m = x_n)\} \in q$ for every $C \in p$, and $\{n \in \mathbb{N} : \exists m \in B \cap D(x_n = y_m)\} \in p$ for every $D \in q$.

Proof of Claim: Suppose $A \notin p$. Since p is an ultrafilter, $\mathbb{N} \setminus A \in p$ and hence $p - \lim_{n\in\mathbb{N}\setminus A}\ x_n = z$. So, without loss of generality, we can assume that $A = \emptyset$.

[1] In the set notation $\{x_n : n \in \mathbb{N}\}$, we shall undertand that $x_n \neq x_m$ whenever $n \neq m$.

But, on the other hand, by Lemma 1.1, $cl(\{x_n : n \in \mathbb{N}\}) \cap \{y_m : m \in \mathbb{N}\} \neq \emptyset$ or $\{x_n : n \in \mathbb{N}\} \cap cl(\{y_m : m \in \mathbb{N}\}) \neq \emptyset$. Then $\{x_n : n \in \mathbb{N}\} \cap \{y_m : m \in \mathbb{N}\} \neq \emptyset$ which is a contradiction. This prove that $A \in p$ and, in a similar way, we can prove that $B \in q$.

Now, let $C \in p$. Let $S_1 = \{U_n : n \in \mathbb{N}\}$. For each $n \in A \setminus C$ choose $a_n \in U_n \setminus \{y_m : m \in \mathbb{N}\}$ and let $a_n = x_n$ in other case. Of course, $z = p - \lim_{n\to\infty} a_n$. Applying the part of the Claim already proved, we obtain that $\{m \in \mathbb{N} : \exists n \in A \cap C(y_m = a_n)\} = \{m \in \mathbb{N} : \exists n \in \mathbb{N}(y_m = a_n)\} \in q$. The proof of $\{n \in \mathbb{N} : \exists m \in B \cap D(x_n = y_m)\} \in p$ is similar.

Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f|_A : A \to B$ is the bijection defined by $f(n) = m$ iff $x_n = y_m$. By the Claim, we have that $\overline{f}(p) = q$ and so $q \leq_{RK} p$. Therefore, by [6, Theorem 9.2], p and q are RK-equivalent. $\square$

We are ready to construct our example.

Example 1.4. Let $\{C_\eta : \eta < \mathfrak{c}\}$ be a family of subsets of $\mathbb{N}^*$ such that:

(1) p and q are RK-incomparable provided that $p \in C_\eta$, $q \in C_\zeta$ where $\eta < \zeta < \mathfrak{c}$; and
(2) $|C_\eta| = 2^{\mathfrak{c}}$ for each $\eta < \mathfrak{c}$.

There is a strongly pseudocompact, ultrapseudocompact space that it is not strongly p-pseudocompact for all $p \in \bigcup_{\eta<\mathfrak{c}} C_\eta$.

Proof: Fix a MAD family $\mathcal{A}$ of size $\mathfrak{c}$ and let $\mathcal{S}$ be a partition of $\{A^* : A \in \mathcal{A}\}$ in subsets of size ω. Enumerate $\bigcup_{\eta<\mathfrak{c}} C_\eta$ by $\{p_\xi : \xi < 2^{\mathfrak{c}}\}$ and $\mathcal{S}$ by $\{S_\eta : \eta < \mathfrak{c}\}$. Now re-enumerate $\mathcal{S}$ by $\{S_\xi : \xi < 2^{\mathfrak{c}}\}$ in such a way that $p_\xi \in C_\eta$ iff $S_\xi = S_\eta$, for $\eta < \mathfrak{c}$ and for $\xi < 2^{\mathfrak{c}}$. For each $\xi < 2^{\mathfrak{c}}$, let $X_\xi = \mathbb{N}^* \setminus \mathcal{B}_{p_\xi}(S_\xi)$. The space will be $X = \bigcap_{\xi<2^{\mathfrak{c}}} X_\xi$ with the topology inherited from $\mathbb{N}^*$. Observe that $A^* \subseteq X$ for all $A \in \mathcal{A}$. Thus, X has dense interior in $\mathbb{N}^*$. Since X contains the subspace of weak P-points of $\mathbb{N}^*$ and this space is ultrapseudocompact [9], we obtain that X is ultrapseudocompact too. We shall prove that X is strongly pseudocompact and it is not strong p-pseudocompact for any $p \in \bigcup_{\eta<\mathfrak{c}} C_\eta$. First, we shall prove that X is not strong p-pseudocompact for any $p \in \bigcup_{\eta<\mathfrak{c}} C_\eta$. Fix $\eta < \mathfrak{c}$ and $p \in C_\eta$. We know that $p = p_\xi$ for some $\xi < 2^{\mathfrak{c}}$. Let $\{V_n : n \in \mathbb{N}\}$ be an enumeration of S_ξ. Suppose that $(x_n)_{n\in\mathbb{N}}$ is any sequence of points such that $x_n \in V_n$ for every $n \in \mathbb{N}$. Then the p-limit point of $(x_n)_{n\in\mathbb{N}}$ belongs to $\mathcal{B}_{p_\xi}(S_\xi)$. This shows that X cannot be strongly p-pseudocompact. To prove that X is strongly pseudocompact, we let $(U_n)_{n\in\mathbb{N}}$ be a sequence of nonempty open subsets of X. Without loss of generality, we may assume that for each $n \in \mathbb{N}$ there is $A_n \in \mathcal{A}$ such that $U_n \subseteq A_n^*$ and U_n is clopen in $\mathbb{N}^*$. We need to consider three cases:

Case I. There is $m \in \mathbb{N}$ such that $|\{n \in \mathbb{N} : A_n = A_m\}| = \omega$. In this case just pick any sequence $(x_n)_{n\in\mathbb{N}}$ in X so that $x_n \in U_n \setminus \{x_i : i < n\}$. Since A_m^* is compact, any accumulation point of $\{x_n : n \in \mathbb{N}$ and $A_n = A_m\}$ lies in $A_m^* \subseteq X$.

Case II. There are an infinite set $B \subseteq \mathbb{N}$ and an injective function $\phi : B \to \mathcal{S}$ such that $U_n \cap (\bigcup \phi(n)) \neq \emptyset$ for every $n \in B$. Now choose a sequence $(x_n)_{n\in\mathbb{N}}$ in X

so that $x_1 \in U_1$ and $x_n \in [U_n \cap \phi(n)] \setminus \{x_i : i < n\}$ for every $n \in B \setminus \{1\}$. Let q be any accumulation point of the set $\{x_n : n \in B\}$ in $\mathbb{N}^*$. Suppose that $q \notin X$. Then there is $\xi < 2^{\mathfrak{c}}$ such that $q \in \mathcal{B}_{p_\xi}(S_\xi)$. Hence, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in $\bigcup S_\xi$ such that $q = p_\xi - \lim_{n\to\infty} y_n$. By Lemma 1.1, we must have that either $cl_{\mathbb{N}^*}(\{x_n : n \in B\}) \cap \{y_n : n \in \mathbb{N}\} \neq \emptyset$ or $\{x_n : n \in B\} \cap cl_{\mathbb{N}^*}(\{y_n : n \in \mathbb{N}\}) \neq \emptyset$. Hence, we obtain that $\{x_n : n \in B\} \cap cl_{\mathbb{N}^*}(\{y_n : n \in \mathbb{N}\}) \neq \emptyset$, but this contradicts the assumption $\phi(n) \cap S_\xi$ is finite for all $n \in B$. Therefore, $q \in X$.

Case *III*. The first two cases do not hold. Then there is $\eta < \mathfrak{c}$ such that $\{V \in S_\eta : \exists n \in \mathbb{N}(U_n \subseteq V)\}$ is infinite and each U_n and each element of S_η contains finitely many U_n's. Without loss of generality, we may assume that for every $n \in \mathbb{N}$ there is $V \in S_\eta$ such that $U_n \subseteq V$. As before, pick $x_1 \in U_1$ and $x_n \in U_n \setminus \{x_i : i < n\}$ for each $n > 1$. Choose $\zeta < \mathfrak{c}$ so that $C_\eta \neq C_\zeta$ and fix $q \in C_\zeta$. Then, $z = q - \lim_{n\to\infty} x_n \in \mathcal{B}_q(\mathcal{S}_\eta)$. Suppose that $z \notin X$. Then there exists $\xi < 2^{\mathfrak{c}}$ such that $z \in \mathcal{B}_{p_\xi}(S_\xi)$. Let $\{y_n : n \in \mathbb{N}\} \subseteq \bigcup S_\xi$ such that $z = p_\xi - \lim\ y_n$. It follows from Lemma 1.1 that $\{x_n : n \in \mathbb{N}\} \cap \{y_n : n \in \mathbb{N}\} \neq \emptyset$, and so $S_\xi = S_\eta$. This implies that $p_\xi \in C_\eta$. Since $\eta \neq \zeta$, we must have that p_ξ and q are *RK*-incomparable. On the other hand, as $z \in \mathcal{B}_q(\mathcal{S}_\eta) \cap \mathcal{B}_{p_\xi}(\mathcal{S}_\xi)$, by Lemma 1.3, q and p_ξ are *RK*-equivalent which is impossible. Therefore, $z \in X$ and it is an accumulation point of the set $\{x_n : n \in \mathbb{N}\}$. This shows that X is strongly pseudocompact. □

So far, we do not know whether or not there is a strongly pseudocompact space that it is not strongly p-pseudocompact for all $p \in \mathbb{N}^*$. This will be true when $\mathbb{N}^* = \bigcup_{\eta<\mathfrak{c}} C_\eta$ and the family $\{C_\eta : \eta < \mathfrak{c}\}$ satisfies the two conditions from the previous example. But we know that there are models of *ZFC* where $\mathbb{N}^* = \bigcup_{\eta<\mathfrak{c}} C_\eta$ does not hold (see for instance the book [13] that describes a model of *ZFC* in which there is a free ultrafilter on $\mathbb{N}^*$ which is *RK*-below any free ultrafilter on $\mathbb{N}$). By using the $2^{\mathfrak{c}}$-many weak *P*-points of $\mathbb{N}^*$ pairwise *RK*-incomparable constructed in [14], we can see that there is, in *ZFC*, a family of C_η's satisfying the conditions of Example 1.4.

2. Pseudocompact groups

I. Protasov [12] has shown that every infinite totally bounded group contains a nonclosed discrete subset. Inspired in the proof of his result we introduce the following notions.

Definition 2.1. We say that a space X is:

(1) *D-pseudocompact* if for every infinite countable family $\{U_n : n \in \mathbb{N}\}$ of nonempty pairwise disjoint open subsets, there is a discrete set $D \subseteq \bigcup_{n\in\mathbb{N}} U_n$ such that $cl(D) \setminus \bigcup_{n\in\mathbb{N}} cl(U_n) \neq \emptyset$ and $|U_n \cap D| \leq \omega$ for all $n \in \mathbb{N}$;

(2) *F-pseudocompact* if for every infinite countable family $\{U_n : n \in \mathbb{N}\}$ of nonempty pairwise disjoint open subsets, there is a non-closed discrete set $D \subseteq \bigcup_{n\in\mathbb{N}} U_n$ such that $|U_n \cap D| < \omega$ for all $n \in \mathbb{N}$.

These two properties are closely related to the strong pseudocompactness which is equivalent to: for every sequence $(U_n)_{n\in\mathbb{N}}$ of nonempty pairwise disjoint open sets, there exists a discrete set D such that $cl(D)\setminus\bigcup_{n\in\mathbb{N}} U_n \neq \emptyset$ and $|D\cap U_n| = 1$ for all $n\in\mathbb{N}$. Clearly, by definition,

strongly pseudocompact $\Rightarrow$ F-pseudocompact $\Rightarrow$ D-pseudocompact.

An example of a pseudocompact non-D-pseudocompact space is given in [9]. Next, we give an example of a D-pseudocompact, non-F-pseudocompact space X.

Example 2.2. There is a D-pseudocompact space X that is not F-pseudocompact.

Proof: Fix a free ultrafilter p on $\mathbb{N}$ and let $\{A_n : n\in\mathbb{N}\}$ be a partition of $\mathbb{N}$ in infinite sets. Define

$$B_n = T(p)\cap A_n^*, \quad \text{and} \quad P_n = \{p-\lim_{n\to\infty} x_n : \{x_n : n\in\mathbb{N}\}\in[\mathcal{D}(B_n)]^\omega\}$$

for each $n\in\mathbb{N}$. Clearly, for every $n\in\mathbb{N}$ and all $\mathcal{S}\in[\mathcal{D}(B_n)]^\omega$, we can choose a point $q(n,\mathcal{S})\in cl(\mathcal{S})\setminus P_n$. Let

$$Q_n = \{q(n,\mathcal{S}) : \mathcal{S}\in[\mathcal{D}(B_n)]^\omega\}, \quad \text{for each} \quad n\in\mathbb{N}.$$

Observe that $\big(\bigcup_{n\in\mathbb{N}} P_n\big)\cap T(p)=\emptyset$. The desired space will be:

$$X = \Big(\bigcup_{n\in\mathbb{N}}(B_n\cup Q_n)\Big)\cup\mathcal{B}_p\big(\{P_n : n\in\mathbb{N}\}\big).$$

To prove that X is D-pseudocompact, let $\{U_i : i\in\mathbb{N}\}$ be a pairwise disjoint family of nonempty open sets. Without loss of generality, assume that $\bigcup_{i\in\mathbb{N}} U_i\subseteq\bigcup_{n\in\mathbb{N}} A_n^*$. We will verify two cases:

Case I. There is $n\in\mathbb{N}$ such that $|\{i\in\mathbb{N} : U_i\cap A_n^*\neq\emptyset\}| = \omega$. In this case there is an infinite subset T of $\mathbb{N}$ such that $U_i\cap A_n^*\neq\emptyset$ iff $i\in T$. As B_n is dense in A_n^*, it is possible to pick some $x_i\in B_n\cap U_i$ for each $i\in T$. Since $\{x_i : i\in T\}\in[\mathcal{D}(B_n)]^\omega$, by definition, $q(n,\{x_i : i\in T\})\in Q_n\setminus\bigcup_{i\in\mathbb{N}} cl(U_i)$.

Case II. The first case does not hold, i.e. $|\{i\in\mathbb{N} : U_i\cap A_n^*\neq\emptyset\}| < \omega$ for each $n\in\mathbb{N}$. Then, $|\{n\in\mathbb{N} : \exists i\in\mathbb{N}(U_i\cap A_n^*\neq\emptyset)\}| = \omega$. In this case there is $\{k_n : n\in\mathbb{N}\}\subseteq\mathbb{N}$ and there is $\{i_n : n\in\mathbb{N}\}\subseteq\mathbb{N}$ such that $k_0<k_1<\ldots<k_n<\ldots$ and $A_{k_n}\cap U_{i_n}\neq\emptyset$ for each $n\in\mathbb{N}$. Choose $\mathcal{S}^n\in[\mathcal{D}(B_{k_n}\cap U_{i_n})]^\omega$ such that $x_n = p-\lim_{n\to\infty}\mathcal{S}^n\in U_{i_n}$ for each $n\in\mathbb{N}$, thus $x_n\in P_{k_n}\cap U_{i_n}$ for each $n\in\mathbb{N}$. Let $z = p-\lim_{n\to\infty} x_n$ and $D=\bigcup_{n\in\mathbb{N}}\mathcal{S}^n$. Then D is a countable discrete subset of $\bigcup_{n\in\mathbb{N}} U_i$ such that $z\in\mathcal{B}_p(\{P_n : n\in\mathbb{N}\})\cap cl(D)$. Since $\mathcal{B}_p\big(\{P_n : n\in\mathbb{N}\}\big)\cap\big(\bigcup_{n\in\mathbb{N}} A_n^*\big)=\emptyset$, $z\in cl(D)\setminus\bigcup_{i\in\mathbb{N}} cl(U_i)$. Therefore, X is D-pseudocompact.

Now, suppose that X is F-pseudocompact and let $U_n = B_n\cup Q_n$ for each $n\in\mathbb{N}$. Then there are a discrete set $D\subseteq\bigcup_{n\in\mathbb{N}} U_n$ and one point $z\in cl(D)\setminus\bigcup_{n\in\mathbb{N}} U_n$ such that $|D\cap U_n|<\omega$ for each $n\in\mathbb{N}$. Thus $z\in\mathcal{B}_p(\{P_n : n\in\mathbb{N}\})$. Let $\{y_i : i\in\mathbb{N}\}\subseteq\{P_n : n\in\mathbb{N}\}$ such that $z = p-\lim_{n\to\infty} y_i$. By Lemma 1.1,

$cl(\{y_i : i \in \mathbb{N}\}) \cap D \neq \emptyset$ or $\{y_i : i \in \mathbb{N}\} \cap cl(D) \neq \emptyset$. Since $cl(\{y_i \in \mathbb{N}\}) \setminus \mathcal{B}_p(\{P_n : n \in \mathbb{N}\}) = \{y_i : i \in \mathbb{N}\}$ and $cl(D) \setminus \mathcal{B}_p(\{P_n : n \in \mathbb{N}\}) = D$, we obtain that $\{y_i : i \in \mathbb{N}\} \cap D \neq \emptyset$ but this is a contradiction because of $\{y_i : i \in \mathbb{N}\} \cap X = \emptyset$. Thus X cannot be F-pseudocompact. $\square$

We could not answer the next question.

Question 2.3. *Is there an F-pseudocompact non-strongly pseudocompact space?*

We can see that in the proof of I. Protasov it is shown that every totally bounded topological group is D-pseudocompact. On the other side, it is well-known that every pseudocompact group is totally bounded. All these remarks make Question 0.2 be natural and interesting. By using the basic idea of the construction of Protasov, we can prove the following.

Theorem 2.4. *Every pseudocompact topological group is F-pseudocompact.*

PROOF: Let G be a pseudocompact group and let $(U_n)_{n\in\mathbb{N}}$ be a sequence of nonempty pairwise disjoint open sets of G. Since G is pseudocompact we can find an accumulation point x for the sequence $(U_n)_{n\in\mathbb{N}}$. Without loss of generality, assume that $x = e$. By the proof of Lemma from [12], we can find a discrete set $\{a_n : n \in \mathbb{N}\}$ such that:

(1) The set $\{a_n a_m^{-1} : n,\ m \in \mathbb{N}$ and $n < m\}$ is discrete,
(2) $\{a_n a_m^{-1} : m \in \mathbb{N}\} \subseteq U_n$ for each $n \in \mathbb{N}$, and
(3) $e \in cl(\{a_n a_m^{-1} : n,\ m \in \mathbb{N}$ and $n < m\})$.

Let $D = \{a_n a_m^{-1} : n,\ m \in \mathbb{N}\ \text{ and }\ n < m \leq 2n\}$. It is evident that $|D \cap U_n| < \omega$ for every $n \in \mathbb{N}$. Now, let U be an open set such that $e \in U$. Fix a symmetric open neighborhood V of e so that $V^2 \subseteq U$. Since G is totally bounded, there is a finite set F such that $\{a_n : n \in \mathbb{N}\} \subseteq VF$. If $i > |F|$, then, there are $n, m \in \mathbb{N}$ and $g \in F$ such that $i \leq n < m \leq 2i \leq 2n$ and $a_n, a_m \in Vg$. So $a_n a_m^{-1} \in D$ and $a_n a_m^{-1} \in (Vg)(Vg)^{-1}$. As $(Vg)(Vg)^{-1} = V^2 \subseteq U$, we obtain that $D \cap U \neq \emptyset$. Since $e \notin D$, D is not closed. Therefore, G is F-pseudocompact. $\square$

When we tried to construct an example of an F-pseudocompact, non-strongly pseudocompact group, we ran into the following obstacle.

Recall that an ultrafilter $p \in \mathbb{N}^*$ is a *Q-point* if for every infinite partition $\{P_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ in finite sets, there is $A \in p$ such that $|A \cap P_n| = 1$ for each $n \in \mathbb{N}$ (we remark that the existence of Q-points is independent from the axioms of ZFC).

Proposition 2.5. *Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of nonempty open sets of X. Suppose that $\{y_m : m \in \mathbb{N}\} \subseteq \bigcup_{n\in\mathbb{N}} U_n$ satisfies that $|U_n \cap \{y_m : m \in \mathbb{N}\}| < \omega$, for all $n \in \mathbb{N}$, and $(y_m)_{m\in\mathbb{N}}$ has a p-limit $x \in X$ for some Q-point p. Then there is a set $\{x_n : n \in \mathbb{N}\} \subseteq \{y_m : m \in \mathbb{N}\}$ such that $U_m \cap \{x_n : n \in \mathbb{N}\} = \{x_m\}$ for each $m \in \mathbb{N}$ and $x \in cl(\{x_n : n \in \mathbb{N}\})$.*

PROOF: Without loss of generality, we can suppose that the sets U_n's are pairwise disjoint. For each $n \in \mathbb{N}$, we let $P_n = \{m : y_m \in U_n\}$. It is evident that the family

$\{P_n : n \in \mathbb{N}\}$ is a partition of $\mathbb{N}$ in finite sets. Since p is a Q-point, there is a set $A \in p$ such that $|A \cap P_n| = 1$ for every $n \in \mathbb{N}$. Define $N(n)$ as the unique natural number in $A \cap P_n$ and let $x_n = y_{N(n)}$. Clearly, $U_m \cap \{x_n : n \in \mathbb{N}\} = \{x_m\}$ for each $m \in \mathbb{N}$. Let $x = p - \lim_{n\to\infty} y_m$. We are going to prove that $x \in cl(\{x_n : n \in \mathbb{N}\})$. Suppose that this assertion is false. Let V be an open set such that $x \in V$ and $V \cap \{x_n : n \in \mathbb{N}\} = \emptyset$. Since $x = p - \lim_{n\to\infty} y_m$,

$$\{m \in \mathbb{N} : y_m \in V\} \setminus A = \{m \in \mathbb{N} : y_m \in V\} \in p,$$

but this is a contradiction because of $A \in p$ and p is an ultrafilter. So, $x \in cl(\{x_n : n \in \mathbb{N}\})$. $\square$

This proposition was the main problem to construct a pseudocompact group that is not strongly pseudocompact. In the way towards a possible construction inside of the Cantor cube $\{0,1\}^{\mathfrak{c}}$ we showed the following.

Given a product $X = \prod_{i\in I} X_i$ and $J \subseteq I$, we define $X_J := \prod_{i\in J} X_i$ and let $\pi_J : X \to X_J$ be the projection map. For $A \subseteq X$ we set $\sup(A) := \{i \in I : \pi_i[A] \neq X_i\}$.

Theorem 2.6. *Let $\{X_i : i \in I\}$ be a family of compact metric spaces and let $X = \prod_{i\in I} X_i$. Then, every pseudocompact dense subspace of X is ultrapseudocompact.*

Proof: Assume that $Y \subseteq X$ is a pseudocompact dense subspace of X. Fix a sequence of open sets $(U_n)_{n\in\mathbb{N}}$ in Y and $p \in \omega^*$. For each $n \in \mathbb{N}$, choose an open set W_n of X such that $U_n = Y \cap W_n$. Let $S = \bigcup_{n\in\mathbb{N}} \sup(W_n)$. For each $n \in \mathbb{N}$ pick $u_n \in U_n$ and let $z_n = \pi_S(u_n)$. Since X_S is compact there exists $z = p - \lim_{n\to\infty} z_n$. Let $V = \{x \in X : \pi_S(x) = z\}$. Note that V is a G_δ set of X. Since Y is a pseudocompact dense subset of the compact space X, Y must be G_δ-dense in X. Hence, we obtain that $V \cap Y \neq \emptyset$. Pick any $y \in V \cap Y$.

Claim: $y \in L(p, (U_n)_{n\in\mathbb{N}})$.

Proof of Claim: Let B be an open set of X with $y \in B$. Then

$$\begin{aligned}\{n \in \mathbb{N} : (B \cap Y) \cap U_n \neq \emptyset\} &= \{n \in \mathbb{N} : (B \cap W_n) \cap Y \neq \emptyset\}\\ &= \{n \in \mathbb{N} : B \cap W_n \neq \emptyset\} \supseteq \{n \in \mathbb{N} : z_n \in \pi_S[W_n] \neq \emptyset\} \in p.\end{aligned}$$

So $y \in L(p, (U_n)_{n\in\mathbb{N}})$. $\square$

Thus, every dense pseudocompact subgroup of a Cantor cube $\{0,1\}^\alpha$, for any uncountable cardinal α, is actually ultrapseudocompact. Indeed, this assertion is also a direct consequence of the fact that every pseudocompact topological group is ultrapseudocompact (for a proof see [10]).

To finish the paper we state a particular case of Question 0.2.

Question 2.7. *Given an uncountable cardinal α, is every dense pseudocompact subgroup of $\{0,1\}^\alpha$ strongly pseudocompact?*

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