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Remarks on Fréchet differentiability of pointwise Lipschitz, cone-monotone and quasiconvex functions

Luděk Zajíček

Abstract. We present some consequences of a deep result of J. Lindenstrauss and D. Preiss on Γ-almost everywhere Fréchet differentiability of Lipschitz functions on c_0 (and similar Banach spaces). For example, in these spaces, every continuous real function is Fréchet differentiable at Γ-almost every x at which it is Gâteaux differentiable. Another interesting consequences say that both cone-monotone functions and continuous quasiconvex functions on these spaces are Γ-almost everywhere Fréchet differentiable. In the proofs we use a general observation that each version of the Rademacher theorem for real functions on Banach spaces (i.e., a result on a.e. Fréchet or Gâteaux differentiability of Lipschitz functions) easily implies by a method of J. Malý a corresponding version of the Stepanov theorem (on a.e. differentiability of pointwise Lipschitz functions). Using the method of separable reduction, we extend some results to several non-separable spaces.

Keywords: cone-monotone function; Fréchet differentiability; Gâteaux differentiability; pointwise Lipschitz function; Γ-null set; quasiconvex function; separable reduction

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1. Introduction and notation

D. Preiss proved in [19] the following very deep theorem.

Theorem P. *Each real Lipschitz function on an Asplund Banach space is Fréchet differentiable at all points of a set D which is uncountable in each ball.*

This theorem motivated a number of related interesting results, see the recent monograph [15].

One of the most interesting open questions in this area asks in which Asplund spaces X there exists an "a.e. version" of Preiss' theorem (i.e., a result which asserts that there exists a non-trivial σ-ideal $\mathcal{I}$ such that the set of all Fréchet non-differentiability points of every Lipschitz function on X belongs to $\mathcal{I}$).

A partial answer was given by J. Lindenstrauss and D. Preiss [14] who defined and applied the important notion of Γ-null sets. A special case (cf. [14, Corollary 3.12], [15, Corollary 6.3.11]) of their main result (which works with vector functions) reads as follows.

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Theorem LP. *Let X be a Banach space. Suppose that X^* is separable and every porous set in X is Γ-null. Then any real Lipschitz function f defined on an open subset G of X is Γ-almost everywhere Fréchet differentiable.*

The main examples of the spaces X which satisfy the assumptions of Theorem LP are subspaces of c_0, the spaces $C(K)$ with K compact countable, and the Tsirelson space ([14, Theorem 4.6], [15, Theorem 10.6.8]).

Theorem LP was generalized to some non-separable spaces:

Theorem C. *Let X be a closed subspace of $c_0(\Delta)$ (where Δ is uncountable) or $X = C(K)$, where K is a scattered compact topological space. Then any real Lipschitz function f defined on an open subset G of X is Γ-almost everywhere Fréchet differentiable.*

For the proofs (which use Theorem LP and separable reduction methods) see [15, (4), p. 45] (for $X = c_0(\Delta)$) and [7, Theorem 6.18] (for all cases).

The following Theorem LPT from [15] is an interesting generalization and strengthening of Preiss' theorem. It works not only with Lipschitz functions, but also with cone-monotone functions.

Recall that a function f on an open subset G of a Banach space X is called *cone-monotone*, if there exists a closed convex cone $K \subset X$ with non-empty interior such that $f(y) \geq f(x)$ whenever $y - x \in K$. It is easy to see (see, e.g., [15, p. 223]) that for every Lipschitz function f on G there exists a functional $x^* \in X^*$ such that the function $g := f + x^*$ is cone-monotone. So each differentiability theorem on cone-monotone functions implies a corresponding differentiability theorem on Lipschitz functions.

Theorem LPT. *Let f be a cone-monotone (or Lipschitz) function on an open subset G of an Asplund space X. Then f is Fréchet differentiable at all points of a set which is non-σ-porous in each ball.*

Theorem LPT immediately follows from [15, Theorem 12.1.3] in the case when X is separable. The proof for non-separable spaces follows by the separable reduction method of [15]. It is sketched in [15, (1), p. 44] and can be easily completed by a standard application of [15, Corollary 3.6.7].

The main results of the present note are Theorem 3.1, Corollary 3.2, Theorem 3.4 and Theorem 3.9 from Section 3.

Theorem 3.1 generalizes "Rademacher's theorems", Theorem LP and Theorem C to corresponding "Stepanov's theorems". More precisely, it says that if X is as in these theorems and f is an arbitrary real function on X, then the set of all points at which f is Lipschitz and is not Fréchet differentiable is Γ-null. (Theorem 3.1 is an immediate consequence of Theorem LP, Theorem C and Proposition 2.1.)

Corollary 3.2 asserts that, if X is as in Theorem LP, then every continuous real function is Fréchet differentiable at Γ-almost every x at which it is Gâteaux differentiable. It follows immediately from Theorem 3.1 via a lemma from [26].

Theorem 3.4 shows that the "cone-monotone analogue" of Theorem LP also holds. It follows easily from Theorem 3.1 via results of J. Duda on Gâteaux differentiability [9] and Lipschitzness [10] of cone-monotone functions.

Theorem 3.4 on differentiability of cone-monotone functions easily implies Corollary 3.5 on differentiability of continuous quasiconvex functions.

Theorem 3.9 and Theorem 3.10 (proved by the method of separable reduction from [15]) generalize Theorem 3.4 and Corollary 3.5 to some non-separable Banach spaces, namely, subspaces $c_0(\Gamma)$ and spaces $C(K)$, where K is a scattered compact topological space.

Section 2 is devoted to a general observation that each version of the Rademacher theorem for real functions on Banach spaces (i.e., a result on a.e. Fréchet or Gâteaux differentiability of Lipschitz functions) easily implies by a method of J. Malý from [16] a corresponding version of Stepanov's theorem (on a.e. differentiability of pointwise Lipschitz functions). This observation is formulated in Proposition 2.1 (see also Remark 2.2).

Notation and some definitions. In the following, by a Banach space we mean a real Banach space. The symbol $B(x,r)$ will denote the open ball with center x and radius r.

Let X, Y be Banach spaces, $G \subset X$ an open set, and $f : G \to Y$ a mapping. We say that f *is Lipschitz at* $x \in G$ if $\limsup_{y\to x} \frac{\|f(y)-f(x)\|}{\|y-x\|} < \infty$. We say that f is *pointwise Lipschitz* if f is Lipschitz at all points of G.

We say that a subset A of a Banach spaces X is (*upper*) *porous at a point* $x \in X$, if there exist $p > 0$ and a sequence $x_n \to x$ such that $B(x_n, p\|x_n - x\|) \cap A = \emptyset$ for each n. We say that a set $A \subset X$ is *porous* if A is porous at each point $x \in A$.

Our main results use (in a Banach space X) the notion of Γ-null sets in X and in the proofs we use the class $\tilde{\mathcal{C}}$ of (small) subsets of X. However, we do not need the (rather complicated) definitions of these classes. For the definition of Γ-null sets see [15] (or [14]) and for the definition of $\tilde{\mathcal{C}}$ [21] (or [25], or [9]).

Note only that the class of Γ-null sets is incomparable with the class of all Aronszajn null sets (i.e., the class of all Gauss null sets) and also with the class of Haar null sets, see [15, Example 5.4.11]. (For information on Aronszajn, Gauss and Haar null sets see [1].)

The sets from $\tilde{\mathcal{C}}$ are "much smaller". Indeed (see [25]),

(1.1) in a separable Banach space X, each set from $\tilde{\mathcal{C}}$ is Γ-null.

Moreover, each set from $\tilde{\mathcal{C}}$ is also Aronszajn null (see [21, Proposition 13]).

Finally note the fact which easily follows from the definition ([15, Definition 5.1.1]) that

(1.2) in a Banach space X, the system of all Γ-null sets is a σ-ideal.

Following [14] and [15], we will write "Γ-almost everywhere" instead of "at all points except for a Γ-null set".

2. Stepanov's theorems via Rademacher's theorems

This section is devoted to a general observation (Proposition 2.1) which shows that each version of the Rademacher theorem for real functions on Banach spaces (i.e., a result on a.e. Fréchet or Gâteaux differentiability of Lipschitz functions) easily implies a corresponding version of Stepanov's theorem (on a.e. differentiability of pointwise Lipschitz functions).

The proof is a straightforward combination of the well-known method of [16] and Stone's theorem on refinements of open covers of metric spaces. Of course, in the case of a separable X we do not need Stone's theorem and thus (after obvious changes) the proof essentially coincides with the proof of J. Malý from [16] (which works with functions on $\mathbb{R}^n$).

Note that we apply Proposition 2.1 also in non-separable spaces, see the proof of Theorem 3.1.

Proposition 2.1. *Let X be a Banach space and $\mathcal{I}$ a σ-ideal of subsets of X. Suppose that*

(R) *each real Lipschitz function on X is Fréchet* (*resp. Gâteaux*) *differentiable except for a set from $\mathcal{I}$.*

Let $G \subset X$ be an open set, f an arbitrary real function on G, and

$$A := A_f := \{x \in G : \ f \ \text{ is Lipschitz but not Fréchet } (\text{resp. Gâteaux}) \text{ differentiable at } \ x\}.$$

Then $A \in \mathcal{I}$.

In particular, each real pointwise Lipschitz function on G is Fréchet (*resp. Gâteaux*) *differentiable except for a set from $\mathcal{I}$.*

Proof: First observe that, without any loss of generality, we can suppose that f is bounded. Indeed, if we define $f_k : G \to Y$ by the equalities $f_k(x) := f(x)$ if $\|f(x)\| \leq k$ and $f_k(x) := 0$ if $\|f(x)\| > k$, it is easy to see that $A_f \subset \bigcup_{k=1}^{\infty} A_{f_k}$. So we suppose that $\|f(x)\| \leq K$, $x \in G$.

For each $n \in \mathbb{N}$, set

$$L_n := \{x \in G : \ \|f(y) - f(x)\| \leq n\|y - x\| \ \text{ whenever } \ \|y - x\| \leq 1/n\}$$

and $A_n := A \cap L_n$. It is easy to see that

$$A = \bigcup_{n=1}^{\infty} A_n. \tag{2.1}$$

Fix an arbitrary $n \in \mathbb{N}$ and consider the open cover $\mathcal{C}$ of G consisting of all open balls with radius $(2n)^{-1}$. By the Stone theorem ([11, Theorem 4.4.1]) the cover $\mathcal{C}$ has an open refinement which is σ-discrete. Write this refinement as $\mathcal{V} = \bigcup_{i=1}^{\infty} \mathcal{V}_i$, where (as in [11]) $\mathcal{V}_i = \{V_{s,i}\}_{s \in S}$ is a discrete family for each $i \in \mathbb{N}$. Note that each family $\mathcal{V}_i$ (from the proof of [11, Theorem 4.4.1]) may be assumed to be not only discrete (i.e., topologically discrete), but also metrically discrete, namely (see [11, (4), p. 280]),

$$\operatorname{dist}(V_{s_1,i}, V_{s_2,i}) \geq 2^{-i}, \quad \text{whenever} \ \ s_1, s_2 \in S, \ s_1 \neq s_2. \tag{2.2}$$

Denote $D_i := \bigcup_{s\in S} V_{s,i}$ and $A_{n,i} := A_n \cap D_i$.

Fix an arbitrary $i \in \mathbb{N}$ and denote $M := A_{n,i}$. Since $\mathcal{V}$ is a cover of G, it is clearly sufficient to prove $M \in \mathcal{I}$. Without any loss of generality we suppose $M \neq \emptyset$. Set $S^* := \{s \in S : V_{s,i} \cap M \neq \emptyset\}$. For each $s \in S^*$ and $x \in V_{s,i}$, set

$$g_{s,i}(x) := \inf\{g(x) : g \ \text{ is Lipschitz with constant } n \text{ on } V_{s,i} \text{ and } f \leq g \text{ on } V_{s,i}\}$$

and

$$h_{s,i}(x) := \sup\{h(x) : h \ \text{ is Lipschitz with constant } n \text{ on } V_{s,i} \text{ and } h \leq f \text{ on } V_{s,i}\}.$$

Consider, for $a \in V_{s,i} \cap M$, the functions

$$g_a(x) := f(a) + n\|x - a\|, \ x \in V_{s,i} \quad \text{ and } \quad h_a(x) := f(a) - n\|x - a\|, \ x \in V_{s,i}.$$

Clearly g_a and h_a are Lipschitz with constant n and, since $M \subset L_n$ and $\operatorname{diam} V_{s,i} \leq 1/n$, we have $h_a \leq f \leq g_a$ on $V_{s,i}$. Consequently we obtain that $g_{s,i}$ and $h_{s,i}$ are finite and Lipschitz with constant n on $V_{s,i}$,

$$h_{s,i} \leq f \leq g_{s,i} \ \text{ on } \ V_{s,i} \quad \text{ and } \quad h_{s,i}(a) = f(a) = g_{s,i}(a) \ \text{ for each } \ a \in M \cap V_{s,i}. \tag{2.3}$$

Define the functions g_i, h_i on $D_i^* := \bigcup_{s\in S^*} V_{s,i}$ by the equalities $g_i(x) := g_{s,i}(x)$ and $h_i(x) := h_{s,i}(x)$ for $x \in V_{s,i}$. We will show that

$$g_i \ \text{ and } \ h_i \ \text{ are Lipschitz on } \ D_i^*. \tag{2.4}$$

To this end, consider arbitrary $x_1 \in D_i^*$, $x_2 \in D_i^*$. Let $x_1 \in V_{s_1,i}$ and $x_2 \in V_{s_2,i}$. If $s_1 = s_2$, then we know that $\|g_i(x_1) - g_i(x_2)\| \leq n\,\|x_1 - x_2\|$. If $s_1 \neq s_2$, then $\|x_1 - x_2\| \geq 2^{-i}$ by (2.2), and so

$$\|g_i(x_1) - g_i(x_2)\| \leq 2K = (2K2^i)2^{-i} \leq (2K2^i)\|x_1 - x_2\|.$$

Since the same inequalities hold also for h_i, (2.4) follows.

So we can (see [18]) extend g_i and h_i to Lipschitz functions g and h defined on all X. Let N_g and N_h be the sets of all points of Fréchet (resp. Gâteaux) non-differentiability of g and h. By (R) we have $N_g \in \mathcal{I}$ and $N_h \in \mathcal{I}$. So it is sufficient to prove $M \subset N_g \cup N_h$. Suppose on the contrary that there exists $a \in M$ such that both g and h are Fréchet (resp. Gâteaux) differentiable at a. Let $a \in V_{s,i}$. Using the facts that $g = g_{s,i}$ and $h = h_{s,i}$ on $V_{s,i}$ and (2.3), we clearly obtain that f is Fréchet (resp. Gâteaux) differentiable at a, which contradicts $a \in A$. $\square$

Remark 2.2. (i) It is easy to see that Proposition 2.1 holds (with the same proof) for an arbitrary notion of differentiability ("A-differentiability") having the following natural property:

(*) If $a \in X$, $h(a) = f(a) = g(a)$, $h \le f \le g$ on a neighbourhood of a and both g and h are A-differentiable at a, then f is A-differentiable at a.

In particular, Proposition 2.1 holds for Hadamard differentiability. Moreover, we can suppose that X is a metric space in which a notion of A-differentiability satisfying (*) is defined.

(ii) It is an open question whether Proposition 2.1 holds in the vector case (i.e., for mappings from X to Y, where X, Y are Banach spaces).

D. Bongiorno [3, pp. 518–519] has shown that the vector version of Proposition 2.1 holds for Gâteaux differentiability if X is separable and

(**) each Lipschitz mapping $f : A \to Y$ (where $A \subset X$) has a Lipschitz extension $f^* : X \to Y$.

She used the method (based on consideration of differentiability points of suitable distance functions) which was probably first used in [2] and was recently used in several articles, e.g. in [8].

Using this main idea, it will be proved in [17] that the vector version of Proposition 2.1 holds for "almost all derivatives" (also for non-separable X), if the condition (**) holds. The proof of this general observation is very simple and provides probably an optimal application of the mentioned idea from [2] and [3].

3. Main results

Proposition 2.1 together with Theorem LP and Theorem C immediately imply the following generalizations of Theorem LP and Theorem C.

Theorem 3.1. *Let X be a Banach space. Suppose that*

(i) *X^* is separable and every σ-porous set in X is Γ-null, or*
(ii) *X is a closed subspace of $c_0(\Delta)$, where Δ is uncountable, or*
(iii) *$X = C(K)$, where K is a scattered compact topological space.*

Let G be an open subset of X and f an arbitrary real function on X. Then the set of all points at which f is Lipschitz and is not Fréchet differentiable is Γ-null.

In particular, each pointwise Lipschitz real function on G is Γ-almost everywhere Fréchet differentiable.

Corollary 3.2. *Let X be a Banach space such that X^* is separable and every σ-porous set in X is Γ-null. Suppose that $f : X \to \mathbb{R}$ is continuous (or, more generally, has the Baire property and its restriction to each line is continuous).*

Then f is Fréchet differentiable at Γ-almost every point x at which it is Gâteaux differentiable.

Proof: By [26, Lemma 3.7] the set M of all points at which f is Gâteaux differentiable but not Lipschitz is σ-directionally porous set. Since by [15, Remark 5.2.4] each σ-directionally porous set is Γ-null, our assertion follows from Theorem 3.1. □

Remark 3.3. It is still possible that the Rademacher theorem for Fréchet differentiability holds (with a σ-ideal $\mathcal{I}$ different from that of Γ-null sets) for real Lipschitz functions on a separable Asplund X which does not satisfy the assumptions of Theorem LP (e.g. for $X = \ell_2$).

If such a theorem exists, the following analogue of Corollary 3.2 also holds:

Each continuous real function f on X is Fréchet differentiable at $\mathcal{I}$-almost every point x at which it is Gâteaux differentiable.

Indeed, we can repeat the proof of Corollary 3.2, using now the fact that $M \in \mathcal{I}$ by [21, Proposition 14].

Recall that Theorem 3.1(i) holds if X is a subspace of c_0, or $X = C(K)$ with K compact countable, or X is the Tsirelson space (see the references after Theorem LP).

To these cases apply also Corollary 3.2 and the following generalization of Theorem LP (together with its Corollary 3.5).

Theorem 3.4. *Let X be a Banach space such that X^* is separable and each porous set in X is Γ-null. Then each cone-monotone function on X is Γ-almost everywhere Fréchet differentiable.*

Proof: Since X is separable, [9, Theorem 15] implies that there exists a set $C \in \tilde{\mathcal{C}}$ such that f is Gâteaux differentiable at all points of $X \setminus C$. Consequently ([10, Lemma 2.5]), f is Lipschitz at all points of $X \setminus C$. Thus Theorem 3.1 implies that there exists a Γ-null set D such that f is Fréchet differentiable at all points of $X \setminus (C \cup D)$. Consequently the assertion of the theorem follows by (1.1) and (1.2). $\square$

By a standard method (see [6] or [4]) we obtain:

Corollary 3.5. *Let X be a Banach space such that X^* is separable and each porous set in X is Γ-null. Then each real continuous quasiconvex function on X is Γ-almost everywhere Fréchet differentiable.*

Proof: Recall that f is quasiconvex if and only if $S_\lambda(f) := \{x \in X : f(x) \leq \lambda\}$ is convex for every $\lambda \in \mathbb{R}$. Set $\overline{\lambda} := \inf\{f(x) : x \in X\}$. If $f(x) > \overline{\lambda}$, then f is cone-monotone on an open neighbourhood of x (see [6, Theorem 3.1] or [4, Proposition 2]). Since X is separable and (1.2) holds, we easily see that Theorem 3.4 implies that f is Γ-almost everywhere Fréchet differentiable on the open set $\{x \in X : f(x) > \overline{\lambda}\}$. So we are done if $S_{\overline{\lambda}} = \{x \in X : f(x) = \overline{\lambda}\} = \emptyset$. If $S_{\overline{\lambda}} \neq \emptyset$, then we distinguish two cases.

If $S_{\overline{\lambda}}$ is nowhere dense, then it is easy to prove that $S_{\overline{\lambda}}$ is a porous set (see the proof of [20, Theorem 2]) and so $S_{\overline{\lambda}}$ is Γ-null.

In the opposite case f is constant (and so Fréchet differentiable) on the nonempty open set $\operatorname{int}(S_{\overline{\lambda}})$. It is almost obvious (considering supporting hyperplanes) that the boundary of $S_{\overline{\lambda}}$ is a porous set, and so it is Γ-null. $\square$

It is an interesting open question whether the assumption that each porous set in X is Γ-null can be omitted in Corollary 3.5 (also the case of a Lipschitz quasiconvex f is open). However, the case of Gâteaux differentiability can be proved easily modifying some details of the proof of Corollary 3.5.

Moreover, it will be proved in a forthcoming article written jointly with J. Tišer that each continuous quasiconvex function on a separable Banach space is Γ-almost everywhere Gâteaux differentiable.

Note that the results of [22] imply that each real continuous quasiconvex function on X is Gâteaux differentiable outside a Haar null set if X is separable *and reflexive*.

Theorem 3.4 generalizes to some non-separable spaces (Theorem 3.9). We will prove this generalization using the separable reduction method of [15], which is based on the notion of a rich family of separable subspaces (see Definition 3.6 below). (Other possibility would be modifying the proof of [7, Theorem 6.18], which uses the separable reduction method based on the set-theoretic notion of an elementary submodel.)

Definition 3.6. Let X be a normed linear space. A family $\mathcal{F}$ of closed separable subspaces of X is called a *rich family* on X if the following holds.

(R1) If $Y_i \in \mathcal{F}$ $(i \in \mathbb{N})$ and $Y_1 \subset Y_2 \subset \dots$, then $\overline{\bigcup\{Y_n : n \in \mathbb{N}\}} \in \mathcal{F}$.

(R2) For each closed separable subspace Y_0 of X there exists $Y \in \mathcal{F}$ such that $Y_0 \subset Y$.

The basic fact ([15, Proposition 3.6.2]) concerning rich families reads as follows.

Fact 1. *Let X be a normed linear space and let $\{\mathcal{F}_n : n \in \mathbb{N}\}$ be rich families of closed separable subspaces of X. Then $\mathcal{F} := \bigcap\{\mathcal{F}_n : n \in \mathbb{N}\}$ is also a rich family of closed separable subspaces of X.*

We will need also the following results of [15].

Fact 2 ([15, Corollary 5.6.2])**.** *Let X be a Banach space and $E \subset X$ a Borel set. Then E is Γ-null in X if and only if there exists a rich family $\mathcal{F}$ on X such that for every $Y \in \mathcal{F}$, $E \cap Y$ is Γ-null in Y.*

Fact 3 ([15, Theorem 3.6.10])**.** *Let X and Z be Banach spaces and $f : X \to Z$ a function. Then there exists a rich family $\mathcal{F}$ on X such that for every $Y \in \mathcal{F}$, f is Fréchet differentiable* (*as a function on X*) *at every $x \in Y$ at which its restriction to Y is Fréchet differentiable* (*as a function on Y*)*.*

Moreover, we will need the following lemmas.

Lemma 3.7. *Let Y be a closed separable subspace of $c_0(\Delta)$, where Δ is uncountable. Then Y is linearly isometric to a closed subspace of c_0.*

Proof: It is easy to show that there exists an infinite countable $C \subset \Delta$ such that $x(s) = 0$ whenever $x \in Y$ and $s \notin C$. Obviously, the mapping $x \mapsto x\restriction_C$ is a linear isometry of Y on a closed subspace of $c_0(C)$. □

Lemma 3.8. *Let $X = C(K)$, where K is a scattered compact topological space. Then there exists a rich family $\mathcal{F}$ on X such that each $Y \in \mathcal{F}$ is linearly isometric to a space $C(L)$, where L is a countable compact.*

PROOF: We can define $\mathcal{F}$ as the family of all closed separable subalgebras of $C(K)$. The fact that $\mathcal{F}$ is a rich family is standard. The proof that each $Y \in \mathcal{F}$ is isometric to a space $C(L)$, where L is a countable compact is more difficult. A proof of this fact is sketched in [13, Proof of Theorem 2.1, p. 263]. A different (more natural) proof, which is implicitly contained in [7], is the following:

Let Y be a closed separable subalgebra of $C(K)$. Write $x \sim y$ if $f(x) = f(y)$ for each $f \in Y$. Let $L := K/_\sim$ be the quotient topological space and $q : K \to L$ the natural quotient mapping (see [11, p. 90]). Using [11, Proposition 2.4.9] it is not difficult to prove that $\sim$ is a closed equivalence relation (in the sense of [11]). So [11, Theorem 3.2.11] implies that L is a compact space. Let $Y^* := \{f^* \in C(L) : f^* \circ q \in Y\}$. Then Y^* is clearly a closed subalgebra of $C(L)$ which contains all constant functions. So the Stone-Weierstrass theorem implies that $Y^* = C(L)$. Let $\eta : C(L) \to Y$ be defined by $\eta(f^*) := f^* \circ q$. Then η is clearly a linear isometry and, using [11, Proposition 2.4.2], it is easy to see that η is surjective.

Consequently $C(L)$ is separable, which implies that L is metrizable (see, e.g., [5, Theorem 6.5]). Since a continuous image of a scattered compact is a scattered compact and a metrizable scattered compact is countable (see [12, Lemmas 12.24 and 12.25]), we obtain that L is a countable compact. $\square$

Theorem 3.9. *Suppose that either*

(i) *X is a subspace of $c_0(\Delta)$, where Δ is uncountable, or*
(ii) *$X = C(K)$, where K is a scattered compact topological space.*

Let G be an open subset of X and let f be a cone-monotone function on G. Then f is Fréchet differentiable Γ-almost everywhere on G.

PROOF: We will suppose that $G = X$; the proof for a general G is essentially the same. Let $N \subset X$ be the set of all Fréchet non-differentiability points of f. The set N is Borel (it is true even for an arbitrary f, see [23, Theorem 2] or [15, Corollary 3.5.5]).

Let $\mathcal{F}_1$ be the rich family on X from Fact 3 (which corresponds to f).

Let f be monotone with respect to a cone C and $x_0 \in \operatorname{int} C$. The family $\mathcal{F}_2$ of all separable subspaces of X which contain x_0 is clearly rich.

Now we define $\mathcal{F}_3$, distinguishing cases (i) and (ii). In the case (i), let $\mathcal{F}_3$ be the family of all separable subspaces of X. In the case (ii), let $\mathcal{F}_3$ be the rich family from Lemma 3.8. Fact 1 implies that $\mathcal{F} := \mathcal{F}_1 \cap \mathcal{F}_2 \cap \mathcal{F}_3$ is a rich family. Now consider an arbitrary $Y \in \mathcal{F}$. Since $Y \in \mathcal{F}_2$, we can easily see that the restriction $f^* := f \restriction_Y$ is a cone-monotone function on Y. Since $Y \in \mathcal{F}_3$, the set N^* of all Fréchet non-differentiability points from Y of f^* is Γ-null in Y by Lemma 3.7, Lemma 3.8 and Theorem 3.4. Since $Y \in \mathcal{F}_1$, we obtain that $N \cap Y = N^*$ is Γ-null in Y. So Fact 2 implies that N is Γ-null in X. $\square$

By the same way (even simpler, without consideration of $\mathcal{F}_2$), but using Corollary 3.5 instead of Theorem 3.4, we clearly obtain:

Theorem 3.10. *Suppose that either*

(i) *X is a subspace of $c_0(\Delta)$, where Δ is uncountable, or*
(ii) *$X = C(K)$, where K is a scattered compact topological space.*

Let f be a continuous quasiconvex function on X. Then f is Fréchet differentiable Γ-almost everywhere on X.

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Charles University, Faculty of Mathematics and Physics, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail: zajicek@karlin.mff.cuni.cz

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