Some evolution equations under the List's flow and their applications

Bingqing Ma

Abstract. In this paper, we consider some evolution equations of generalized Ricci curvature and generalized scalar curvature under the List's flow. As applications, we obtain L^2-estimates for generalized scalar curvature and the first variational formulae for non-negative eigenvalues with respect to the Laplacian.

Keywords: List's flow; eigenvalue; scalar curvature

Classification: Primary 53C44; Secondary 53C21

1. Introduction

Let $(M^n, g(t))$, $(t \in [0,T))$ be a solution to the following List's flow which was introduced by B. List [5]:

$$\begin{cases} \frac{\partial}{\partial t} g = -2\mathrm{Ric} + 2\alpha\, d\varphi \otimes d\varphi, \\ \varphi_t = \Delta\varphi, \end{cases} \tag{1.1}$$

where $\alpha > 0$ is a constant and φ is a smooth function on M^n. Δ denotes the Laplace-Beltrami operator given by $g(t)$. Throughout this paper, we always assume that M^n is compact without boundary. The motivation to study the system (1.1) stems from its connection to general relativity. In [7], Lott and Sesum obtain some long time behavior of the List's flow when $\alpha = 2$. Li [4] studied the eigenvalues and entropies under the harmonic-Ricci flow, he derived some monotonicity formulas for eigenvalues of Laplacian. Wang [10] proved some differential Harnack inequalities about the coupled Ricci flow which is closely related to List's flow. In [2], Fang also obtained some differential Harnack inequalities associated with the List's flow which generalize the results of Cao and Hamilton in [1]. For the study of List's flow and some developments, see [6], [5], [8], [3] and the references therein.

Let $h_{ij} = R_{ij} - \alpha\, \varphi_i\varphi_j$ be a symmetric two-tensor which we call the *generalized Ricci curvature*. Then (1.1) becomes

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -2h_{ij}, \\ \varphi_t = \Delta\varphi. \end{cases} \tag{1.2}$$

The research of the author is supported by NSFC (No. 11171368, 11371018).

In this paper, we consider the following evolution equation:

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -2\tilde{h}_{ij}, \\ \varphi_t = \Delta\varphi, \end{cases} \tag{1.3}$$

where $\tilde{h}_{ij} = h_{ij} - (ar+b)g_{ij}$. Here a, b are two constants,

$$r = \frac{\int_M (\mathrm{tr_g} h)\, dV_g}{\int_M dV_g}$$

is the average of the trace of two-tensor h_{ij} and $\mathrm{tr_g} h = g^{ij}h_{ij} = R - \alpha|\nabla\varphi|^2$ which is called the *generalized scalar curvature*. Note that if $a = \frac{1}{n}$ and $b = 0$, then the volume of M^n is a constant for all t. In this case, the equation (1.3) with $a = \frac{1}{n}$ and $b = 0$ is called the normalized List's flow.

The rest of this paper is organized as follows: In Section 2, we first derive some evolution equations of generalized Ricci curvature and generalized scalar curvature under the List's flow. As applications, we obtain L^2-estimates for generalized scalar curvature and the first variational formulae for non-negative eigenvalues with respect to the Laplacian.

2. Some evolution equations

We first recall the following lemma (see Lemma 1.4 in [5]):

Lemma 2.1 ([5])**.** *Let $g(t)$ be a solution to the evolution equation*

$$\frac{\partial}{\partial t} g_{ij} = v_{ij}$$

on M^n. Then the following evolution equations hold:

(1) $\frac{\partial}{\partial t} g^{ij} = -v^{ij}$, *where* $v^{ij} = g^{ik}g^{jl}v_{kl}$;
(2) $\frac{\partial}{\partial t} R_{ij} = -\frac{1}{2}\Delta v_{ij} - \frac{1}{2}(\mathrm{tr_g} v)_{ij} + \frac{1}{2}g^{kl}(v_{ik,jl} + v_{jk,il})$, *where* $\mathrm{tr_g} v = g^{ij}v_{ij}$;
(3) $\frac{\partial}{\partial t} R = -\Delta(\mathrm{tr_g} v) + g^{ij}g^{kl}(v_{ik,lj} - R_{ik}v_{jl})$;
(4) $\frac{\partial}{\partial t} dV_g = \frac{1}{2}(\mathrm{tr_g} v)\, dV_g$,

where $,i$ denotes the covariant derivative in the direction of i.

Using Lemma 2.1 above, we can prove the following results:

Lemma 2.2. *Let $g(t)$ be a solution to the evolution equation* (1.3) *on M^n. Then the following evolution equations hold:*

(1) $(\frac{\partial}{\partial t} - \Delta)h_{ij} = -g^{kl}R_{ik}h_{jl} - g^{kl}R_{jk}h_{il} - 2R_{lijk}h^{lk} + 2\alpha(\Delta\varphi)\varphi_{ij}$;
(2) $(\frac{\partial}{\partial t} - \Delta)(\mathrm{tr_g} h) = 2|h_{ij}|^2 + 2\alpha(\Delta\varphi)^2 - 2(ar+b)\mathrm{tr_g} h$.

Proof: Using Lemma 2.1, we have

$$\frac{\partial}{\partial t} R_{ij} = -\frac{1}{2}\Delta v_{ij} - \frac{1}{2}(\mathrm{tr_g} v)_{ij} + \frac{1}{2}g^{kl}(v_{ik,jl} + v_{jk,il}). \tag{2.1}$$

Taking $v_{ij} = -2\tilde{h}_{ij} = -2[h_{ij} - (ar+b)g_{ij}]$, we have

$$\begin{aligned}-\frac{1}{2}\Delta v_{ij} &= \Delta h_{ij} = \Delta R_{ij} - \alpha\,\Delta(\varphi_i\varphi_j)\\ &= \Delta R_{ij} - \alpha\,(\Delta\varphi_i)\varphi_j - \alpha\,\varphi_i(\Delta\varphi_j) - 2\alpha\varphi_{ik}\varphi_{jk}.\end{aligned}$$

Here and hereafter we use moving frames in all calculations. The second term of the right hand side in (2.1) gives

$$-\frac{1}{2}(\mathrm{tr_g} v)_{ij} = [R - \alpha|\nabla\varphi|^2 - n(ar+b)]_{ij} = R_{,ij} - 2\alpha\varphi_{ki}\varphi_{kj} - 2\alpha\varphi_k\varphi_{kji}.$$

Note that

$$\begin{aligned}\frac{1}{2}v_{ik,jk} &= -h_{ik,jk}\\ &= -R_{ik,jk} + \alpha(\varphi_i\varphi_k)_{,jk}\\ &= -(R_{ik,kj} + R_{lk}R_{lijk} + R_{il}R_{lkjk})\\ &\quad + \alpha[\varphi_{ijk}\varphi_k + (\Delta\varphi)\varphi_{ij} + \varphi_{ik}\varphi_{jk} + \varphi_i\varphi_{jkk}]\\ &= -\frac{1}{2}R_{,ij} - R_{lk}R_{lijk} - R_{ik}R_{jk}\\ &\quad + \alpha[\varphi_{ikj}\varphi_k + \varphi_k\varphi_l R_{lijk} + (\Delta\varphi)\varphi_{ij} + \varphi_{ik}\varphi_{jk} + \varphi_i\varphi_{jkk}].\end{aligned}$$

Hence, we obtain from (2.1) that

$$\begin{aligned}\frac{\partial}{\partial t}R_{ij} &= \Delta R_{ij} - 2R_{ik}R_{jk} - 2R_{lk}R_{lijk} + 2\alpha\varphi_k\varphi_l R_{lijk}\\ &\quad + 2\alpha(\Delta\varphi)\varphi_{ij} - 2\alpha\varphi_{ki}\varphi_{kj}.\end{aligned} \tag{2.2}$$

By a direct calculation, we have

$$\begin{aligned}\frac{\partial}{\partial t}(\varphi_i\varphi_j) &= (\varphi_t)_i\varphi_j + \varphi_i(\varphi_t)_j\\ &= (\Delta\varphi)_i\varphi_j + \varphi_i(\Delta\varphi)_j\\ &= \Delta(\varphi_i)\varphi_j - R_{ik}\varphi_k\varphi_j + \varphi_i\Delta(\varphi_j) - R_{jk}\varphi_k\varphi_i\\ &= \Delta(\varphi_i\varphi_j) - 2\varphi_{ik}\varphi_{jk} - R_{ik}\varphi_k\varphi_j - R_{jk}\varphi_k\varphi_i.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial t}h_{ij} &= \frac{\partial}{\partial t}R_{ij} - \alpha\frac{\partial}{\partial t}(\varphi_i\varphi_j)\\ &= \Delta R_{ij} - 2R_{ik}R_{jk} - 2R_{lk}R_{lijk} + 2\alpha\varphi_k\varphi_l R_{lijk}\\ &\quad + 2\alpha(\Delta\varphi)\varphi_{ij} - 2\alpha\varphi_{ki}\varphi_{kj} - \alpha\Delta(\varphi_i\varphi_j) + 2\alpha\varphi_{ik}\varphi_{jk}\\ &\quad + \alpha R_{ik}\varphi_k\varphi_j + \alpha R_{jk}\varphi_k\varphi_i\\ &= \Delta h_{ij} - R_{ik}h_{jk} - R_{jk}h_{ik} - 2R_{lijk}h_{lk} + 2\alpha(\Delta\varphi)\varphi_{ij}.\end{aligned}$$

On the other hand, using Lemma 2.1, it holds that

$$\begin{aligned}
\frac{\partial}{\partial t}(\mathrm{tr_g}h) &= \frac{\partial}{\partial t}R - \alpha\frac{\partial}{\partial t}(|\nabla\varphi|^2)\\
&= 2\Delta R - 4\alpha|\nabla^2\varphi|^2 - 4\alpha\varphi_j\varphi_{jii}\\
&\quad - \Delta R + 2\alpha[2\varphi_j\varphi_{jii} + (\Delta\varphi)^2 + |\nabla^2\varphi|^2 - R_{ij}\varphi_i\varphi_j]\\
&\quad + 2R_{ik}h_{ik} - 2(ar+b)R\\
&\quad - \alpha[\Delta(|\nabla\varphi|^2) - 2|\nabla^2\varphi|^2 - 2\alpha|\nabla\varphi|^4 - 2(ar+b)|\nabla\varphi|^2]\\
&= \Delta(\mathrm{tr_g}h) + 2|h_{ij}|^2 + 2\alpha(\Delta\varphi)^2 - 2(ar+b)\mathrm{tr_g}h,
\end{aligned}$$

where we used

$$\begin{aligned}
h_{ik,ki} &= \frac{1}{2}\Delta R - \alpha[2\varphi_j\varphi_{jii} + (\Delta\varphi)^2 + |\nabla^2\varphi|^2 - R_{ij}\varphi_i\varphi_j],\\
\frac{\partial}{\partial t}(|\nabla\varphi|^2) &= (\frac{\partial}{\partial t}g^{ij})\varphi_i\varphi_j + g^{ij}\frac{\partial}{\partial t}(\varphi_i\varphi_j)\\
&= \Delta(|\nabla\varphi|^2) - 2|\nabla^2\varphi|^2 - 2\alpha|\nabla\varphi|^4 - 2(ar+b)|\nabla\varphi|^2.
\end{aligned}$$

We complete the proof of Lemma 2.2. □

3. L^2-estimates and some applications

For any a, b, we will prove that the nonnegativity of $\mathrm{tr_g}h(t)$ is preserved by the evolution equation (1.3).

Theorem 3.1. *Let $g(t)$ be a solution to the evolution equation* (1.3) *on a compact manifold M^n. If $\mathrm{tr_g}h \geq 0$ holds at $t = 0$, then also for all $t > 0$ as long as the solution exists.*

Proof: Let

$$\beta = e^{2\int_0^t [ar(s)+b]\,ds}(\mathrm{tr_g}h).$$

Then

$$\begin{aligned}
(\frac{\partial}{\partial t} - \Delta)\beta &= e^{2\int_0^t [ar(s)+b]\,ds}[2(ar+b)(\mathrm{tr_g}h) + \frac{\partial}{\partial t}(\mathrm{tr_g}h) - \Delta(\mathrm{tr_g}h)]\\
&= e^{2\int_0^t [ar(s)+b]\,ds}[2|h_{ij}|^2 + 2\alpha(\Delta\varphi)^2]\\
&\geq 0,
\end{aligned}$$

where the second equality used Lemma 2.2. From the maximum principle of parabolic equations, we obtain that $\mathrm{tr_g}h(t)$ remains nonnegative if $\mathrm{tr_g}h(0)$ is nonnegative under the initial metric $g(0)$. □

Next, we begin with an elementary lemma.

Lemma 3.1. *Let $g(t)$ be a solution to the evolution equation* (1.3) *on a compact manifold M^n. Then for any $f \in C^2(\mathbb{R}_+ \times M^n)$, it holds that*

$$
\begin{aligned}
\frac{d}{dt}\int_M f\,dV_g = {}& \int_M (\frac{\partial}{\partial t} - \Delta) f\,dV_g \\
& - \int_M (\mathrm{tr_g} h) f\,dV_g + n(ar+b)\int_M f\,dV_g
\end{aligned}
\tag{3.1}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}\int_M |\nabla f|^2\,dV_g = -2\int_M |\nabla^2 f|^2\,dV_g - \int_M (\mathrm{tr_g} h)|\nabla f|^2\,dV_g \\
&\quad - 2\alpha\int_M \langle\nabla\varphi,\nabla f\rangle^2\,dV_g \\
&\quad - 2\int_M (\Delta f)(\frac{\partial}{\partial t} - \Delta) f\,dV_g + (n-2)(ar+b)\int_M |\nabla f|^2\,dV_g.
\end{aligned}
\tag{3.2}
$$

Proof: By (4) of Lemma 2.1, we have

$$\frac{\partial}{\partial t} dV_g = -(\mathrm{tr_g}\tilde{h})\,dV_g = [-\mathrm{tr_g} h + n(ar+b)]\,dV_g.$$

Therefore,

$$
\begin{aligned}
\frac{d}{dt}\int_M f\,dV_g &= \int_M \{\frac{\partial}{\partial t} f + [-\mathrm{tr_g} h + n(ar+b)] f\}\,dV_g \\
&= \int_M \frac{\partial}{\partial t} f\,dV_g - \int_M (\mathrm{tr_g} h) f\,dV_g + n(ar+b)\int_M f\,dV_g \\
&= \int_M (\frac{\partial}{\partial t} - \Delta) f\,dV_g - \int_M (\mathrm{tr_g} h) f\,dV_g + n(ar+b)\int_M f\,dV_g,
\end{aligned}
$$

which completes the proof of (3.1).

According to the Bochner formula, we have

$$\Delta|\nabla f|^2 = 2|\nabla^2 f|^2 + 2R^{ij} f_i f_j + 2\langle\nabla\Delta f, \nabla f\rangle. \tag{3.3}$$

By virtue of (1) of Lemma 2.1, we get

$$\frac{\partial}{\partial t}|\nabla f|^2 = 2\tilde{h}^{ij} f_i f_j + 2\langle\nabla(\frac{\partial}{\partial t} f), \nabla f\rangle,$$

so that

$$
\begin{aligned}
&(\frac{\partial}{\partial t}-\Delta)|\nabla f|^2\\
&\quad=-2|\nabla^2 f|^2+2\langle\nabla(\frac{\partial}{\partial t}-\Delta)f,\nabla f\rangle+2(\tilde{h}^{ij}-R^{ij})f_i f_j \qquad (3.4)\\
&\quad=-2|\nabla^2 f|^2+2\langle\nabla(\frac{\partial}{\partial t}-\Delta)f,\nabla f\rangle-2\alpha\langle\nabla\varphi,\nabla f\rangle^2-2(ar+b)|\nabla f|^2
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d}{dt}\int_M|\nabla f|^2\,dV_g&=\int_M\{\frac{\partial}{\partial t}|\nabla f|^2+[-\mathrm{tr_g}h+n(ar+b)]|\nabla f|^2\}\,dV_g\\
&=\int_M\{(\frac{\partial}{\partial t}-\Delta)|\nabla f|^2+[-\mathrm{tr_g}h+n(ar+b)]|\nabla f|^2\}\,dV_g\\
&=-2\int_M|\nabla^2 f|^2\,dV_g-\int_M(\mathrm{tr_g}h)|\nabla f|^2\,dV_g-2\alpha\int_M\langle\nabla\varphi,\nabla f\rangle^2\,dV_g\\
&\quad-2\int_M(\Delta f)(\frac{\partial}{\partial t}-\Delta)f\,dV_g+(n-2)(ar+b)\int_M|\nabla f|^2\,dV_g.
\end{aligned}
$$

We complete the proof of Lemma 3.1. □

Theorem 3.2. *Let $g(t)$ be a solution to the evolution equation* (1.3) *on a compact manifold M^n. Then for any $f\in C^2(\mathbb{R}_+\times M^n)$,*

$$
\begin{aligned}
\frac{d}{dt}\int_M|\nabla f|^2\,dV_g\le&-\int_M(\mathrm{tr_g}h)|\nabla f|^2\,dV_g+(n-2)(ar+b)\int_M|\nabla f|^2\,dV_g\\
&+\frac{n}{8}\int_M[(\frac{\partial}{\partial t}-\Delta)f]^2\,dV_g.
\end{aligned}\tag{3.5}
$$

Proof: Using the Cauchy inequality, we have

$$|\nabla^2 f|^2\ge\frac{1}{n}(\Delta f)^2,$$

so that by (3.2)

$$
\begin{aligned}
\frac{d}{dt}\int_M|\nabla f|^2\,dV_g\le&-\frac{2}{n}\int_M(\Delta f)^2\,dV_g-\int_M(\mathrm{tr_g}h)|\nabla f|^2\,dV_g\\
&-2\int_M(\Delta f)(\frac{\partial}{\partial t}-\Delta)f\,dV_g+(n-2)(ar+b)\int_M|\nabla f|^2\,dV_g\\
\le&-\int_M(\mathrm{tr_g}h)|\nabla f|^2\,dV_g+(n-2)(ar+b)\int_M|\nabla f|^2\,dV_g\\
&+\frac{n}{8}\int_M[(\frac{\partial}{\partial t}-\Delta)f]^2\,dV_g,
\end{aligned}
$$

which completes the proof of Theorem 3.2. □

Together with the evolution for $\mathrm{tr_g}h$ in Lemma 2.2, one can establish the following

Theorem 3.3. *Let $g(t)$ be a solution to the evolution equation* (1.3) *on a compact manifold M^n. Then*

$$\begin{aligned} \frac{d}{dt}\int_M |\nabla(\mathrm{tr_g}h)|^2\,dV_g \leq &-\int_M (\mathrm{tr_g}h)|\nabla(\mathrm{tr_g}h)|^2\,dV_g \\ &+(n-2)(ar+b)\int_M |\nabla(\mathrm{tr_g}h)|^2\,dV_g \\ &+\frac{n}{8}\int_M [2|h_{ij}|^2+2\alpha(\Delta\varphi)^2-2(ar+b)\mathrm{tr_g}h]^2\,dV_g. \end{aligned} \tag{3.6}$$

Next, we give the evolution equation of eigenvalues with respect to the Laplacian.

Theorem 3.4. *Let $g(t)$ be a solution to the evolution equation* (1.2) *on a compact manifold M^n.*

(1) *If $\lambda(t)$ denotes the non-negative eigenvalue of the Laplacian on M^n, then*

$$\frac{d}{dt}\lambda = 2\int_M h^{ij}f_if_j\,dV_g - \int_M (\mathrm{tr_g}h)|\nabla f|^2\,dV_g + \lambda\int_M (\mathrm{tr_g}h)f^2\,dV_g.$$

In particular, if $\mathrm{tr_g}h\geq 0$ holds at $t=0$, then

$$\frac{d}{dt}\lambda \leq 2\int_M h^{ij}f_if_j\,dV_g + \lambda\int_M (\mathrm{tr_g}h)f^2\,dV_g.$$

(2) *If $\lambda(t)$ denotes the non-negative eigenvalue of $\Delta-\frac{1}{2}\mathrm{tr_g}h$ on M^n, then*

$$\frac{d}{dt}\lambda = 2\int_M h^{ij}f_if_j\,dV_g + \int_M f^2[|h_{ij}|^2+\alpha(\Delta\varphi)^2]\,dV_g.$$

In particular, if $h_{ij}(t)\geq 0$, then the eigenvalue of the operator $-\Delta+\frac{1}{2}\mathrm{tr_g}h$ are nondecreasing.

(3) *If $\lambda(t)$ denotes the non-negative eigenvalue of $\Delta+\frac{1}{4}\mathrm{tr_g}h$ on M^n, then*

$$\frac{d}{dt}\lambda \leq \int_M [\frac{n-1}{2n}(\mathrm{tr_g}h)^2+\frac{5}{2}\lambda(\mathrm{tr_g}h)]f^2\,dV_g + 2\lambda^2.$$

Remark 3.1. In particular, taking $\alpha=2$, then (2) in Theorem 3.4 becomes Theorem 1.10 in [4].

Remark 3.2. We remark that the second and third inequalities of p. 160 in [9] is slight incorrect. The right version of the third inequality of p. 160 in [9] should be

$$\begin{aligned}\frac{d}{dt}\lambda = &-2\int_M |\nabla^2 f|^2\, dV_g + (2\varphi - 1)\int_M R|\nabla f|^2\, dV_g\\ &+\int_M (2\varphi S - \frac{2}{3}\varphi AR + \varphi' R) f^2\, dV_g\\ &-\varphi\int_M R(R - 4\varphi R - \frac{2}{3}A + 2\lambda) f^2\, dV_g\\ &+\lambda\int_M R(R - 4\varphi R - \frac{2}{3}A + 2\lambda) f^2\, dV_g.\end{aligned}$$

As a result, for $n = 3$, the estimate $\frac{d}{dt}\lambda \le \frac{19}{8}\lambda^2$ of Theorem 6.4 in [9] cannot be obtained.

Proof: Let us consider an eigenfunction f associated with the eigenvalue λ, that is,

$$-(\Delta + V)f = \lambda f, \qquad \int_M f^2\, dV_g = 1$$

where V is a potential function to be chosen later. Then, we obtain from (3.2) that

$$\begin{aligned}\frac{d}{dt}\int_M |\nabla f|^2\, dV_g = &-2\int_M |\nabla^2 f|^2\, dV_g - \int_M (\mathrm{tr_g} h)|\nabla f|^2\, dV_g\\ &-2\alpha\int_M \langle\nabla\varphi, \nabla f\rangle^2\, dV_g\\ &+2\int_M (V+\lambda)^2 f^2\, dV_g + \int_M (V+\lambda)\frac{\partial f^2}{\partial t}\, dV_g\\ &+(n-2)(ar+b)\int_M |\nabla f|^2\, dV_g.\end{aligned} \tag{3.7}$$

By virtue of the divergence theorem,

$$\int_M |\nabla f|^2\, dV_g = \int_M f(-\Delta)f\, dV_g = \int_M (V+\lambda) f^2\, dV_g,$$

so that

$$\begin{aligned}\frac{d}{dt}\int_M |\nabla f|^2\, dV_g = &\int_M f^2 \frac{\partial}{\partial t}(V+\lambda)\, dV_g + \int_M (V+\lambda)\frac{\partial f^2}{\partial t}\, dV_g\\ &+\int_M [-\mathrm{tr_g} h + n(ar+b)](V+\lambda) f^2\, dV_g.\end{aligned} \tag{3.8}$$

Thus, (3.8) combines with (3.7) to give

$$
\begin{aligned}
\frac{d}{dt}\lambda = &-2\int_M |\nabla^2 f|^2\, dV_g - \int_M (\mathrm{tr_g} h)|\nabla f|^2\, dV_g - 2\alpha \int_M \langle \nabla\varphi, \nabla f\rangle^2\, dV_g \\
&+ 2\int_M (V+\lambda)^2 f^2\, dV_g - \int_M f^2 \frac{\partial V}{\partial t}\, dV_g \\
&+ \int_M [(\mathrm{tr_g} h) - 2(ar+b)](V+\lambda) f^2\, dV_g.
\end{aligned} \tag{3.9}
$$

Choose $V = c\,\mathrm{tr_g} h$, where c is a constant to be determined. Then

$$
\frac{\partial V}{\partial t} = c\frac{\partial}{\partial t}\mathrm{tr_g} h
$$

so that

$$
\int_M f^2 \frac{\partial V}{\partial t}\, dV_g = c\int_M f^2 \frac{\partial}{\partial t}\mathrm{tr_g} h\, dV_g.
$$

Using integration by parts again and by use of Lemma 2.2, we have

$$
\begin{aligned}
\int_M (\mathrm{tr_g} h)|\nabla f|^2\, dV_g &= \int_M (\mathrm{tr_g} h)(c\,\mathrm{tr_g} h + \lambda) f^2\, dV_g + \frac{1}{2}\int_M f^2 \Delta(\mathrm{tr_g} h)\, dV_g \\
&= \int_M (\mathrm{tr_g} h)(c\,\mathrm{tr_g} h + \lambda) f^2\, dV_g + \frac{1}{2}\int_M f^2 \frac{\partial}{\partial t}\mathrm{tr_g} h\, dV_g \\
&\quad - \int_M f^2 [|h_{ij}|^2 + \alpha(\Delta\varphi)^2 - (ar+b)\mathrm{tr_g} h]\, dV_g.
\end{aligned}
$$

In other words,

$$
\begin{aligned}
-\int_M f^2 \frac{\partial V}{\partial t}\, dV_g &= -c\int_M f^2 \frac{\partial}{\partial t}\mathrm{tr_g} h\, dV_g \\
&= -2c\int_M (\mathrm{tr_g} h)|\nabla f|^2\, dV_g + 2c\int_M (\mathrm{tr_g} h)(c\,\mathrm{tr_g} h + \lambda) f^2\, dV_g \\
&\quad - 2c\int_M f^2 [|h_{ij}|^2 + \alpha(\Delta\varphi)^2 - (ar+b)\mathrm{tr_g} h]\, dV_g.
\end{aligned}
$$

It follows that

$$\begin{aligned}\frac{d}{dt}\lambda &= -2\int_M |\nabla^2 f|^2\, dV_g - \int_M (\mathrm{tr_g} h)|\nabla f|^2\, dV_g - 2\alpha\int_M \langle\nabla\varphi, \nabla f\rangle^2\, dV_g\\ &\quad + 2\int_M (V+\lambda)^2 f^2\, dV_g - \int_M f^2\frac{\partial V}{\partial t}\, dV_g\\ &\quad + \int_M [(\mathrm{tr_g} h) - 2(ar+b)](V+\lambda) f^2\, dV_g\\ &= -2\int_M |\nabla^2 f|^2\, dV_g - (1+2c)\int_M (\mathrm{tr_g} h)|\nabla f|^2\, dV_g - 2\alpha\int_M \langle\nabla\varphi, \nabla f\rangle^2\, dV_g\\ &\quad + \int_M [(1+4c)(\mathrm{tr_g} h) + 2\lambda - 2(ar+b)](c\,\mathrm{tr_g} h + \lambda) f^2\, dV_g\\ &\quad - 2c\int_M f^2[|h_{ij}|^2 + \alpha(\Delta\varphi)^2 - (ar+b)\mathrm{tr_g} h]\, dV_g.\end{aligned}$$

Integrating (3.3) yields

$$\begin{aligned}-2\int_M |\nabla^2 f|^2\, dV_g &- 2\alpha\int_M \langle\nabla\varphi, \nabla f\rangle^2\, dV_g\\ &= \int_M (2R^{ij} f_i f_j + 2\langle\nabla\Delta f, \nabla f\rangle)\, dV_g\\ &\quad - 2\alpha\int_M \langle\nabla\varphi, \nabla f\rangle^2\, dV_g\\ &= 2\int_M h^{ij} f_i f_j\, dV_g - 2\int_M (c\,\mathrm{tr_g} h + \lambda)^2 f^2\, dV_g.\end{aligned} \tag{3.10}$$

Therefore, we obtain

$$\begin{aligned}\frac{d}{dt}\lambda &= 2\int_M h^{ij} f_i f_j\, dV_g - (1+2c)\int_M (\mathrm{tr_g} h)|\nabla f|^2\, dV_g\\ &\quad + \int_M [(1+2c)(\mathrm{tr_g} h) - 2(ar+b)](c\,\mathrm{tr_g} h + \lambda) f^2\, dV_g\\ &\quad - 2c\int_M f^2[|h_{ij}|^2 + \alpha(\Delta\varphi)^2 - (ar+b)\mathrm{tr_g} h]\, dV_g.\end{aligned} \tag{3.11}$$

Now we are in the position to complete the proof of Theorem 3.4. Notice that under the evolution equation (1.2), we have $a = b = 0$.

If we choose $c = 0$, then (3.11) reduces to

$$\frac{d}{dt}\lambda = 2\int_M h^{ij} f_i f_j\, dV_g - \int_M (\mathrm{tr_g} h)|\nabla f|^2\, dV_g + \lambda\int_M (\mathrm{tr_g} h) f^2\, dV_g$$

and (1) of Theorem 3.4 follows.

If we choose $c = -\frac{1}{2}$, then (3.11) reduces to

$$\frac{d}{dt}\lambda = 2\int_M h^{ij} f_i f_j \, dV_g + \int_M f^2[|h_{ij}|^2 + \alpha(\Delta\varphi)^2] \, dV_g. \tag{3.12}$$

Hence, we obtain (2) of Theorem 3.4.

If we choose $c = \frac{1}{4}$, then (3.11) reduces to

$$\begin{aligned} \frac{d}{dt}\lambda = {} & 2\int_M h^{ij} f_i f_j \, dV_g - \frac{3}{2}\int_M (\mathrm{tr_g} h)|\nabla f|^2 \, dV_g \\ & + \frac{3}{2}\int_M (\mathrm{tr_g} h)(\frac{1}{4}\,\mathrm{tr_g} h + \lambda) f^2 \, dV_g \\ & - \frac{1}{2}\int_M f^2[|h_{ij}|^2 + \alpha(\Delta\varphi)^2] \, dV_g. \end{aligned} \tag{3.13}$$

It follows from (3.10) that

$$2\int_M h^{ij} f_i f_j \, dV_g \le 2\int_M (\frac{1}{4}\,\mathrm{tr_g} h + \lambda)^2 f^2 \, dV_g.$$

Thus, we deduce from (3.13) the following

$$\begin{aligned} \frac{d}{dt}\lambda \le {} & 2\int_M (\frac{1}{4}\,\mathrm{tr_g} h + \lambda)^2 f^2 \, dV_g \\ & + \frac{3}{2}\int_M (\mathrm{tr_g} h)(\frac{1}{4}\,\mathrm{tr_g} h + \lambda) f^2 \, dV_g - \frac{1}{2}\int_M f^2 |h_{ij}|^2 \, dV_g. \end{aligned} \tag{3.14}$$

Applying the Cauchy inequality

$$|h_{ij}|^2 \ge \frac{1}{n}(\mathrm{tr_g} h)^2$$

into (3.14) gives

$$\frac{d}{dt}\lambda \le \int_M [\frac{n-1}{2n}(\mathrm{tr_g} h)^2 + \frac{5}{2}\lambda(\mathrm{tr_g} h)] f^2 \, dV_g + 2\lambda^2.$$

We complete the proof of Theorem 3.4. □

Acknowledgment. The author would like to express their thanks to the referee for his/her valuable comments, which made this paper more readable.

References

[1] Cao X.D., Hamilton R., *Differential Harnack estimates for time-dependent heat equations with potentials*, Geom. Funct. Anal. **19** (2009), 989–1000.

[2] Fang S.W., *Differential Harnack inequalities for heat equations with potentials under the Bernhard List's flow*, Geom. Dedicata **161** (2012), 11–22.

[3] Ma B.Q., Huang G.Y., *Lower bounds for the scalar curvature of noncompact gradient solitons of List's flow*, Arch. Math. **100** (2013), 593–599.

[4] Li Y., *Eigenvalues and entropys under the harmonic-Ricci flow*, arXiv:1011.1697, to appear in Pacific J. Math.

[5] List B., *Evolution of an extended Ricci flow system*, PhD Thesis, AEI Potsdam, `http://www.diss.fu-berlin.de/2006/180/index.html` (2006).

[6] List B., *Evolution of an extended Ricci flow system*, Comm. Anal. Geom. **16** (2008), 1007–1048.

[7] Lott J., Sesum N., *Ricci flow on three-dimensional manifolds with symmetry*, arXiv:1102.4384, to appear in Comm. Math. Helv.

[8] Müller R., *Ricci flow coupled with harmonic map flow*, Ann. Sci. Éc. Norm. Supér. **45** (2012), 101–142.

[9] Qian Z.M., *Ricci flow on a 3-manifold with positive scalar curvature*, Bull. Sci. Math. **133** (2009), 145–168.

[10] Wang L.F., *Differential Harnack inequalities under a coupled Ricci flow*, Math. Phys. Anal. Geom. **15** (2012), 343–360.

Department of Mathematics, Henan Normal University, Xinxiang 453007, P.R. China

E-mail: bqma@henannu.edu.cn

(*Received* January 9, 2013, *revised* October 15, 2013)