Semicontinuous integrands as jointly measurable maps

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Abstract. Suppose that $(X,\mathcal{A})$ is a measurable space and Y is a metrizable, Souslin space. Let $\mathcal{A}^u$ denote the universal completion of $\mathcal{A}$. For $x \in X$, let $\underline{f}(x,\cdot)$ be the lower semicontinuous hull of $f(x,\cdot)$. If $f : X \times Y \to \overline{\mathbb{R}}$ is $(\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$-measurable, then $\underline{f}$ is $(\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$-measurable.

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Let $(X,\mathcal{A})$ be a measurable space. For every bounded measure μ on $(X,\mathcal{A})$, let $\mathcal{A}^\mu$ denote the completion of $\mathcal{A}$ with respect to μ. Let

$$\mathcal{A}^u := \bigcap \{\mathcal{A}^\mu : \mu \text{ is a bounded measure on } (X,\mathcal{A})\}.$$

The σ-algebra $\mathcal{A}^u$ is called the *universal completion* of $\mathcal{A}$.

Let Y be a topological space, and let $\mathcal{B}(Y)$ represent the σ-algebra of Borel subsets of Y. The space Y is said to be *Souslin* if it is Hausdorff and there exist a Polish space P and a continuous surjection from P to Y.

Given $f : X \times Y \to \overline{\mathbb{R}}$, define the map $\underline{f} : X \times Y \to \overline{\mathbb{R}}$ by

$$\underline{f}(x,y) := \sup_{V_y} \inf_{z \in V_y} f(x,z),$$

where V_y ranges over all neighborhoods of y. For each $x \in X$, $\underline{f}(x,\cdot)$ is the *lower semicontinuous hull* of $f(x,\cdot)$. If Y is metrizable, $\underline{f}$ can be expressed as

$$\underline{f}(x,y) = \sup_{n \in \mathbb{N}} \inf_{z \in N_{\frac{1}{n}}(y)} f(x,z),$$

where $N_{\frac{1}{n}}(y)$ represents the open $\frac{1}{n}$-neighborhood of y.

Theorem. *Suppose that $(X,\mathcal{A})$ is a measurable space and Y is a metrizable, Souslin space. Suppose further that the map $f : X \times Y \to \overline{\mathbb{R}}$ is $(\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$-measurable. Then $\underline{f}$ is $(\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$-measurable.*

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Proof: Define $g^n : X \times Y \to \overline{\mathbb{R}}$ by

$$g^n(x, y) := \inf_{z \in N_{\frac{1}{n}}(y)} f(x, z).$$

We first show that g^n is $(\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$-measurable for each n.

Let

$$D^n := \left\{(x, y, z) \in X \times Y \times Y : z \in N_{\frac{1}{n}}(y)\right\}.$$

The map g^n is $(\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$-measurable if for $a \in \mathbb{R}$,

$$\{(x, y) \in X \times Y : g^n(x, y) < a\} \in \mathcal{A}^u \otimes \mathcal{B}(Y). \tag{1}$$

Given $a \in \mathbb{R}$ we have

$$\{(x, y) \in X \times Y : g^n(x, y) < a\} = \mathrm{Proj}_{X \times Y}(E^n), \tag{2}$$

where

$$E^n := \{(x, y, z) \in D^n : f(x, z) < a\}$$

and $\mathrm{Proj}_{X\times Y}(E^n)$ represents the projection of E^n onto $X \times Y$. Thus, to establish (1) it suffices to show that $\mathrm{Proj}_{X\times Y}(E^n)$ belongs to $\mathcal{A}^u \otimes \mathcal{B}(Y)$.

Because Y is a Souslin space, Y is a Lindelöf space, and since Y is in addition metrizable, Y is separable. Because Y is separable, there is a countable, dense subset Q of Y. Let $\{y^1, y^2, \dots\}$ be an enumeration of this set. For $\alpha > 0$ and $y \in Y$, define

$$A^{(\alpha,y)} := \{(x, z) \in X \times N_\alpha(y) : f(x, z) < a\}.$$

Let $\mathrm{Proj}_X(A^{(\alpha,y)})$ be the projection of $A^{(\alpha,y)}$ onto X. Let $\mathbb{Q}$ denote the set of rational numbers in $(0, \frac{1}{n})$. Define

$$S^n := \left\{(\alpha, \beta) \in \mathbb{Q} \times \mathbb{Q} : \alpha + \beta \le \tfrac{1}{n}\right\}.$$

We have

$$\mathrm{Proj}_{X\times Y}(E^n) = \bigcup_{(m,(\alpha,\beta))\in\mathbb{N}\times S^n} \left[\mathrm{Proj}_X(A^{(\alpha,y^m)}) \times N_\beta(y^m)\right]. \tag{3}$$

To see this, observe that given $(x, y) \in \mathrm{Proj}_{X\times Y}(E^n)$, there exists z such that $(x, y, z) \in D^n$ (*i.e.*, $(x, y, z) \in X \times Y \times Y$ and $z \in N_{\frac{1}{n}}(y)$) and $f(x, z) < a$. Let d be a compatible metric on Y, and fix

$$\epsilon \in \left(0, \tfrac{1}{3}\left(\tfrac{1}{n} - d(y, z)\right)\right).$$

For $y' \in N_\epsilon(y)$ we have

$$d(y',z) \le d(y',y) + d(y,z) < \epsilon + d(y,z) < \tfrac{1}{3}\left(\tfrac{1}{n} - d(y,z)\right) + d(y,z),$$

so there is a rational number

$$\beta \in \left(\tfrac{1}{3}\left(\tfrac{1}{n} - d(y,z)\right), \tfrac{1}{2}\left(\tfrac{1}{n} - d(y,z)\right)\right)$$

such that $d(y',z) < \beta + d(y,z)$ for all $y' \in N_\epsilon(y)$, and hence there is a rational number

$$\alpha \in \left(\beta + d(y,z), \tfrac{1}{2}\left(\tfrac{1}{n} - d(y,z)\right) + d(y,z)\right)$$

such that $d(y',z) < \alpha$ for all $y' \in N_\epsilon(y)$. Consequently, since by denseness of Q in Y one may choose m such that $y^m \in N_\epsilon(y)$, we have $z \in N_\alpha(y^m)$. It follows that $(x,z) \in X \times N_\alpha(y^m)$ and $f(x,z) < a$ (so that $x \in \mathrm{Proj}_X(A^{(\alpha,y^m)})$) and, since

$$d(y,y^m) < \epsilon < \tfrac{1}{3}\left(\tfrac{1}{n} - d(y,z)\right) < \beta,$$

we have $y \in N_\beta(y^m)$. We conclude that $(x,y) \in \mathrm{Proj}_X(A^{(\alpha,y^m)}) \times N_\beta(y^m)$ with $(\alpha,\beta) \in \mathbb{Q} \times \mathbb{Q}$ and

$$\alpha + \beta \le \tfrac{1}{2}\left(\tfrac{1}{n} - d(y,z)\right) + d(y,z) + \tfrac{1}{2}\left(\tfrac{1}{n} - d(y,z)\right) \le \tfrac{1}{n}.$$

Conversely, if $(x,y) \in \mathrm{Proj}_X(A^{(\alpha,y^m)}) \times N_\beta(y^m)$ for some $(m,(\alpha,\beta)) \in \mathbb{N} \times S^n$, then there exists z such that $(x,z) \in X \times N_\alpha(y^m)$ and $f(x,z) < a$. In addition,

$$d(y,z) \le d(y,y^m) + d(y^m,z) < \beta + \alpha \le \tfrac{1}{n}.$$

Consequently, $(x,y,z) \in X \times Y \times Y$ and $z \in N_{\frac{1}{n}}(y)$ (so that $(x,y,z) \in D^n$) and $f(x,z) < a$, which implies that $(x,y) \in \mathrm{Proj}_{X\times Y}(E^n)$.

Because f is $(\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$-measurable, we have $A^{(\alpha,y)} \in \mathcal{A}^u \otimes \mathcal{B}(Y)$ for every $\alpha > 0$ and $y \in Y$. Therefore, because Y is a Souslin space, the measurable projection theorem (*e.g.*, Sainte-Beuve [6, Theorem 4]) gives $\mathrm{Proj}_X(A^{(\alpha,y)}) \in \mathcal{A}^u$ for $\alpha > 0$ and $y \in Y$.[1] In light of (3), therefore, we conclude that $\mathrm{Proj}_{X\times Y}(E^n) \in \mathcal{A}^u \otimes \mathcal{B}(Y)$.

Because $\mathrm{Proj}_{X\times Y}(E^n) \in \mathcal{A}^u \otimes \mathcal{B}(Y)$, g^n is $(\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$-measurable (recall (2) and (1)). It only remains to observe that

$$\underline{f}(x,y) = \sup_{n\in\mathbb{N}} \inf_{z \in N_{\frac{1}{n}}(y)} f(x,z) = \sup_{n\in\mathbb{N}} g^n(x,y),$$

so $\underline{f}$ is $(\mathcal{A}^u \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$-measurable. □

In the remainder of the paper we present an application of the above result. Let $(X,\mathcal{A},\mu)$ be a finite measure space with $\mathcal{A} = \mathcal{A}^u$. Let Y be a metrizable Lusin space (*i.e.*, a metrizable topological space which is homeomorphic to a Borel subset

[1] For the case when Y is Polish, the measurable projection theorem can also be found in Cohn [5, Proposition 8.4.4].

of a compact metrizable space). A *transition probability* with respect to $(X, \mathcal{A})$ and $(Y, \mathcal{B}(Y))$ is a function $\sigma : \mathcal{B}(Y) \times X \to [0, 1]$ satisfying the following:

- $\sigma(\cdot|x)$ is a probability measure on $(Y, \mathcal{B}(Y))$ for every $x \in X$;
- $\sigma(B|\cdot)$ is $(\mathcal{A}, \mathcal{B}([0, 1]))$-measurable for every $B \in \mathcal{B}(Y)$.

The set of transition probabilities with respect to $(X, \mathcal{A})$ and $(Y, \mathcal{B}(Y))$ is denoted by $\mathcal{S}$.

A *normal integrand* on $X \times Y$ is a map $f : X \times Y \to \overline{\mathbb{R}}$ satisfying the following:

- $f(x, \cdot)$ is lower semicontinuous on Y for every $x \in X$;
- f is $(\mathcal{A} \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$-measurable.

Let $L_1(X, \mathcal{A}, \mu)$ represent the set of $(\mathcal{A}, \mathcal{B}(\mathbb{R}))$-measurable functions $\xi : X \to \mathbb{R}$ such that

$$\int_X |\xi(x)|\mu(dx) < \infty.$$

The set of all normal integrands f on $X \times Y$ for which there exists $\xi \in L_1(X, \mathcal{A}, \mu)$ such that $\xi(x) \leq f(x, y)$ for all $(x, y) \in X \times Y$ is denoted by $\mathcal{F}$.

For $f \in \mathcal{F}$, the functional $I_f : \mathcal{S} \to \overline{\mathbb{R}}$ is defined by

$$I_f(\sigma) := \int_X \int_Y f(x, y)\sigma(dy|x)\mu(dx).$$

The *narrow topology* on $\mathcal{S}$ is the coarsest topology that makes the functionals in $\{I_f : f \in \mathcal{F}\}$ lower semicontinuous. This topology has been studied by Balder [1], [2], [3] and applied to the theory of games with incomplete information (*e.g.*, Balder [2] and Carbonell-Nicolau and McLean [4]).

Suppose that the map $f : X \times Y \to \overline{\mathbb{R}}$ is $(\mathcal{A} \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$-measurable. Suppose further that there exists $\xi \in L_1(X, \mathcal{A}, \mu)$ such that $\xi(x) \leq f(x, y)$ for all $(x, y) \in X \times Y$. Then $\underline{f}$ satisfies $\varphi(x) \leq \underline{f}(x, y)$ for all $(x, y) \in X \times Y$ and for some $\varphi \in L_1(X, \mathcal{A}, \mu)$. In addition, $\underline{f}(x, \cdot)$ is lower semicontinuous on Y for every $x \in X$, and, by virtue of Theorem, $\underline{f}$ is $(\mathcal{A} \otimes \mathcal{B}(Y), \mathcal{B}(\overline{\mathbb{R}}))$-measurable. Consequently, $\underline{f} \in \mathcal{F}$. It follows that if $\mathcal{S}$ is endowed with the narrow topology, for each $\epsilon > 0$ and every $\sigma \in \mathcal{S}$ there exists an open set V in $\mathcal{S}$ containing σ such that

$$I_{\underline{f}}(\nu) \geq I_{\underline{f}}(\sigma) - \epsilon, \quad \text{for all} \quad \nu \in V.$$

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