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Products of small modules

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Abstract. Module is said to be small if it is not a union of strictly increasing infinite countable chain of submodules. We show that the class of all small modules over self-injective purely infinite ring is closed under direct products whenever there exists no strongly inaccessible cardinal.

Keywords: small module; self-injectivity; von Neumann regular ring

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It is easy to verify that every finitely generated module M satisfies the natural compactness condition that the covariant functor $\mathrm{Hom}(M, -)$ commutes with all direct sums of modules. Nevertheless, there are known large classes of infinitely generated modules satisfying this condition, for instance, every uncountable union of a chain of finitely generated modules forms an infinitely generated example. Every module M satisfying the functorial compactness condition is called small in this paper.

It is well known that a finite direct sum of small modules is small in general and infinite direct sum of arbitrary nonzero modules is not small. Nevertheless, the case of product of small modules is rather more complicated. In the work [9] it is proved that over each ring which contains a right ideal isomorphic to a 2-generated free module (hence there exists a right ideal isomorphic to infinitely generated free module) every injective module is small. As the class of all injective modules is closed under all direct products, this observation leads to the natural question formulated explicitly in [2, Remark 3.2] whether there exists a ring R such that the class of all small right modules over R is closed under direct products.

The main objective of the present paper is to give a partial answer to this question, dependently on a model of set theory. We use for this purpose mainly tools and methods developed in the works [1], [5], [6], [8], [9], [11], [13], which study properties of classes of all small modules for some particular classes of rings.

Recall that a ring is called *right steady* if every small (right) module is finitely generated. Obviously, a ring over which every product of small modules is small is very far from being steady. Before we start searching rings over which small right modules are closed under direct products, we prove first that we may restrict our consideration to the case of simple self-injective regular rings (Proposition 2.3). Our main result (Theorem 3.4) proves necessary condition on set theory which holds true if over a right self-injective right purely infinite ring there exists non-small product of small modules. As this set theoretical condition contradicts to the hypothesis that there is no strongly inaccessible cardinal, which is consistent with ZFC, we can easily see that under the hypothesis of non-existence of a strongly inaccessible cardinal the class of all small modules is closed under products (Theorem 3.5).

1. Preliminaries

Throughout the paper, a *ring* R means an associative ring with unit, a *module* is a right R-module and an *ideal* means a two-sided ideal. We say that $R \subseteq Q$ is a ring extension if R is a subring of Q, note that Q has a natural structure of R-algebra. Moreover, $E(M)$ denotes an injective envelope of an arbitrary module M. We say that a module M is (less than, at most) κ-generated if the least cardinality of any set of generators is (less than, at most) κ and we write $\mathrm{gen}(M) = \kappa$ ($< \kappa$, $\leq \kappa$).

As we have remarked, a module M is said to be *small* whenever the natural $\mathbf{Z}$-monomorphism $\bigoplus_i \mathrm{Hom}(M, N_i) \to \mathrm{Hom}(M, \bigoplus_i N_i)$ is surjective for every system of modules N_i. We will usually deal with the following equivalent condition of smallness:

Lemma 1.1 ([8, Lemma 1.2])**.** *A module is small iff it is not a union of a strictly increasing infinite countable chain of submodules.*

We will use freely an easy consequence of Lemma 1.1 that any factor of small module is small. Observation that every union of an uncountable strictly increasing chain of finitely generated modules forms an infinitely generated small module naturally leads to the useful definition of a λ-*reducing* module for an infinite cardinal λ as a module M such that every at most λ-generated submodule is contained in some finitely generated submodule of M. Recall that the classes of all small as well as λ-reducing modules are closed under homomorphic images and finite (direct) sums [12, Proposition 1.3].

The following elementary observations about λ-reducing modules are used freely in the sequel.

Lemma 1.2. *Let $\lambda \leq \kappa$ be infinite cardinals and M an infinitely generated κ-reducing module. Then:*

(i) *M is small and λ-reducing,*
(ii) $\mathrm{gen}(M) > \kappa$,
(iii) *M contains a κ^+-generated κ-reducing submodule,*
(iv) *M contains an ω_1-generated ω-reducing submodule.*

For a module M we define singular submodule $Z(M_R) := \{m \in M \mid \mathrm{rann}_R(m) \trianglelefteq R\}$ where rann denotes an annihilator and submodule $U \trianglelefteq V$ means that U is an *essential* submodule of V, i.e. $U \cap W = 0$ implies $W = 0$ for a submodule W of V.

We say that a ring R is *right non-singular*, if $Z(R_R) = 0$, R is called (*von Neumann*) *regular* if for every $x \in R$ there exists $y \in R$ such that $x = xyx$, and R is *right self-injective*, provided it is injective as a right module over itself. We observe that simple rings form examples of non-singular rings. As a fact we state a deep statement about their maximal right rings of quotients. For a definition of maximal right ring of quotients and other properties of this notion we refer to [7].

Proposition 1.3 ([7, Proposition XII.2.1]). *The maximal right ring of quotients $Q_{\max}(R)$ of a right non-singular ring R is regular and right self-injective and it is injective as a right R-module.*

Description and examples of self-injective regular rings are given in [3, Chapters 9, 10].

Finally, recall several set-theoretical notions and facts which we will need in the final part of this paper. A *filter* on a set X is a nonempty family of nonempty subsets of X closed under finite intersections and supersets. An *ultrafilter* on X is a filter which is not properly contained in any other filter on X. We say that a filter $\mathcal{F}$ is λ-*complete*, if $\bigcap \mathcal{G} \in \mathcal{F}$ for every subsystem $\mathcal{G} \subseteq \mathcal{F}$ such that $|\mathcal{G}| < \lambda$ and $\mathcal{F}$ is *countably complete*, if it is ω_1-complete. A cardinal λ is said to be *measurable* if there exists a λ-complete nonprincipal ultrafilter on λ and it is *Ulam-measurable* if there exists a countably complete nonprincipal ultrafilter on λ. A regular cardinal κ is *strongly inaccessible* if $2^\lambda < \kappa$ for each $\lambda < \kappa$.

Theorem 1.4. *The following holds.*

(i) *Every Ulam-measurable cardinal is greater or equal to the first measurable cardinal.*
(ii) *Every measurable cardinal is strongly inaccessible.*
(iii) *It is consistent with ZFC that there is no strongly inaccessible cardinal.*

PROOF: (i) [10, Theorem 2.43.]. (ii) [10, Theorem 2.44.]. (iii) [4, Corollary IV.6.9]. □

2. Non-singular rings with (DS-P)

We say that a ring R satisfies the condition (DS-P) if every product of an arbitrary family of small R-modules is small. Let us start with an easy observation which states correspondence between small modules over a ring and over its extension.

Lemma 2.1. *Let $R \subseteq Q$ be a ring extension, M be a Q-module and Q_R be small as an R-module. Then M is a small Q-module if and only if it is a small R-module.*

PROOF: Assume that M is a small Q-module. Let $M = \bigcup_{i<\omega} M_i$ for a countable chain of R-submodules $M_0 \subseteq M_1 \subseteq \dots$. We put $N_i = \{m \in M \mid mQ \subseteq M_i\}$ for each $i < \omega$. Obviously, $N_0 \subseteq N_1 \subseteq \dots$ forms a chain of Q-submodules of M and $N_i \subseteq M_i$ for every $i < \omega$. Let $m \in M$. Since $(mQ)_R$ is a homomorphic image of the small R-module Q_R, there exists n such that $mQ \subseteq M_n$. Thus $M = \bigcup_{i<\omega} N_i$.

Now, by the hypothesis there exists $n < \omega$ such that $N_n = M$, hence $M_n = M$ and M is small R-module by Lemma 1.1.

The converse is clear, because every Q-module has also a natural structure of an R-module. $\square$

The next assertion describes closure properties of the class of all rings satisfying (DS-P).

Lemma 2.2. *Let R satisfy* (DS-P).

(i) *Every injective right R-module is small.*
(ii) *If R is a right non-singular ring with the maximal right ring of quotients Q, then Q satisfies* (DS-P).
(iii) *Every factor ring of R satisfies* (DS-P).

PROOF: (i) Let E_R be an injective R-module. Then there exists a cardinal κ and a surjective homomorphism $\pi : R^{(\kappa)} \to E$. Since the canonical embedding $R^{(\kappa)} \to R^{\kappa}$ is injective, π can be extended to an epimorphism $R^{\kappa} \to E$ by the injectivity of E. Since $(R_R)^{\kappa}$ is small by the hypothesis, the module E is a homomorphic image of a small module and therefore small as well.

(ii) By Proposition 1.3 Q_R is injective, so by (i) it is small as an R-module. Thus every product of small Q-modules is small as an R-module by the hypothesis and Lemma 2.1, hence it is a small Q-module.

(iii) Since every (small) module over any factor ring has a natural structure of a (small) R-module, the assertion is clear. $\square$

Now, we are able to show that searching of rings satisfying (DS-P) may be restricted to the case of simple self-injective regular rings.

Proposition 2.3. *If a ring R satisfies* (DS-P) *and I is a maximal two-sided ideal, then R/I is* (*right*) *non-singular and $Q_{\max}(R/I)$ is a non-artinian right self-injective simple ring satisfying* (DS-P).

PROOF: As R/I is simple, it is (right) non-singular, hence $Q_{\max}(R/I)$ is right self-injective by Proposition 1.3. By applying Lemma 2.2(ii), $Q_{\max}(R/I)$ satisfies (DS-P), hence it is non-artinian. Finally, let J be a nonzero ideal of $Q_{\max}(R/I)$. Since R is essential in $Q_{\max}(R/I)_R$, the intersection $R/I \cap J$ is a nonzero ideal of R. Thus $1 \in R/I \subseteq J$ and $J = Q_{\max}(R/I)$. $\square$

Corollary 2.4. *If R is simple ring satisfying* (DS-P), *then $Q_{\max}(R/I)$ satisfies* (DS-P) *as well.*

3. Self-injective rings

We say that a ring R is *right purely infinite* if there is a right ideal $K \leq R$ such that $K \simeq R^{(\omega)}$ as right R-modules, i.e., there is an exact sequence $0 \to R^{(\omega)} \to R$ in **Mod**-R.

It is easy to see that the endomorphism ring of an infinite-dimensional vector space forms an example of a right purely infinite regular ring. Recall that there exist right purely infinite simple regular self-injective rings [3, Example 10.11]. Moreover, note that every simple self-injective ring which is not directly finite is purely infinite by [3, Proposition 10.21]

First recall a key fact about the smallness of injective modules.

Lemma 3.1 ([9, Example 2.8])**.** *Every injective module over a right purely infinite ring is small.*

Lemma 3.2. *Let κ be an infinite cardinal, R be a right purely infinite self-injective ring and $(M_\alpha \mid \alpha < \kappa)$ be a system of R-modules.*

(i) *If M_α is ω_1-reducing for every $\alpha < \kappa$, then $\prod_{\alpha<\kappa} M_\alpha$ is ω_1-reducing as well.*
(ii) *If $\kappa = \omega$, then $\prod_{\alpha<\omega} M_\alpha / \bigoplus_{\alpha<\omega} M_\alpha$ is ω_1-reducing.*
(iii) *The product of any system of finitely generated modules is ω_1-reducing.*

Proof: Put $M = \prod_{\alpha<\kappa} M_\alpha$. For any product $\prod_{\alpha<\kappa} M_\alpha$ denote by $\nu_\alpha : M_\alpha \to \prod_{\alpha<\kappa} M_\alpha$ the natural embedding and $\pi_\alpha : \prod_{\alpha<\kappa} M_\alpha \to M_\alpha$ the natural projection.

Similarly we define ν_J and π_J for any subset J of κ.

(i) Note that $\prod_{\alpha<\kappa} R^{(n_\alpha)} \cong R^\kappa$ is injective for all finite n_α, hence ω_1-reducing by Lemma 3.1. Fix a countable set $D := \{m_n \mid n < \omega\} \subseteq M$. By hypothesis on M_α, for each $\alpha < \kappa$ there is some finitely generated submodule F_α of M_α such that $\{\pi_\alpha(m_n) \mid n < \omega\} \subseteq F_\alpha$ and there is some n_α such that we can write F_α as a factormodule of a finitely generated free R-module $R^{(n_\alpha)}$. Hence $D \subseteq \prod_{\alpha<\kappa} F_\alpha$ and the exact sequence $\prod_{\alpha<\kappa} R^{(n_\alpha)} \to \prod_{\alpha<\kappa} F_\alpha \to 0$ shows that the middle term is a factor-module of an ω_1-reducing R-module, hence it is itself ω_1-reducing. Then there exists a finitely generated submodule F of $\prod_{\alpha<\kappa} F_\alpha$ such that $D \subseteq F (\subseteq M)$.

(ii) Put $S = \bigoplus_{\alpha<\omega} M_\alpha$. Fix a countable set $D := \{m_n \mid n < \omega\} \subseteq M$ and for each $\alpha < \omega$ define (a finitely generated) R-module $G_\alpha = \sum_{j\leq\alpha} \pi_\alpha(m_j)R$. Observe that $D \subseteq S + \prod_{\alpha<\omega} G_\alpha$. By (i) $\prod_{\alpha<\omega} G_\alpha$ is ω_1-reducing, hence a factor-module $\prod_{\alpha<\omega} G_\alpha + S/S$ is also ω_1-reducing. Then there exists a finitely generated module $F \subseteq \prod_{\alpha<\omega} G_\alpha (\subseteq M)$ such that $m_n + S \in F + S/S$ for all $n < \omega$.

(iii) As finitely generated modules are ω_1-reducing, (iii) is a direct consequence of (ii). □

Let $\mathcal{I}$ be a non-empty system of subsets of a set X. We recall that $\mathcal{I}$ is said to be *an ideal* if it is closed under subsets (i.e. if $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$) and under finite unions, (i.e. if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$). $\mathcal{I}$ is *a prime ideal* if it is a proper ideal and for all subsets A, B of X, $A \cap B \in \mathcal{I}$ implies $A \in \mathcal{I}$ or $B \in \mathcal{I}$. If $Y \subseteq X$, the system $P(Y)$ of all subsets of Y forms an ideal on X which is called *principal*. We say that the set $\mathcal{I} \mid Y = \{Y \cap A \mid A \in \mathcal{I}\}$ is *a trace* of $\mathcal{I}$ on Y.

It is easy to see that the trace of an ideal is also an ideal and that $\mathcal{I}$ is a prime ideal if and only if for every $A \subseteq X$, $A \in \mathcal{I}$ or $X \backslash A \in \mathcal{I}$. Moreover, a principal prime ideal on X is of the form $P(X \setminus \{x\})$ for some $x \in X$. Note that there is a dual one-to-one correspondence between ultrafilters and prime ideals on X defined by $\mathcal{I} \mapsto \mathcal{P}(X)\backslash\mathcal{I}$ for an ideal $\mathcal{I}$.

Lemma 3.3. *Let R be a right purely infinite right self-injective ring and let $(M_\alpha \mid \alpha \in I)$ be a family of small modules. Let $M = \prod_{\alpha\in I} M_\alpha$ be the direct product and assume that M is not small, namely $M = \bigcup_{n<\omega} N_n$ for a countable strictly increasing chain of submodules $(N_n \mid n < \omega)$. Denote $\mathcal{A}_n = \{J \subseteq I \mid \prod_{\alpha\in J} M_\alpha \subseteq N_n\}$ and $\mathcal{A} = \bigcup_{n<\omega} \mathcal{A}_n$. Then the following holds:*

(i) *$\mathcal{A}_n$ is an ideal for each n,*
(ii) *$\mathcal{A}$ is closed under countable unions of sets,*
(iii) *there exists $n < \omega$ for which $\mathcal{A} = \mathcal{A}_n$,*
(iv) *there exists a subset $I_0 \subseteq I$ such that the trace of $\mathcal{A}$ on I_0 is a non-principal prime ideal.*

Proof: (i) Obviously $\emptyset \in \mathcal{A}_n$ and because M is not small, $I \notin \mathcal{A}_n$. The closure of $\mathcal{A}_n$ under subsets is obvious by the definition. The closure of $\mathcal{A}_n$ under finite unions follows from the decomposition $\prod_{\alpha\in J\cup K} M_\alpha = \prod_{\alpha\in J} M_\alpha \oplus \prod_{\alpha\in K\backslash J} M_\alpha \subseteq N_n$.

(ii) First we show that $\mathcal{A}$ is closed under countable unions of pairwise disjoint sets. Let $K_j \in \mathcal{A}$ be pairwisely disjoint subsets of I for all $j < \omega$. We show that there exists $k < \omega$ such that $K_j \in \mathcal{A}_k$ for each $j < \omega$. Assume by contradiction that for every $n < \omega$ there exists (possibly distinct) $i(n)$ such that $K_{i(n)} \notin \mathcal{A}_n$. Hence there is $f_n \in \prod_{\alpha\in K_{i(n)}} M_\alpha$ for which $\nu_{K_{i(n)}}(f_n) \notin N_n$. Since $\prod_{j<\omega} f_j R = \bigcup_{n<\omega}(\prod_j f_j R \cap N_n)$ is small by Lemma 3.2(iii) there is $k < \omega$ such that $\nu_{K_{i(k)}}(f_k) \in \prod_{j<\omega} f_j R \subseteq N_k$, a contradiction.

Put $P_j = \prod_{\alpha\in K_j} M_\alpha$ for $j < \omega$. We have proved that there is some $k < \omega$ such that $P_j \subseteq N_k$ and it follows that $\bigoplus_{j<\omega} P_j \subseteq N_k$. Let $P = \prod_{j<\omega} P_j = \prod_j \prod_{\alpha\in K_j} M_\alpha$. As $P/\bigoplus_{j<\omega} P_j$ is small by Lemma 3.2(i) there exists some $l \geq k$ such that $P = \bigcup_{j<\omega}(P \cap N_j) \subseteq N_l$.

Now let J_j, $j < \omega$ be any system of subsets of I and put $J_0 = K_0$ and $J_i = K_i \backslash \bigcup_{j<i} K_j$ for $i > 0$. So $\bigcup_{j<\omega} J_j = \bigcup_{j<\omega} K_j$ and by the preceding we get the result.

(iii) Assume that $\mathcal{A} \neq \mathcal{A}_n$ for every n. Then there exists a sequence $(J_j \in \mathcal{A}\backslash\mathcal{A}_j \mid j \in \omega)$. By (ii) $J := \bigcup_{j<\omega} J_j \in \mathcal{A}$ and there is some $n < \omega$ such that $J \in \mathcal{A}_n$. Since $J_j \subseteq J \in \mathcal{A}_n$ for each $j < \omega$, we obtain a contradiction.

(iv) We will show that there exists $I_0 \subseteq I$ such that for every $K \subseteq I_0, K \in \mathcal{A}$ or $I_0\backslash K \in \mathcal{A}$. Assume that such I_0 does not exist. Then we may construct a countably infinite sequence of disjoint sets $(K_i \mid i < \omega)$ where K_i are non-empty for $i > 0$ in the following way: Put $K_0 = \emptyset$ and $J_0 = I_0$. There exist disjoint sets $J_{i+1}, K_{i+1} \subset J_i$ such that $J_i = J_{i+1} \cup K_{i+1}$ where $J_{i+1}, K_{i+1} \notin \mathcal{A}$. Now, for each $n \geq 1$ there exists $g_n \in \prod_{\alpha\in K_n} M_\alpha$ such that $\nu_{K_n}(g_n) \notin N_n$ which contradicts to the fact that $\prod_{n<\omega} g_n R \subseteq N_m$ for some $m < \omega$ (cf. the proof of (ii)).

Finally, assume that the trace of $\mathcal{A}$ on I_0 is principal. Since it is a prime ideal, there exists $i \in I_0$ such that $\mathcal{A} \mid I_0 = P(I_0 \setminus \{i\})$. Thus $I_0 \setminus \{i\} \in \mathcal{A}_n$. Now $\prod_{j \in I_0 \setminus \{i\}} M_j \subseteq N_n$ for some n and $\{i\} \in \mathcal{A}$, so $I_0 \in \mathcal{A} \mid I_0$ a contradiction. □

Theorem 3.4. *Let R be a right self-injective right purely infinite ring. Then the following holds.*

(i) *A countable product of small R-modules is small.*

(ii) *If there exists a system $(M_\alpha \mid \alpha < \kappa)$ of small R-modules such that the product $\prod_{\alpha<\kappa} M_\alpha$ is not small, then there exists an uncountable cardinal $\lambda < \kappa$ and a countable complete ultrafilter on λ.*

Proof: (i) It follows immediately from Lemma 3.3(iii).

(ii) Suppose that $M = \prod_{\alpha \in I} M_\alpha$ is not a small module. Then by Lemma 3.3(iv) there exists $I_0 \subseteq I$ and a non-principal prime ideal $\mathcal{A}_0$ on I_0 which is closed under countable unions of sets by Lemma 3.3(ii). If we define $\mathcal{F} = \{I_0 \setminus A \mid A \in \mathcal{A}_0\}$ then $\mathcal{F}$ forms a countable complete non-principal ultrafilter on I_0. □

Before we prove our main result, which combines the last theorem and set-theoretical assertions, note that its hypothesis is consistent with ZFC by Theorem 1.4(iii).

Theorem 3.5. *Let R be a non-artinian right self-injective, right purely infinite ring. If we assume that there is no strongly inaccessible cardinal, then the class of all small R-modules is closed under direct products.*

Proof: If the product of an uncountable system of small modules is not small, then by Theorem 3.4(ii) there exists a countable complete ultrafilter on λ. Hence there exists a measurable cardinal $\mu \leq \lambda$ by Theorem 1.4(i), which is strongly inaccessible by Theorem 1.4(ii). □

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