Paratopological (topological) groups with certain networks

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Abstract. In this paper, we discuss certain networks on paratopological (or topological) groups and give positive or negative answers to the questions in [13]. We also prove that a non-locally compact, k-gentle paratopological group is metrizable if its remainder (in the Hausdorff compactification) is a Fréchet-Urysohn space with a point-countable cs*-network, which improves some theorems in [Liu C., *Metrizability of paratopological* (*semitopological*) *groups*, Topology Appl. **159** (2012), 1415–1420], [Liu C., Lin S., *Generalized metric spaces with algebraic structures*, Topology Appl. **157** (2010), 1966–1974].

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1. Introduction

Recall that a *topological group* G is a group G with a (Hausdorff) topology such that the product map of $G \times G$ into G is jointly continuous and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous. A *paratopological group* G is a group G with a topology such that the product map of $G \times G$ into G is jointly continuous. A *semitopological group* is a group with a topology such that the product map of $G \times G$ into G is separately continuous. A *quasitopological group* is a semitopological group and the inverse map is continuous.

Let X be a topological space and F is a subset of X, F is called a *sequential neighborhood* of x in X if every sequence converging to x is eventually in F. F is a *sequentially open* subset of X if F is a sequential neighborhood of x for each $x \in F$.

Definition 1.1. Let $\mathscr{P} = \bigcup_{x \in X} \mathscr{P}_x$ be a cover of a space X such that for each $x \in X$, (a) if $U, V \in \mathscr{P}_x$, then $W \subset U \cap V$ for some $W \in \mathscr{P}_x$; (b) the family $\mathscr{P}_x$ is a network of x in X, i.e., $x \in \bigcap \mathscr{P}_x$, and if $x \in U$ with U open in X, then $P \subset U$ for some $P \in \mathscr{P}_x$.

(1) The family $\mathscr{P}$ is called a *sn-network* (*sequential neighborhood network*) for X [12] if each element of $\mathcal{P}_x$ is sequential neighborhood of x for all $x \in X$. X is called *snf-countable* if X has a *sn*-network $\mathcal{P}$ such that each $\mathcal{P}_x$ is countable.

(2) The family $\mathscr{P}$ is called a *so-network* (*sequentially open network*) [12] for X if each element of $\mathcal{P}_x$ is a sequentially open neighborhood of X. X is called *sof-countable* if X has a *so*-network $\mathcal{P}$ such that each $\mathscr{P}_x$ is countable.

(3) Fix $x \in X$, $\mathcal{P}_x$ is said to be a *strong so-network* at x if $\mathcal{P}_x$ is a so-network at x, and for any sequential open set W with $x \in W$, there is a $P \in \mathcal{P}_x$ such that $x \in P \subset W$.

(4) The family $\mathscr{P}$ is called a *weak base* [1] for X if for every $A \subset X$, the set A is open in X whenever for each $x \in A$ there exists $P \in \mathcal{P}_x$ such that $P \subset A$. X is called *weakly first-countable* if for each $x \in X$, $\mathscr{P}_x$ is countable.

We can see that first-countable $\rightarrow$ sof-countable $\rightarrow$ snf-countable; first-countable $\rightarrow$ weakly first-countable $\rightarrow$ snf-countable. A sequential, snf-countable (sof-countable) space is weakly first-countable (first-countable).

In this paper, we consider the following questions.

Question 1.2 ([13, Question 4.1]). *Let G be a snf-countable semitopological group or quasitopological group. Is G sof-countable?*

Question 1.3 ([13, Question 4.3]). *Let G be a topological group. Is σG a topological group?*

Question 1.4 ([13, Question 4.5]). *Is every snf-countable topological group an $\aleph$-space?*

Question 1.5 ([13, Question 4.6]). *Does every snf-countable ω-narrow topological group have a countable sn-network?*

Question 1.6 ([13, Question 4.12]). *Let G be a paratopological group with a G_δ-diagonal. If G is a wM-space, is it metrizable?*

We shall give positive answers to Question 1.6 (when G is regular) and negative answers to Questions 1.2, 1.4, 1.5. Ordman and Smith-Thomas [18] gave an example that the sequential coreflection of a topological group is not a topological group, it implies the answer of Question 1.3 is negative, we present another example for Question 1.3 and give a sufficient and necessary condition for σG to be a topological group in terms of strong so-network.

By a remainder of a space X we mean the subspace $bX \setminus X$ of a Hausdorff compactification bX of X. Arhangel'skii [2] proved that if the remainder of a Hausdorff compactification of a non-locally compact topological group G has a point-countable base, then G and bG are separable and metrizable. It is natural to ask if Arhangel'skii's result is still valid for a paratopological group. The author [15] proved that Arhangel'skii theorem is valid for a k-gentle paratopological group. We could improve the above result by replacing "point-countable base" with "Fréchet-Urysohn space with a point-countable cs*-network".

All spaces are Hausdorff unless stated otherwise. The notations $\mathbb{N}, \mathbb{Q}, \mathbb{R}$ denote natural numbers, rational numbers and real numbers respectively. The letter e denotes the neutral element of a group. $F(X)$ is a free group on X. Readers may refer to [2], [7], [10] for notations and terminology not explicitly given here.

2. Main results

Let X be a topological space, a function $d : X \times X \to \mathbb{R}^+$ is a *symmetric* on the set X if for $x, y \in X$

(1) $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$.

A space X is said to be *symmetrizable* if there is a symmetric d on X satisfying the following condition: $U \subset X$ is open if and only if for each $x \in U$, there exists $\epsilon > 0$ with $B(x, \epsilon) \subset U$. Here $B(x, \epsilon) = \{y \in x : d(x, y) \in \epsilon\}$.

Example 2.1. There is a separable, snf-countable quasitopological group that is not sof-countable.

Proof: Let $G = \mathbb{R}^2$ with usual addition "+", then $(G, +)$ is a group. Define $d : G \times G \to \mathbb{R}^+ \cup \{0\}$ as follows:

$$d((x,y),(x',y')) = \begin{cases} |x-x'|, & x \neq x', y = y'; \\ |y-y'|, & x = x', y \neq y'; \\ 0, & x = x', y = y'; \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to check that $d(x, y)$ is a symmetric and $(G, +)$ is a separable, quasitopological group. G is weakly first-countable, in fact, for each $x \in G$, let $\mathcal{P}_x = \{B(x, 1/n) : n \in \mathbb{N}\}$, where $B(x, 1/n) = \{y \in G : d(x, y) < 1/n\}$.

It is easy to see that (0,0) is a cluster point of $\{(r_1, r_2) : r_1, r_2 \in \mathbb{Q}^+\}$, where $\mathbb{Q}^+ = \{r \in \mathbb{Q} : r > 0\}$. If G is first-countable, then there is a sequence $\{s_n : n \in \mathbb{N}\} \subset \{(r_1, r_2) : r_1, r_2 \in \mathbb{Q}^+\}$ such that $s_n \to (0, 0)$. $d(s_n, (0, 0)) \to 0$ by [10, Lemma 9.3]. This is a contradiction since $d(s_n, (0, 0)) = 1$. Hence G is not first-countable. Therefore, G is not sof-countable since a sof-countable sequential space is first-countable. □

The proof of the following proposition is based on the idea in [13].

Proposition 2.2. *Let G be a paratopological group satisfying the condition* (w)*: for any two sn-networks $\{U_\alpha(e) : \alpha \in \Gamma\}$, $\{V_\beta(e) : \beta \in \Gamma\}$ at e and for any $\alpha \in \Gamma$, there exists $\beta \in \Gamma$ such that $V_\beta(e) \subset U_\alpha(e)$. Then there is a so-network $\{W_\alpha(e) : \alpha \in \Gamma\}$ at e and for each $\alpha \in \Gamma$, there exists $\beta \in \Gamma$ such that $W_\beta(e)W_\beta(e) \subset W_\alpha(e)$.*

Proof: Since G is a paratopological group, $\{U_\alpha(e)U_\alpha(e) : \alpha \in \Gamma\}$ is still a sn-network at e. Let $W_\alpha(e) = \{x \in U_\alpha(e) : xU_\beta(e) \subset U_\alpha(e)$ for some $\beta \in \Gamma\} \subset U_\alpha(e)$. So $e \in W_\alpha(e)$ for each α, then $\{W_\alpha(e) : \alpha \in \Gamma\}$ is a network at e and satisfies the condition (a) in Definition 1.1, in fact, for any $W_\alpha(e), W_\beta(e)$, let $U_\gamma(e) \subset U_\alpha(e) \cap U_\beta(e)$, $W_\gamma(e) = \{x \in U_\gamma(e) : xU_\delta(e) \subset U_\gamma(e)\}$, then $W_\gamma(e) \subset W_\alpha(e) \cap W_\beta(e)$. We prove that each $W_\alpha(e)$ is sequentially open. For $y \in W_\alpha(e)$ and $\{y_n\}$ is a sequence converging to y, $yU_\beta(e) \subset U_\alpha(e)$. By the condition (w), we choose $\gamma \in \Gamma$ such that $U_\gamma(e)U_\gamma(e) \subset U_\beta(e)$. $(yU_\gamma(e))U_\gamma(e) \subset yU_\beta(e) \subset U_\alpha(e)$, which implies $yU_\gamma(e) \subset W_\alpha(e)$. Since $yU_\gamma(e)$ is a sequential neighborhood of y, then $\{y_n\}$ is eventually in $yU_\gamma(e)$, hence $\{y_n\}$ is eventually in $W_\alpha(e)$ and $W_\alpha(e)$ is sequentially open. For $\alpha \in \Gamma$, choose $\beta \in \Gamma$ so that $U_\beta(e)U_\beta(e) \subset U_\alpha(e)$. For $y, z \in W_\beta(e) = \{x \in U_\beta(e) : xU_\gamma(e) \subset U_\beta(e)$ for some $\gamma \in \Gamma\} \subset U_\beta(e)$ we have $yU_\gamma(e) \subset U_\beta(e)$, $zU_\gamma(e) \subset U_\beta(e)$, then $yzU_\gamma(e) \subset yU_\gamma(e)zU_\gamma(e) \subset U_\beta(e)U_\beta(e) \subset U_\alpha(e)$, that implies $yz \in W_\alpha(e)$, and hence $W_\beta(e)W_\beta(e) \subset W_\alpha(e)$. □

Lemma 2.3. *Let $\{U_n : n \in \mathbb{N}\}$ be a decreasing countable network at x and W be sequential neighborhood of x, then there exists $n_0 \in \mathbb{N}$ such that $U_{n_0} \subset W$.*

Proof: Suppose not, $U_n \setminus W \neq \emptyset$ and pick $x_n \in U_n \setminus W$. Then $x_n \to x$ and $\{x_n\} \cap W = \emptyset$. This is a contradiction since W is a sequential neighborhood of x. □

Note that if G is snf-countable, we may assume G has a decreasing countable sn-network. By Lemma 2.3, a snf-countable paratopological group satisfies the condition (w).

Corollary 2.4 ([13, Theorem 3.4])**.** *Every snf-countable paratopological group G is sof-countable.*

Since a weakly first-countable space is a sequential snf-countable space and a sequential sof-countable space is first-countable, we have the following.

Corollary 2.5. *Let G be a weakly first-countable paratopological group. Then G is first-countable.*

Definition 2.6. Let (X, τ) be a space. A *sequential closure topology* σ_τ [8] on X is defined as follows: $O \in \sigma_\tau$ if and only if O is a sequentially open subset in (X, τ). The topological space (X, σ_τ) is denoted by σX.

Obviously, σX is a sequential space for any space X. If G is a topological group, it is easy to see that σG is a quasitopological group.

Theorem 2.7. *Let G be a paratopological group. Then σG is a paratopological group if and only if G has a strong so-network $\mathcal{P}_e$ at e satisfying the condition $(*)$: for each $P_1 \in \mathcal{P}_e$, there is a $P_2 \in \mathcal{P}_e$ such that $P_2P_2 \subset P_1$.*

Proof: Necessity: Let $\{V_\alpha(e) : \alpha \in \Gamma\}$ be the local base at e in σG, and let W be a sequentially open neighborhood of G with $e \in W$, then W is open in σG, there is a $V_\beta(e) \in \{V_\alpha(e) : \alpha \in \Gamma\}$ such that $V_\beta(e) \subset W$. Since $\{V_\alpha(e) : \alpha \in \Gamma\}$ is a so-network at e in G, then $\{V_\alpha(e) : \alpha \in \Gamma\}$ is a strong so-network at e in G. Since σG is a paratopological group and $\{V_\alpha(e) : \alpha \in \Gamma\}$ is the local base at e, it is easy to see that the condition $(*)$ is satisfied.

Sufficiency: Suppose G has a strong so-network $\{V_\alpha(e) : \alpha \in \Gamma\}$ at e such that for each $V_\alpha(e)$, $V_\beta(e)V_\beta(e) \subset V_\alpha(e)$. Fix $a, b \in G$, and let U be an open neighborhood (in σG) of ab. Since $(ab)^{-1}U$ is a sequentially open neighborhood of e in G, there is a $V \in \{V_\alpha(e) : \alpha \in \Gamma\}$ such that $V \subset (ab)^{-1}U$, then $abV \subset U$. Let $W, W' \in \{V_\alpha(e) : \alpha \in \Gamma\}$ such that $WW \subset V$, $W' \subset W$ and $W'b \subset bW$ (note that $e \in bWb^{-1}$ is sequentially open in G). Then aW', bW' are open neighborhoods of a, b in σG respectively, $aW'bW' \subset abWW' \subset abWW \subset abV \subset U$. □

Corollary 2.8. *Let G be a topological group. Then σG is a topological group if and only if G has a strong so-network $\mathcal{P}_e$ at e satisfying the condition $(*)$: for each $P_1 \in \mathcal{P}_e$, there is a $P_2 \in \mathcal{P}_e$ such that $P_2P_2 \subset P_1$.*

Corollary 2.9 ([13, Theorem 4.4])**.** *Let G be a snf-countable topological group. Then σG is a topological group.*

PROOF: By Proposition 2.2, G has a countable so-network $\mathcal{P}_e$ at e satisfying the condition $(*)$: for each $P_1 \in \mathcal{P}_e$, there is a $P_2 \in \mathcal{P}_e$ such that $P_2P_2 \subset P_1$. We also can see that $\mathcal{P}_e$ is a strong so-network at e by Lemma 2.3. Then σG is a topological group by Corollary 2.8. □

Proposition 2.10. *Let $F(X)$ be a free topological group on a sequential space X. Then $\sigma F(X)$ is a topological group if and only if $F(X)$ is a sequential space.*

PROOF: Sufficiency is obvious.

Necessity: Suppose $F(X)$ is not sequential, then the topology on $\sigma F(X)$ is strictly finer than the topology on $F(X)$ and the topology on X as a subspace of $\sigma F(X)$ is compatible with the original topology on X (note that X is sequential). However, the topology on $F(X)$ is the finest group topology on $F(X)$ that generates on X its original topology [6, Corollary 7.1.8]. Hence $\sigma F(X)$ is not a topological group. □

Remark: Usually, the sequential coreflection of a topological group need not to be a topological group. Let S_{ω_1} be the space obtained by identifying all limit points of the topological sum of ω_1 convergent sequences. Then S_{ω_1} is Fréchet-Urysohn. Let $F(S_{\omega_1})$ be the free topological group on S_{ω_1}, by [6, Theorem 7.1.13 (b)], $F(S_{\omega_1})$ contains a closed copy of $S_{\omega_1}\times S_{\omega_1}$. Since $S_{\omega_1}\times S_{\omega_1}$ is not a sequential space [9], then $F(S_{\omega_1})$ is not a sequential space, hence its sequential coreflection $\sigma F(S_{\omega_1})$ is not a topological group by Proposition 2.10.

A subset B of a paratopological group G is called ω-narrow in G if, for each neighborhood U of the neutral element of G, there is a countable subset F of G such that $B \subset FU \cap UF$.

Let $X = \Pi_{i\in I}X_i$ be the product of spaces X_i, with $i \in I$. A standard base of the *ω-box topology* on X consists of the ω-cubes $B = \Pi_{i\in I}B_i$, where each B_i is open in X_i (and, clearly, the number of indices $i \in I$ with $B_i \neq X_i$ is countable).

Example 2.11. There is a Lindelöf (hence, ω-narrow), snf-countable, zero-dimensional topological group G such that G^n is topologically isomorphic to G, $w(G) = c$ and G does not have a σ-locally finite network.

PROOF: Let $D = \{0, 1\}$ be the discrete topological group with operation "addition". In the product group ΠD^c, consider the subgroup $G = \sigma\Pi D^c = \{x \in \Pi D^c : |supp\,(x)| < \omega\}$, where $supp\,(x)$ denotes the set $\{\alpha \in \omega_1 : x(\alpha) \neq 0\}$. Endow G with ω-box topology $\mathcal{T}$. Then $(G, +, \mathcal{T})$ is a zero-dimensional topological group. It is proved in [6, Example 4.4.11] that G is a Lindelöf topological P-group, G^n is topologically isomorphic to G and $w(G) = c$.

Claim. Every countable subset of G does not have a cluster point.

Suppose not, then there is a countable subset A of G such that $a \in \overline{A \setminus \{a\}}$ for some $a \in G$. Put $J = \cup\{supp\,(x) : x \in A \setminus \{a\}\}$, then J is a countable subset of ω_1. Let $V = \Pi Y_i \cap G$, where $p(Y_i) = D$ if $i \notin J \cup supp\,(a)$; $p(Y_i) = \{1\}$ if $i \in supp\,(a)$; $p(Y_i) = \{0\}$ if $i \in J \setminus supp\,(a)$. V is an open neighborhood of a since $V = \Pi Y_i$ is open in ΠD^c that is endowed with ω-box topology. It is easy to see that $V \cap A = \emptyset$. This is a contradiction.

1) G is snf-countable.

By Claim, there is no non-trivial convergent sequence in G, $\{x\}$ is a sequential neighborhood of $x \in G$, hence G is snf-countable.

2) G does not have σ-locally finite network.

Suppose that G has a σ-locally finite network. Since G is a Lindelöf space, G is a cosmic space (i.e. G has a countable network). Hence G is hereditarily separable. This is a contradiction since $|G| > \omega$ and every countable subset of G is discrete by Claim. □

Remark: The topological group G in Example 2.11 is neither an $\aleph$-space nor a cosmic space (i.e. a space with countable network). Hence the answers for Questions 1.4, 1.5 are negative. However, the group G in Example 2.11 is not separable. Note that a separable topological group is ω-narrow [6, Corollary 3.4.8], it is natural to ask if there is a Lindelöf, separable, snf-countable topological group that is not a σ-space.

In what follows, we construct a Lindelöf, separable, snf-countable topological group that is not a σ-space.

Simon [19] proved the following:

Theorem 2.12. *There is a countable dense subset A of ΠD^c such that $|\overline{H}| = 2^c$ for any infinite subset $H \subset A$.*

The following proposition comes from a discussion with Arhangel'skii.

Proposition 2.13. *There is a Lindelöf, separable space Y satisfying the following:*

(1) *Y is not a σ-space (i.e. a space having no σ-locally finite network);*
(2) *every compact subset of Y is finite;*
(3) *Y^n is Lindelöf for each $n \in \mathbb{N}$.*

Proof: Let $A(\Pi D^c) = X \cup X_1$ be the Alexandroff duplicate of $X = \Pi D^c$, where X_1 is a copy of X, and let G be the Lindelöf topological group of Example 2.11. Since G is zero-dimensional and $w(G) = c$, then G is homeomorphic to a subspace of $X = \Pi D^c$ by [7, Theorem 6.2.16]. By Theorem 2.12, we can choose a countable dense subset A of X such that $|\overline{H}| = 2^c$ for any infinite subset $H \subset A$. Let $A_1 \subset X_1$ be a copy of A, and let $Y = G \cup A_1$. Note that G is a Lindelöf space that is not a σ-space and A_1 is countable, then Y is a Lindelöf, separable space that is not a σ-space. We prove each compact subset of Y is finite. Let K be a compact subset of Y, then $K \cap G$ is compact in Y since G is a closed subset of Y. By Claim in Example 2.11, $K \cap G$ is finite. If $K \cap A_1$ is infinite, by Theorem 2.12, $|\overline{K \cap A_1}| = 2^c$. $\overline{K \cap A_1} \subset K \subset Y$, then $\overline{K \cap A_1} \cap G \subset K \cap G$ is an infinite compact subset of G. This is a contradiction. So $K \cap A_1$ is finite, therefore K is finite.

Note that G^n is Lindelöf for each n and A_1 is countable, it is easy to see that Y^n is a union of countably many Lindelöf subspaces, hence Y^n is Lindelöf for each n. □

Theorem 2.14. *There is a Lindelöf, separable, snf-countable topological group that is not a σ-space.*

Proof: Let Y be the space in Proposition 2.13, and let $F(Y)$ be the free topological group on Y. Since Y is separable and Y^n is Lindelöf for each n, $F(Y)$ is also Lindelöf and separable by [6, Corollary 7.1.18, Theorem 7.1.13]. $F(Y)$ is not a σ-space since Y is not a σ-space. We prove that each compact subset of $F(Y)$ is finite. Let K be a compact subset of $F(Y)$. Since Y is Dieudonné-complete, by [5, Corollary 1.8], there exist a compact $Z \subset Y$ and $n \in \mathbb{N}$ such that K is a continuous image of a subspace in Z^n. Z is finite since each compact subset of Y is finite, Z^n is also finite, hence K is finite and $F(Y)$ is snf-countable. □

A space X is a q-space if X has a g-function satisfying: for $x \in X$, if $x_n \in g(n, x)$, then $\{x_n\}$ has a cluster point in X. A space X is a wM-space if there exists a sequence $(\mathcal{U}_n)$ of open covers of X such that if $x_n \in st^2(x, \mathcal{U}_n)$ for each $n \in \mathbb{N}$, then the set $\{x_n : n \in \mathbb{N}\}$ has a cluster point in X.

Theorem 2.15. *Let G be a regular paratopological group in which each singleton is a G_δ-set. If G is a wM-space, then G is metrizable.*

Proof: Since G is a wM-space, then G is a q-space. Moreover, G is first-countable since a regular q-space in which each singleton is a G_δ-set is first-countable [17], hence G has a regular G_δ-diagonal [14]. Therefore, G is metrizable since a wM-space with a regular G_δ-diagonal is metrizable [20]. □

Remark: Theorem 2.15 gives a positive answer to Question 1.6 when G is regular and T_1. But the author doesn't know if we can replace "paratopological group" with "semitopological group" in Theorem 2.15.

Next, we discuss remainder of a paratopological group in its Hausdorff compactification. Arhangel'skii [2] proved the following.

Theorem 2.16 ([2]). *Let G be a non-locally compact topological group and the remainder $Y = bG \setminus G$ have a point-countable base. Then G and bG are separable and metrizable.*

Let $f : X \to Y$ be a map. The map f is called *k-gentle* [4] if for each compact subset F of X the image $f(F)$ is also compact. A paratopological group is called k-gentle if the inverse map $x \to x^{-1}$ is k-gentle. Liu and Lin [16] improved the result by replacing "point-countable base" with "pseudo open s-image of a space with a point-countable base". On the other hand, the author also proved the following theorem on k-gentle, paratopological group.

Theorem 2.17 ([15]). *Let G be a non-locally compact, k-gentle paratopological group and the remainder $Y = bG \setminus G$ have a point-countable base. Then G and bG are separable and metrizable.*

Next, we are able to improve both theorems in [15], [16].

A family $\mathcal{S}$ of subsets of a space X is said to be a *κ-sensor* [3] at $x \in X$ if, for each open neighborhood $O(x)$ of x and each open set U such that $x \in \overline{U}$, there exists $P \in \mathcal{S}$ satisfying the following conditions: $P \subset O(x)$ and $x \in \overline{U \cap P}$.

If there exists a countable κ-sensor at x, the space is said to be *countably κ-sensitive* at x [3].

The family $\mathscr{P}$ is called a *cs*-network* for X [12] if, whenever $x \in X$ and a sequence S converges to $x \in U$ with U open, there exists $P \in \mathcal{P}$ such that $x \in P \subset U$ and P contains a subsequence of S.

Tanaka [21] proved that a space X is a pseudo open s-image of a space with a point-countable base if and only if X is a Fréchet-Urysohn space with a point-countable cs*-network.

Lemma 2.18. *Let X be a Fréchet-Urysohn space with a point-countable cs*-network. Then X is of countably κ-sensitive at each $x \in X$.*

Proof: X is a Fréchet-Urysohn space with a point-countable cs*-network $\mathcal{P}$. Fix $x \in X$, an open neighborhood $O(x)$ of x and an open set U with $x \in \overline{U}$. Let $\mathcal{P}_x = \{P \in \mathcal{P} : x \in P\}$, $|\mathcal{P}_x| \leq \omega$. Since X is Fréchet-Urysohn, there is a sequence $S \subset U$ converging to x. $\mathcal{P}_x$ is a cs*-network at x, then there is $P \in \mathcal{P}_x$ such that $x \in P \subset O(x)$ and P contains a subsequence S_1 of S. $x \in \overline{S_1} \subset \overline{P \cap U}$. Hence $\mathcal{P}_x$ is a countable κ-sensor at x. $\square$

Theorem 2.19. *Let G be a non-locally compact, k-gentle paratopological group. If the remainder $Y = bG \setminus G$ is a pseudo open s-image of a space with a point-countable base, then G and bG are separable and metrizable.*

Proof: By [4, Theorem 4.4], Y is either Lindelöf or pseudocompact. If Y is Lindelöf, then G is a topological group [4, Corollary 4.5]. Hence G and bG are separable and metrizable by [16, Theorem 5.2].

If Y is pseudocompact, by Lemma 2.18, Y is of countably κ-sensitive at each $x \in Y$. Then Y is first-countable by [3, Theorem 1.5]. Y has a point-countable base since a first-countable, quotient s-image of a space with a point-countable base has a point-countable base [11]. Therefore, G and bG are separable and metrizable by Theorem 2.17. $\square$

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