Dihedral-like constructions of automorphic loops

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Abstract. Automorphic loops are loops in which all inner mappings are automorphisms. We study a generalization of the dihedral construction for groups. Namely, if $(G,+)$ is an abelian group, $m \geq 1$ and $\alpha \in \mathrm{Aut}(G)$, let $\mathrm{Dih}(m,G,\alpha)$ be defined on $\mathbb{Z}_m \times G$ by

$$(i,u)(j,v) = (i \oplus j,\, ((-1)^j u + v)\alpha^{ij}).$$

The resulting loop is automorphic if and only if $m = 2$ or ($\alpha^2 = 1$ and m is even). The case $m = 2$ was introduced by Kinyon, Kunen, Phillips, and Vojtěchovský. We present several structural results about the automorphic dihedral loops in both cases.

Keywords: dihedral automorphic loop; automorphic loop; inner mapping group; multiplication group; nucleus; commutant; center; commutator; associator subloop; derived subloop

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1. Introduction

A set Q with a binary operation $(\cdot)$ is *a loop* if for every $x \in Q$ the right and left translations $R_x,\ L_x : Q \longrightarrow Q,\ yR_x = y \cdot x,\ yL_x = x \cdot y$ are bijections of Q, and if there is a neutral element $1 \in Q$ such that $1 \cdot x = x \cdot 1 = x$ for every $x \in Q$.

Let Q be a loop. The group generated by R_x and L_x for all $x \in Q$ is called the *multiplication group* of Q and it is denoted by $\mathrm{Mlt}(Q)$. The subgroup of $\mathrm{Mlt}(Q)$ stabilizing the neutral element of Q is called the *inner mapping group* of Q and it is denoted by Inn(Q). It is well known that the inner mapping group Inn(Q) is the permutation group generated by

$$R_{x,y} = R_x R_y R_{xy}^{-1}, \quad T_x = R_x L_x^{-1}, \quad L_{x,y} = L_x L_y L_{yx}^{-1},$$

where $x, y \in Q$.

A loop is *automorphic* (also known as *A-loop*) if $\mathrm{Inn(Q)} \leq \mathrm{Mlt}(Q)$, that is, if every inner mapping of Q is an automorphism of Q. Note that groups are automorphic loops, but the converse is certainly not true.

Automorphic loops were first studied in 1956 by Bruck and Paige [3]. Structure theory for commutative and general automorphic loops was developed in

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[1], [5], [6]. In this paper, we generalize a construction of dihedral automorphic loops introduced by Kinyon, Kunen, Phillips and Vojtěchovský [1], where they focused on the special case $m = 2$. Here, we consider for an integer $m \geq 1$, an abelian group $(G, +)$ and an automorphism α of G the loop $\mathrm{Dih}(m, G, \alpha)$ defined on $\mathbb{Z}_m \times G$ by

$$(i, u)(j, v) = (i \oplus j,\ (s_j u + v)\alpha^{ij}), \tag{1}$$

where $s_j = (-1)^{j \bmod m}$, and where we interpret α^{ij} as ordinary integral exponent. To make the multiplication formula unambiguous we demand that $i, j \in \{0, 1, \ldots, m-1\}$. Then we have $\alpha^i \alpha^j = \alpha^{i+j}$. There are several observations that we will use without reference for $i, j \in \mathbb{Z}_m$, $u \in G$:

- $s_i(s_j u) = (s_i s_j)u$, so we can write this as $s_i s_j u$,
- $s_i s_j u = s_j s_i u$,
- $s_i(u\alpha) = (s_i u)\alpha$,
- $s_i s_j = s_{i \oplus j}$, when m is even.

Note that with $m = 1$ the multiplication (1) reduces to $(i, u)(j, v) = (i+j, u+v)$, so $\mathrm{Dih}(1, G, \alpha) = \mathbb{Z}_1 \times G = G$. *We will thus assume throughout the paper that* $m > 1$.

This paper is organized as follows: Section 2 presents definitions and preliminary results about A-loops. We recall without proofs some facts from [1]. In Section 3 we determine all parameters m, G, α that yield automorphic loops. In Section 4 we show how to obtain the nuclei, the commutant and the center. In Section 5 we calculate the associator subloop $\mathrm{A}(Q)$ and the derived subloop Q'.

2. Definitions and preliminary results

In this section we introduce relevant definitions of loop theory [2], and we present some results on automorphic loops.

Definition 2.1. The *dihedral group* of order $2n$, denoted by D_{2n}, is the group generated by two elements x and y with presentation $x^2 = y^n = 1$ and $x \cdot y = y^{n-1} \cdot x$.

The group D_{2n} is isomorphic to $\mathrm{Dih}(2, \mathbb{Z}_n, 1)$. The generalized dihedral group $\mathrm{D}_{2n}(G)$ is isomorphic to $\mathrm{Dih}(2, G, 1)$.

Since it suffices to check the automorphic condition on the generators of Inn(Q), we see that a loop Q is an automorphic loop if and only if, for all $x, y, u, v \in Q$,

$$(uv)R_{x,y} = uR_{x,y} \cdot vR_{x,y}, \tag{A_r}$$

$$(uv)L_{x,y} = uL_{x,y} \cdot vL_{x,y}, \tag{A_ℓ}$$

$$(uv)T_x = uT_x \cdot vT_x. \tag{A_m}$$

In fact it is not necessary to verify all of the conditions (A_r), (A_ℓ) and (A_m):

Proposition 2.2 ([4])**.** *Let Q be a loop satisfying (A_m) and (A_ℓ). Then Q is automorphic.*

Definition 2.3. The *commutant* of a loop Q, denoted by $C(Q)$, is the set of all elements that commute with every element of Q. In symbols,

$$C(Q) = \{a \in Q : x \cdot a = a \cdot x, \forall\, x \in Q\}.$$

Definition 2.4. The *left, right, and middle nucleus* of a loop Q are defined, respectively, by

$$\begin{aligned} N_\lambda(Q) &= \{a \in Q :\ ax \cdot y = a \cdot xy, \forall x, y \in Q\}, \\ N_\rho(Q) &= \{a \in Q :\ xy \cdot a = x \cdot ya, \forall x, y \in Q\}, \\ N_\mu(Q) &= \{a \in Q :\ xa \cdot y = x \cdot ay, \forall x, y \in Q\}. \end{aligned}$$

The nucleus of Q is defined as $N(Q) = N_\lambda(Q) \cap N_\rho(Q) \cap N_\mu(Q)$.

Each of the nuclei is a subloop. All nuclei are in fact groups.

Definition 2.5. The center $Z(Q)$ of a loop Q is the set of all elements of Q that commute and associate with all other elements of Q. It can be characterized as

$$Z(Q) = C(Q) \cap N(Q).$$

Definition 2.6. Let S be a subloop of a loop Q. Then S is *normal* if for all $a, b \in Q$

$$aS = Sa,\ (aS)b = a(Sb),\ a(bS) = (ab)S.$$

The center is always a normal subloop of Q.

Proposition 2.7 ([3])**.** *Let Q be an automorphic loop. Then:*

(i) $N_\lambda(Q) = N_\rho(Q) \subseteq N_\mu(Q)$;
(ii) *each nucleus is normal in Q.*

Definition 2.8. Let Q be a loop and $x, y, z \in Q$. The commutator $[x, y]$ is the unique element of Q satisfying the equation

$$x \cdot y = (y \cdot x) \cdot [x, y].$$

The associator $[x, y, z]$ is the unique element of Q satisfying the equation

$$(x \cdot y) \cdot z = (x \cdot (y \cdot z)) \cdot [x, y, z].$$

Definition 2.9. The *associator subloop* of a loop Q, denoted by $A(Q)$, is the smallest normal subloop of Q containing all associators $[x, y, z]$ of Q. Equivalently, $A(Q)$ is the smallest normal subloop of Q such that $Q/A(Q)$ is associative.

Definition 2.10. The *derived subloop* of a loop Q, denoted by Q', is the smallest normal subloop of Q containing all commutators $[x, y]$ and all associators $[x, y, z]$ of Q. Equivalently, Q' is the smallest normal subloop of Q such that Q/Q' is a commutative group.

3. Parameters that yield automorphic loops

For an abelian group $(G,+)$ denote by $2G$ the subgroup $2G=\{u+u;\ u\in G\}$. Note that if $\alpha\in\mathrm{Aut}(G)$ then the restriction $\alpha\restriction_{2G}$ of α to $2G$ is an automorphism of $2G$.

Lemma 3.1. *Let $Q=\mathrm{Dih}(m,G,1)$. Then Q is a group iff m is even or $2G=0$.*

Proof: With $\alpha=1$ the multiplication formula (1) becomes $(i,u)(j,v)=(i\oplus j,s_ju+v)$. We have

$$(i,u)(j,v)\cdot(k,w)=(i\oplus j,s_ju+v)(k,w)=(i\oplus j\oplus k,s_k(s_ju+v)+w),$$
$$(i,u)\cdot(j,v)(k,w)=(i,u)(j\oplus k,s_kv+w)=(i\oplus j\oplus k,s_{j\oplus k}u+s_kv+w),$$

so Q is a group iff

$$s_ks_ju=s_{j\oplus k}u \tag{2}$$

for every $j,k\in\mathbb{Z}_m$ and every $u\in G$.

If $2G=0$ then $u=-u$ and (2) holds. If m is even then $s_ks_j=s_{j\oplus k}$ and (2) holds again. Conversely, suppose that (2) holds. If m is even, we are done, so suppose that m is odd. With $k=1$, $j=m-1$ the identity (2) yields $-u=u$, or $2G=0$. □

3.1 Middle inner mappings. Recall that $yT_x=x\backslash(yx)$.

Lemma 3.2. *Let $Q=\mathrm{Dih}(m,G,\alpha)$ and (i,u), $(j,v)\in Q$. Then*

$$(j,v)T_{(i,u)}=(j,s_iv+(1-s_j)u). \tag{3}$$

Proof: Note that $(j,v)T_{(i,u)}=(k,w)$ iff $(j,v)(i,u)=(i,u)(k,w)$ iff $(j\oplus i,(s_iv+u)\alpha^{ij})=(i\oplus k,(s_ku+w)\alpha^{ik})$. We deduce $k=j$, and extend the chain of equivalences with $(s_iv+u)\alpha^{ij}=(s_ju+w)\alpha^{ij}$ iff $s_iv+u=s_ju+w$ iff $w=s_iv+(1-s_j)u$. □

Lemma 3.3. *Let $Q=\mathrm{Dih}(m,G,\alpha)$ and $(i,u)\in Q$. Then $T_{(i,u)}\in\mathrm{Aut}(Q)$ iff*

$$(1-s_{j\oplus k})u=(1-s_js_k)u\alpha^{jk} \tag{4}$$

for every $j,k\in\mathbb{Z}_m$.

Proof: We will use (3) without reference. We have

$$\begin{aligned}((j,v)(k,w))T_{(i,u)}&=(j\oplus k,(s_kv+w)\alpha^{jk})T_{(i,u)}\\&=(j\oplus k,s_i(s_kv+w)\alpha^{jk}+(1-s_{j\oplus k})u)\\&=(j\oplus k,s_is_kv\alpha^{jk}+s_iw\alpha^{jk}+(1-s_{j\oplus k})u),\end{aligned}$$

while

$$(j,v)T_{(i,u)}\cdot(k,w)T_{(i,u)}=(j,s_iv+(1-s_j)u)\cdot(k,s_iw+(1-s_k)u)$$

$$
\begin{aligned}
&= (j \oplus k, [s_k(s_i v + (1 - s_j)u) + s_i w + (1 - s_k)u]\alpha^{jk}) \\
&= (j \oplus k, s_k s_i v\alpha^{jk} + s_i w\alpha^{jk} + [(s_k - s_k s_j + 1 - s_k)u]\alpha^{jk}) \\
&= (j \oplus k, s_k s_i v\alpha^{jk} + s_i w\alpha^{jk} + (1 - s_k s_j)u\alpha^{jk}),
\end{aligned}
$$

so $T_{(i,u)} \in \mathrm{Aut}(Q)$ iff (4) holds for every $j, k \in \mathbb{Z}_m$. □

Let us call a loop Q satisfying (A_m) a *middle automorphic loop*.

Proposition 3.4. *Let $Q = \mathrm{Dih}(m, G, \alpha)$.*

(i) *If $m = 2$ then Q is a middle automorphic loop.*
(ii) *If $m > 2$ is odd then Q is a middle automorphic loop iff $2G = 0$.*
(iii) *If $m > 2$ is even then Q is a middle automorphic loop iff $\alpha^2\restriction_{2G} = 1_{2G}$.*

Proof: Consider $T_{(i,u)}$. Suppose that $m = 2$. A quick inspection of all cases $j, k \in \{0, 1\}$ shows that (4) always holds.

Suppose that $m > 2$ is odd. With $j = 2$ and $k = m - 1$, condition (4) becomes $(1 - s_{2\oplus(m-1)})u = (1 - s_2 s_{m-1})u\alpha^{2(m-1)}$, or $2u = 0$, so we certainly must have $2G = 0$ for every $T_{(i,u)}$ to be an automorphism. Conversely, when $2G = 0$ then (4) reduces to $0 = 0$.

Suppose that $m > 2$ is even. Then (4) becomes $(1 - s_{j\oplus k})u = (1 - s_{j\oplus k})u\alpha^{jk}$. When $j \oplus k$ is even then this becomes $0 = 0$. Suppose that $j \oplus k$ is odd. Then one of j, k is odd and the other is even, so that jk is even, and (4) becomes $2u = (2u)\alpha^{2\ell}$ for some ℓ. With $j = 2$, $k = 1$ we obtain $2u = (2u)\alpha^2$, which is equivalent to $\alpha^2\restriction_{2G} = 1_{2G}$. Conversely, when $\alpha^2\restriction_{2G} = 1_{2G}$ then (4) holds. □

3.2 Left inner mappings. Recall that $zL_{x,y} = (yx)\backslash(y(xz))$.

Lemma 3.5. *Let $Q = \mathrm{Dih}(m, G, \alpha)$ and (i, u), (j, v), $(k, w) \in Q$. Then*

$$
\begin{aligned}
(5) \quad (k, w)L_{(j,v),(i,u)} = (k, s_{j\oplus k}u\alpha^{i(j\oplus k)-(i\oplus j)k} &+ s_k v\alpha^{jk+i(j\oplus k)-(i\oplus j)k} \\
&+ w\alpha^{jk+i(j\oplus k)-(i\oplus j)k} - s_k s_j u\alpha^{ij} - s_k v\alpha^{ij}).
\end{aligned}
$$

Proof: The following conditions are equivalent:

$$
\begin{aligned}
(k, w)L_{(j,v),(i,u)} &= (\ell, x), \\
(i, u) \cdot (j, v)(k, w) &= (i, u)(j, v) \cdot (\ell, x), \\
(i, u)(j \oplus k, (s_k v + w)\alpha^{jk}) &= (i \oplus j, (s_j u + v)\alpha^{ij})(\ell, x), \\
(i \oplus j \oplus k, (s_{j\oplus k}u + (s_k v + w)\alpha^{jk})\alpha^{i(j\oplus k)}) &= (i \oplus j \oplus \ell, (s_\ell(s_j u + v)\alpha^{ij} + x)\alpha^{(i\oplus j)\ell}).
\end{aligned}
$$

We deduce that $\ell = k$ and the result follows upon solving for x in the equation

$$
(s_{j\oplus k}u + (s_k v + w)\alpha^{jk})\alpha^{i(j\oplus k)} = (s_k(s_j u + v)\alpha^{ij} + x)\alpha^{(i\oplus j)k}. \qquad \square
$$

Lemma 3.6. *Let $Q = \mathrm{Dih}(m, G, \alpha)$ and (i, u), $(j, v) \in Q$. Then $L_{(j,v),(i,u)} \in \mathrm{Aut}(Q)$ iff*

$$\begin{aligned}
&s_\ell s_{j\oplus k} u\alpha^{i(j\oplus k)-(i\oplus j)k+k\ell} + s_\ell s_k v\alpha^{jk+i(j\oplus k)-(i\oplus j)k+k\ell} \\
&\quad + s_\ell w\alpha^{jk+i(j\oplus k)-(i\oplus j)k+k\ell} - s_\ell s_k s_j u\alpha^{ij+k\ell} - s_\ell s_k v\alpha^{ij+k\ell} \\
&\quad + s_{j\oplus\ell} u\alpha^{i(j\oplus\ell)-(i\oplus j)\ell+k\ell} + s_\ell v\alpha^{j\ell+i(j\oplus\ell)-(i\oplus j)\ell+k\ell} \\
&\quad + x\alpha^{j\ell+i(j\oplus\ell)-(i\oplus j)\ell+k\ell} - s_\ell s_j u\alpha^{ij+k\ell} - s_\ell v\alpha^{ij+k\ell} \\
&= s_{j\oplus k\oplus\ell} u\alpha^{i(j\oplus k\oplus\ell)-(i\oplus j)(k\oplus\ell)} + s_{k\oplus\ell} v\alpha^{j(k\oplus\ell)+i(j\oplus k\oplus\ell)-(i\oplus j)(k\oplus\ell)} \\
&\quad + s_\ell w\alpha^{k\ell+j(k\oplus\ell)+i(j\oplus k\oplus\ell)-(i\oplus j)(k\oplus\ell)} + x\alpha^{k\ell+j(k\oplus\ell)+i(j\oplus k\oplus\ell)-(i\oplus j)(k\oplus\ell)} \\
&\quad - s_{k\oplus\ell} s_j u\alpha^{ij} - s_{k\oplus\ell} v\alpha^{ij}
\end{aligned} \tag{6}$$

for every $k, \ell \in \mathbb{Z}_m$ and every $w, x \in G$.

Proof: This follows from Lemma 3.5, upon comparing $(k, w)L_{(j,v),(i,u)} \cdot (\ell, x)L_{(j,v),(i,u)}$ with $((k, u)(\ell, x))L_{(j,v),(i,u)}$. □

Let us call a loop satisfying (A_ℓ) a *left automorphic loop*. We deduce that $Q = \mathrm{Dih}(m, G, \alpha)$ is a left automorphic loop iff (6) holds for every i, j, k, $\ell \in \mathbb{Z}_m$ and every u, v, w, $x \in G$. We show that this very complicated condition is equivalent to two comparatively simple conditions, which we then analyze separately.

First, setting $u = v = w = 0$ and letting x range over G in (6) yields the condition

$$\alpha^{j\ell+i(j\oplus\ell)-(i\oplus j)\ell+k\ell} = \alpha^{k\ell+j(k\oplus\ell)+i(j\oplus k\oplus\ell)-(i\oplus j)(k\oplus\ell)}.$$

With $\ell = 0$ this further simplifies to

$$\alpha^{ij} = \alpha^{jk+i(j\oplus k)-(i\oplus j)k},$$

which is equivalent to

$$\alpha^{ij+(i\oplus j)k} = \alpha^{i(j\oplus k)+jk}. \tag{7}$$

Suppose that (7) holds for every i, j, k. Then the automorphisms at w in (6) agree since

$$\begin{aligned}
\alpha^{jk+i(j\oplus k)-(i\oplus j)k} &= \alpha^{ij+(i\oplus j)k-(i\oplus j)k} = \alpha^{ij} = \alpha^{ij+(i\oplus j)(k\oplus\ell)-(i\oplus j)(k\oplus\ell)} \\
&= \alpha^{i(j\oplus k\oplus\ell)+j(k\oplus\ell)-(i\oplus j)(k\oplus\ell)}.
\end{aligned}$$

Focusing on x in (6), the following conditions are equivalent:

$$\begin{aligned}
\alpha^{j\ell+i(j\oplus\ell)-(i\oplus j)\ell+k\ell} &= \alpha^{k\ell+j(k\oplus\ell)+i(j\oplus k\oplus\ell)-(i\oplus j)(k\oplus\ell)}, \\
\alpha^{j\ell+i(j\oplus\ell)-(i\oplus j)\ell} &= \alpha^{j(k\oplus\ell)+i(j\oplus k\oplus\ell)-(i\oplus j)(k\oplus\ell)}, \\
\alpha^{j\ell+i(j\oplus\ell)+(i\oplus j)(k\oplus\ell)} &= \alpha^{j(k\oplus\ell)+i(j\oplus k\oplus\ell)+(i\oplus j)\ell}, \\
\alpha^{ij+(i\oplus j)\ell+(i\oplus j)(k\oplus\ell)} &= \alpha^{ij+(i\oplus j)(k\oplus\ell)+(i\oplus j)\ell},
\end{aligned}$$

where we have used (7) twice in the last step. Since the last identity is trivially true, we see that (7) implies that the automorphisms at x in (6) agree, too. Let us now focus on v in (6). Using (7), the following conditions are equivalent:

$$\begin{aligned}
&s_\ell s_k v\alpha^{jk+i(j\oplus k)-(i\oplus j)k+k\ell} - s_\ell s_k v\alpha^{ij+k\ell} + s_\ell v\alpha^{j\ell+i(j\oplus\ell)-(i\oplus j)\ell+k\ell} - s_\ell v\alpha^{ij+k\ell}\\
&= s_{k\oplus\ell} v\alpha^{j(k\oplus\ell)+i(j\oplus k\oplus\ell)-(i\oplus j)(k\oplus\ell)} - s_{k\oplus\ell} v\alpha^{ij},\\
&s_\ell s_k v\alpha^{ij+(i\oplus j)k-(i\oplus j)k+k\ell} - s_\ell s_k v\alpha^{ij+k\ell} + s_\ell v\alpha^{ij+(i\oplus j)\ell-(i\oplus j)\ell+k\ell} - s_\ell v\alpha^{ij+k\ell}\\
&= s_{k\oplus\ell} v\alpha^{ij+(i\oplus j)(k\oplus\ell)-(i\oplus j)(k\oplus\ell)} - s_{k\oplus\ell} v\alpha^{ij}.
\end{aligned}$$

Upon canceling several α^{n-n} and the automorphism α^{ij} present in all summands, we see that the above is equivalent to

$$s_\ell s_k v\alpha^{k\ell} - s_\ell s_k v\alpha^{k\ell} + s_\ell v\alpha^{k\ell} - s_\ell v\alpha^{k\ell} = s_{k\oplus\ell} v - s_{k\oplus\ell} v,$$

which is trivially true. Hence (7) implies that the automorphisms at v in (6) agree, too. Finally, we focus on u in (6). Note that the equality

$$\alpha^{i(j\oplus\ell)-(i\oplus j)\ell} = \alpha^{ij-j\ell}$$

immediately follows from (7). Using this identity, the following conditions are equivalent:

$$\begin{aligned}
&s_\ell s_{j\oplus k} u\alpha^{i(j\oplus k)-(i\oplus j)k+k\ell} - s_\ell s_k s_j u\alpha^{ij+kl} + s_{j\oplus\ell} u\alpha^{i(j\oplus\ell)-(i\oplus j)\ell+k\ell} - s_\ell s_j u\alpha^{ij+k\ell}\\
&= s_{j\oplus k\oplus\ell} u\alpha^{i(j\oplus(k\oplus\ell))-(i\oplus j)(k\oplus\ell)} - s_{k\oplus\ell} s_j u\alpha^{ij},\\
&s_\ell s_{j\oplus k} u\alpha^{ij-jk+k\ell} - s_\ell s_k s_j u\alpha^{ij+k\ell} + s_{j\oplus\ell} u\alpha^{ij-j\ell+k\ell} - s_\ell s_j u\alpha^{ij+k\ell}\\
&= s_{j\oplus k\oplus\ell} u\alpha^{ij-j(k\oplus\ell)} - s_{k\oplus\ell} s_j u\alpha^{ij},\\
&s_\ell s_{j\oplus k} u\alpha^{-jk+k\ell} - s_\ell s_k s_j u\alpha^{k\ell} + s_{j\oplus\ell} u\alpha^{-j\ell+k\ell} - s_\ell s_j u\alpha^{k\ell}\\
&= s_{j\oplus k\oplus\ell} u\alpha^{-j(k\oplus\ell)} - s_{k\oplus\ell} s_j u.
\end{aligned}$$

Upon rearranging, we obtain the identity

$$\begin{aligned}
(8)\quad &s_\ell s_{j\oplus k} u\alpha^{-jk+k\ell} + s_{j\oplus\ell} u\alpha^{-j\ell+k\ell} + s_{k\oplus\ell} s_j u\\
&\qquad = s_\ell s_k s_j u\alpha^{k\ell} + s_\ell s_j u\alpha^{k\ell} + s_{j\oplus k\oplus\ell} u\alpha^{-j(k\oplus\ell)}.
\end{aligned}$$

We have proved:

Lemma 3.7. *Let $Q = \mathrm{Dih}(m, G, \alpha)$. Then Q is left automorphic iff* (7) *and* (8) *hold for every i, j, k, $\ell \in \mathbb{Z}_m$ and every $u \in G$.*

Let us now analyze the two conditions (7) and (8).

Lemma 3.8. *Let $Q = \mathrm{Dih}(m, G, \alpha)$. If $m = 2$ then* (7) *holds. If $m > 2$ then* (7) *holds iff $\alpha^m = 1$.*

Proof: Consider the condition

$$ij + (i \oplus j)k = i(j \oplus k) + jk. \tag{9}$$

When $m = 2$ then (9) holds by a quick inspection of the cases, and thus (7) holds as well. Suppose that $m > 2$. With $i = j = 1$, $k = m - 1$ the condition (9) reduces to $1 + 2(m - 1) = 1 \cdot 0 + m - 1$, or $m = 0$, thus if (7) holds then $\alpha^m = 1$. Conversely, if $\alpha^m = 1$, then (7) holds because (9) is valid modulo m. □

Lemma 3.9. *Let $Q = \mathrm{Dih}(m, G, \alpha)$. If (7) and (8) hold then $\alpha^{m-2} = 1$.*

Proof: When $m = 2$ the conclusion is trivially true. Let us therefore assume that $m > 2$ and, using Lemma 3.8, that $\alpha^m = 1$. Let $k = 1$, $j = \ell = m - 1$. Then (8) becomes

$$\begin{aligned} s_{m-1}u\alpha^{-(m-1)+(m-1)} + s_{m-2}u\alpha^{-(m-1)^2+(m-1)} + s_{m-1}u \\ = -u\alpha^{m-1} + u\alpha^{m-1} + s_{m-1}u, \end{aligned}$$

or, equivalently,

$$s_{m-1}u = -s_{m-2}u\alpha^{-(m-1)^2+(m-1)}.$$

Since $s_{m-1} = -s_{m-2}$ and $(m - 1)^2 \equiv 1 \pmod m$, the last identity is equivalent to $u = u\alpha^{-1+m-1} = u\alpha^{m-2}$, or to $\alpha^{m-2} = 1$. □

Lemma 3.10. *Let $Q = \mathrm{Dih}(m, G, \alpha)$ be a left automorphic loop.*

(i) *If $m > 2$ is even then $\alpha^2 = 1$.*
(ii) *If $m > 2$ is odd then $\alpha = 1$.*

Proof: By Lemma 3.7, Q satisfies (7) and (8). Suppose that $m > 2$. Then Lemma 3.8 implies $\alpha^m = 1$ and Lemma 3.9 implies $\alpha^{m-2} = 1$. Thus $\alpha^2 = 1$. If m is also odd then $\alpha^2 = 1$ and $\alpha^m = 1$ imply $\alpha = 1$. □

Lemma 3.11. *Let $Q = \mathrm{Dih}(m, G, \alpha)$.*

(i) *If $m = 2$ then (8) holds.*
(ii) *If m is even and $\alpha^2 = 1$ then (8) holds.*
(iii) *If $m > 2$ is odd and $\alpha = 1$ then (8) implies $2G = 0$.*

Proof: Suppose that $m = 2$. We can then reduce all subscripts modulo 2 in (8) and use $s_i s_j = s_{i+j}$. Hence (8) becomes

$$\begin{aligned} s_{\ell+j+k}u\alpha^{-jk+k\ell} + s_{j+\ell}u\alpha^{-j\ell+k\ell} + s_{k+\ell+j}u \\ = s_{\ell+k+j}u\alpha^{k\ell} + s_{\ell+j}u\alpha^{k\ell} + s_{j+k+\ell}u\alpha^{-j(k\oplus\ell)} \end{aligned} \tag{10}$$

where all subscripts are reduced modulo 2. When j is even (that is, $j = 0$), (10) becomes

$$s_{k+\ell}u\alpha^{k\ell} + s_\ell u\alpha^{k\ell} + s_{k+\ell}u = s_{k+\ell}u\alpha^{k\ell} + s_\ell u\alpha^{k\ell} + s_{k+\ell}u,$$

a valid identity. If j is odd and k is even, (10) becomes

$$-s_\ell u - s_\ell u\alpha^{-\ell} - s_\ell u = -s_\ell u - s_\ell u - s_\ell u\alpha^{-\ell},$$

clearly true. If j, k are odd and ℓ is even, (10) becomes

$$u\alpha^{-1} - u + u = u - u + u\alpha^{-1},$$

again true. Finally, if j, k, ℓ are odd, (10) becomes

$$-u + u - u = -u\alpha + u\alpha - u,$$

which holds trivially.

Suppose that m is even and $\alpha^2 = 1$. Then we can reduce all subscripts and superscripts in (8) modulo 2, and we proceed as in case (i).

For the rest of the proof let $m > 2$ be odd and suppose that $\alpha = 1$. Then (8) becomes

$$s_\ell s_{j\oplus k}u + s_{j\oplus\ell}u + s_{k\oplus\ell}s_j u = s_\ell s_k s_j u + s_\ell s_j u + s_{j\oplus k\oplus\ell}u.$$

With $j = m-1$ and $k = \ell = 1$ we obtain $-u + u + u = u - u - u$, or $2u = 0$. □

Proposition 3.12. *Let $Q = \mathrm{Dih}(m, G, \alpha)$.*

(i) *If $m = 2$ then Q is left automorphic.*
(ii) *If $m > 2$ is even then Q is left automorphic iff $\alpha^2 = 1$.*
(iii) *If $m > 2$ is odd then Q is left automorphic iff $\alpha = 1$ and $2G = 0$, in which case Q is a group.*

Proof: We will use Lemma 3.7 without reference.

Suppose that $m = 2$. Then (7) holds by Lemma 3.8 and (8) holds by Lemma 3.11.

Suppose that $m > 2$ is even. If Q is left automorphic then $\alpha^2 = 1$ by Lemma 3.10. Conversely, suppose that $\alpha^2 = 1$. Then (8) holds by Lemma 3.11. Since also $\alpha^m = 1$, (7) holds by Lemma 3.8.

Finally, suppose that $m > 2$ is odd. If Q is left automorphic then $\alpha = 1$ by Lemma 3.10. By Lemma 3.11, $2G = 0$. Conversely, suppose that $\alpha = 1$ and $2G = 0$. Then Q is a group by Lemma 3.1, so certainly also a left automorphic loop. □

3.3 Main result.

Theorem 3.13. *Let $m > 1$ be an integer, G an abelian group and α an automorphism of G. Let $Q = \mathrm{Dih}(m, G, \alpha)$ be defined by (1).*

(i) *If $m = 2$ then Q is automorphic.*
(ii) *If $m > 2$ is even then Q is automorphic iff $\alpha^2 = 1$.*
(iii) *If $m > 2$ is odd then Q is automorphic iff $\alpha = 1$ and $2G = 0$, in which case Q is a group.*

Proof: The claim follows from Propositions 2.2, 3.4 and 3.12. □

From now on we will refer to loops $Q = \mathrm{Dih}(m, G, \alpha)$ that are automorphic (equivalently, that satisfy the conditions of Theorem 3.13) as *dihedral automorphic loops*. Since nonassociative examples of dihedral automorphic loops are obtained only when $m = 2$ or when $m > 2$ is even and $\alpha^2 = 1$, we will from now on safely write $s_i s_j = s_{i+j} = s_{i\oplus j}$, and we do not have to reduce exponents of α modulo m.

Remark 3.14. If in the multiplication formula (1) we also reduce the exponent of α (that is, we have $(i,u)\cdot(j,v) = (i \oplus j, (s_j u + v)\alpha^{ij \pmod m}))$, then the resulting loop $\mathrm{Dih}_{red}(m, G, \alpha)$ is not necessarily isomorphic to $\mathrm{Dih}(m, G, \alpha)$. However, it can be shown that $\mathrm{Dih}_{red}(m, G, \alpha) = \mathrm{Dih}(m, G, \alpha)$ whenever one of the loops is automorphic. See [7] for details.

4. Nuclei, commutant and center

In this section we calculate the nuclei, the commutant and the center of dihedral automorphic loops satisfying $\alpha^2 = 1$. (So we do not always cover the case $m = 2$, $\alpha^2 \neq 1$.)

Lemma 4.1. *Let $Q = \mathrm{Dih}(m, G, \alpha)$ be a dihedral automorphic loop such that $\alpha^2 = 1$. If $\alpha = 1$ then $N_\mu(Q) = Q$, else $N_\mu(Q) = \langle 2\rangle \times G$.*

Proof: If $\alpha = 1$ then Q is a group and thus $N_\mu(Q) = Q$. Suppose that $\alpha \neq 1$. Note that in automorphic loops (that satisfy (7) by Lemma 3.7) the formula of Lemma 3.5 simplifies to

$$\begin{aligned}(11)\quad (k,w)L_{(j,v),(i,u)} &= (k, s_{j+k}u\alpha^{ij-jk} + s_k v\alpha^{ij} + w\alpha^{ij} - s_k s_j u\alpha^{ij} - s_k v\alpha^{ij})\\ &= (k, s_{j+k}u\alpha^{ij-jk} - s_k s_j u\alpha^{ij} + w\alpha^{ij}).\end{aligned}$$

Since $(j,v) \in N_\mu(Q)$ iff $(k,w) = (k,w)L_{(j,v),(i,u)}$ for all (i,u), (k,w), we conclude that $(j,v) \in N_\mu(Q)$ iff

$$(12)\qquad s_{j+k}u\alpha^{ij-jk} - s_k s_j u\alpha^{ij} + w\alpha^{ij} = w$$

for all (i,u), $(k,w) \in Q$. With $u = 0$, $i = 1$ this reduces to $w\alpha^j = w$, so $\alpha^j = 1$ is necessary. Because $\alpha \neq 1 = \alpha^2$, we obtain $j \in \langle 2\rangle$. Conversely, if $j \in \langle 2\rangle$ then (12) holds thanks to $s_{j+k} = s_k s_j$ (since m is even). □

For an abelian group G and $\alpha \in \mathrm{Aut}(G)$, let $G_2 = \{u \in G : |u| \leq 2\}$, $\mathrm{Fix}(\alpha) = \{u \in G : u = u\alpha\}$ and $\mathrm{Fix}(\alpha)_2 = G_2 \cap \mathrm{Fix}(\alpha)$.

Lemma 4.2. *Let $Q = \mathrm{Dih}(m, G, \alpha)$ be a dihedral automorphic loop with $\alpha^2 = 1$. If $\alpha = 1$ then $N(Q) = N_\lambda(Q) = N_\rho(Q) = Q$, else $N(Q) = N_\lambda(Q) = N_\rho(Q) = \langle 2\rangle \times \mathrm{Fix}(\alpha)$.*

Proof: Recall that $N(Q) = N_\lambda(Q) = N_\rho(Q) \leq N_\mu(Q)$ in all automorphic loops. We are again done if $\alpha = 1$, so suppose that $\alpha \neq 1$. Note that $(i,u) \in N_\lambda(Q)$

iff $(k,w)L_{(j,v),(i,u)} = (k,w)$ for all (j,v), $(k,w) \in Q$. We deduce from (11) that $(i,u) \in N_\lambda(Q)$ iff (12) holds for all (j,v), (k,w).

If $(i,u) \in N_\lambda(Q)$ then $i \in \langle 2\rangle$ by Lemma 4.1, so (12) reduces to $s_{j+k}u\alpha^{-jk} - s_{j+k}u = 0$, i.e., $u\alpha^{-jk} = u$ for all j, k. With $j = k = 1$ we see that $u \in \mathrm{Fix}(\alpha)$. Conversely, if $u \in \mathrm{Fix}(\alpha)$ and $i \in \langle 2\rangle$ then (12) clearly holds. □

Recall that the commutant $C(Q)$ is not necessarily a (normal) subloop of a loop Q.

Lemma 4.3. *Let $Q = \mathrm{Dih}(m,G,\alpha)$ be a dihedral automorphic loop such that $\alpha^2 = 1$. Then:*

(i) *if $\exp(G) \le 2$ then $C(Q) = Q$;*
(ii) *if $\exp(G) > 2$ then $C(Q) = \langle 2\rangle \times G_2$.*

In either case, $C(Q) \trianglelefteq Q$.

Proof: By Lemma 3.2, $(i,u) \in C(Q)$ iff

$$s_i v + (1 - s_j)u = v \tag{13}$$

holds for all $(j,v) \in Q$. If $\exp(G) = 2$ then (13) holds. If $\exp(G) > 2$ then (13) holds for all (j,v) iff $i \in \langle 2\rangle$ and $u \in G_2$. Hence if $\exp(G) > 2$ then $C(Q) = \langle 2\rangle \times G_2$.

Note that $\langle 2\rangle \times G$ is a group. Thus, to show $C(Q) \le Q$, we only need to check that $C(Q)$ is closed under multiplication and inverses, and this is clear from the multiplication formula.

If $(j,v) \in C(Q)$ then, by Lemma 3.2, $(j,v)T_{(i,u)} \in \{(j,\pm v)\} \in C(Q)$. If $(k,w) \in C(Q)$ then, by (11), $(k,w)L_{(j,v),(i,u)} = (k, s_j u\alpha^{ij} - s_j u\alpha^{ij} + w\alpha^{ij}) \in \{(k,w),(k,w\alpha)\} \in C(Q)$. The proof is similar for right inner mappings. Hence $C(Q) \trianglelefteq Q$. □

Lemma 4.4. *Let $Q = \mathrm{Dih}(m,G,\alpha)$ be a dihedral automorphic loop such that m is even and $\alpha^2 = 1$. Then:*

(i) *if $\exp(G) \le 2$ and $\alpha = 1$ then $Z(Q) = Q$;*
(ii) *if $(\exp(G) \le 2$ and $\alpha \ne 1)$ or $\exp(G) > 2$ then $Z(Q) = \langle 2\rangle \times \mathrm{Fix}(\alpha)_2$.*

Proof: Suppose that $\alpha = 1$. Then Q is a group and $Z(Q) = C(Q)$. If $\exp(G) \le 2$ then $Z(Q) = Q$ by Lemma 4.3. If $\exp(G) > 2$ then $C(Q) = \langle 2\rangle \times G_2 = \langle 2\rangle \times \mathrm{Fix}(\alpha)_2$, by Lemma 4.3.

Now suppose that $\alpha \ne 1 = \alpha^2$. If $\exp(G) \le 2$ then $C(Q) = Q$ and $Z(Q) = N(Q) = \langle 2\rangle \times \mathrm{Fix}(\alpha)_2 = \langle 2\rangle \times \mathrm{Fix}(\alpha)$ by Lemma 4.2. If $\exp(G) > 2$ then $Z(Q) = N(Q) \cap C(Q) = \langle 2\rangle \times \mathrm{Fix}(\alpha)_2$, by Lemmas 4.2 and 4.3. □

Proposition 4.5. *Let Q be a dihedral automorphic loop with $\alpha \ne 1 = \alpha^2$. Then $Q/Z(Q) \cong \mathrm{Dih}(2, G/H, \beta)$, where $H = \mathrm{Fix}(\alpha)_2$ and $\beta \in \mathrm{Aut}(G/H)$ is defined by $(u + H)\beta = u\alpha + H$. Moreover, $\beta^2 = 1$.*

Proof: By Lemma 4.4, $Z(Q) = \langle 2\rangle \times \mathrm{Fix}(\alpha)_2$. The mapping β is well-defined (if $u + H = v + H$ then $u - v \in H \subseteq \mathrm{Fix}(\alpha)$, $u\alpha - v\alpha = (u - v)\alpha = u - v \in H$,

$u\alpha + H = v\alpha + H$) and obviously a surjective homomorphism. Since α fixes elements of H pointwise, we have $u + H \in \ker\beta$ iff $u \in H$, so $\beta \in \mathrm{Aut}(G/H)$.

Consider $f : Q \to \mathrm{Dih}(2, G/H, \beta)$ defined by $(i, u)f = (i \bmod 2, u + H)$. Since

$$\begin{aligned}(i,u)f(j,v)f &= (i \bmod 2, u+H)(j \bmod 2, v+H)\\ &= ((i+j) \bmod 2, (s_j(u+H)+(v+H))\beta^{ij})\\ &= ((i+j) \bmod 2, (s_ju+v)\alpha^{ij}+H)\\ &= (i+j, (s_ju+v)\alpha^{ij})f = ((i,u)(j,v))f,\end{aligned}$$

f is a homomorphism, obviously onto $\mathrm{Dih}(2, G/H, \beta)$. Finally, $\ker(f) = \langle 2\rangle \times H = Z(Q)$. □

Corollary 4.6. *Every dihedral automorphic loop* $\mathrm{Dih}(m, G, \alpha)$ *with* $\alpha \neq 1 = \alpha^2$ *is a central extension of an elementary abelian 2-group by a dihedral automorphic loop of the form* $\mathrm{Dih}(2, K, \beta)$ *with* $\beta^2 = 1$ *and* K *isomorphic to a factor of* G.

As an application of the results in this section, let us have a look at central nilpotency of dihedral automorphic loops. Let $Q = \mathrm{Dih}(m, G, \alpha)$ be a dihedral automorphic loop with $\alpha^2 = 1$ and m even.

If $\alpha = 1$ and $\exp(G) \leq 2$ then $Z(Q) = Q$ by Lemma 4.4. If $\alpha = 1$ and $\exp(G) > 2$ then Q is a group and $Z(Q) = \langle 2\rangle \times \mathrm{Fix}(\alpha)_2 = \langle 2\rangle \times G_2$, and since $(i,u)Z(Q)\cdot(j,v)Z(Q) = (i\oplus j, s_ju+v)(\langle 2\rangle\times G_2) = ((i+j) \bmod 2, s_ju+v)Z(Q)$, we see that $Q/Z(Q)$ is isomorphic to the generalized dihedral group $\mathrm{Dih}(2, G/G_2, 1)$.

Now suppose that $\alpha \neq 1 = \alpha^2$. Then $Q/Z(Q) \cong \mathrm{Dih}(2, G/H, \beta)$, where $H = \mathrm{Fix}(\alpha)_2$ and $\beta^2 = 1$. If $H \neq 1$, we proceed by induction, else $G/H = G$, $\beta = \alpha$ and $Z(Q/Z(Q)) = 1$.

Example 4.7. If G is an abelian group of odd order and $\alpha \in \mathrm{Aut}(G)$ such that $\alpha \neq 1 = \alpha^2$ then $Z(\mathrm{Dih}(2, G, \alpha)) = 1$.

Suppose that $|G| = 2^n$ and $\alpha \in \mathrm{Aut}(G)$ is such that $\alpha \neq 1 = \alpha^2$. Since the involution α fixes the neutral element of G and permutes the subgroup G_2 of even order (a divisor of $|G|$), we have $H = \mathrm{Fix}(\alpha)_2 \neq 1$. Thus $Q/Z(Q) = \mathrm{Dih}(2, G/H, \beta)$ and $2^\ell = |G/H| < |G|$. By induction, Q is centrally nilpotent of class $\leq n$.

Finally suppose that $G = \mathbb{Z}_{2^n}$, $\alpha \in \mathrm{Aut}(G)$ and $1 = \alpha^2$. Whether $\alpha = 1$ or not, we have $Q/Z(Q) = \mathrm{Dih}(2, G/H, \beta)$ for $H = \mathrm{Fix}(\alpha)_2 = \{0, 2^{n-1}\}$ and some $\beta \in \mathrm{Aut}(G/H)$ satisfying $\beta^2 = 1$, because 2^{n-1} is the unique element of order 2 in G. By induction, Q has nilpotence class n.

5. Commutators and associators

Recall that in a loop Q, the commutator $[x, y]$ is defined as $(yx)\backslash(xy)$, and the associator $[x, y, z]$ as $(x \cdot yz)\backslash(xy \cdot z)$.

Lemma 5.1. *In a loop* $Q = \mathrm{Dih}(m, G, \alpha)$ *we have*

$$[(i,u),(j,v)] = (0, ((s_j-1)u + (1-s_i)v)\alpha^{ij}) \tag{14}$$

for (i,u), $(j,v)\in Q$.

Proof: Let $(k,w)=[(i,u),(j,v)]$, so $(i,u)(j,v)=(j,v)(i,u)\cdot(k,w)$, hence,

$$\begin{aligned}(i\oplus j,(s_ju+v)\alpha^{ij})&=(j\oplus i,(s_iv+u)\alpha^{ij})\cdot(k,w)\iff\\(i\oplus j,s_ju\alpha^{ij}+v\alpha^{ij})&=(i\oplus j\oplus k,(s_ks_iv\alpha^{ij}+s_ku\alpha^{ij}+w)\alpha^{(i\oplus j)k}).\end{aligned}$$

We deduce $k=0$, and can rewrite the above expression as $w=(s_j-1)u\alpha^{ij}+(1-s_i)v\alpha^{ij}$. □

Proposition 5.2. *Let* $Q=\mathrm{Dih}(m,G,\alpha)$ *be a dihedral automorphic loop with* $\alpha^2=1$. *Then*

$$\langle [x,y]:x,y\in Q\rangle=\{[x,y]:x,y\in Q\}=0\times 2G$$

is a normal subloop of Q.

Proof: First, using Lemma 5.1 and looking at all cases $i,j \pmod 2$, it is easy to see that $[(i,u),(j,v)]\in 0\times 2G$. Second, $[(1,0),(0,v)]=(0,2v)$. This shows that $\{[x,y]:x,y\in Q\}=0\times 2G$. It is easy to see from (1) that $0\times 2G$ is a subloop of Q. Finally, to show that $0\times 2G$ is normal in Q, we calculate, using Lemmas 3.2, 3.5 and an analog of Lemma 3.5:

$$\begin{aligned}(0,2w)L_{(j,v),(i,u)}&=(0,2w\alpha^{1+ij}),\\(0,2w)T_{(i,u)}&=(0,2s_iw),\\(0,2w)R_{(j,v),(i,u)}&=(0,2w\alpha^{ij}).\end{aligned}$$

□

Lemma 5.3. *In a dihedral automorphic loop* $Q=\mathrm{Dih}(m,G,\alpha)$ *with* $\alpha^2=1$ *we have*

$$[(i,u),(j,v),(k,w)]=(0,(s_{j+k}u(1-\alpha^{-jk})\alpha^{ij}+w(1-\alpha^{ij}))\alpha^{(i\oplus j)k})\tag{15}$$

for $(i,u),(j,v),(k,w)\in Q$.

Proof: When m is odd and $\alpha=1$ then Q is a group and (15) yields $[(i,u),(j,v),(k,w)]=1$. The case when m is even and $\alpha^2=1$ follows by straightforward calculation, but since the identity (7) is involved, we give all the details: let $(\ell,x)=[(i,u),(j,v),(k,w)]$ so

$$\begin{aligned}&(i,u)(j,v)\cdot(k,w)=((i,u)\cdot(j,v)(k,w))(\ell,x),\\&(i\oplus j,(s_ju+v)\alpha^{ij})\cdot(k,w)=((i,u)\cdot(j\oplus k,(s_kv+w)\alpha^{jk}))(\ell,x),\\&(i\oplus j\oplus k,[(s_{k+j}u+s_kv)\alpha^{ij}+w]\alpha^{(i\oplus j)k})\\&\ =(i\oplus j\oplus k,[s_{j+k}u+s_kv\alpha^{jk}+w\alpha^{jk}]\alpha^{i(j\oplus k)})(\ell,x),\end{aligned}$$

$$
\begin{aligned}
&(i \oplus j \oplus k, s_{k+j} u\alpha^{ij+(i\oplus j)k} + s_k v\alpha^{ij+(i\oplus j)k} + w\alpha^{(i\oplus j)k}) \\
&= (i \oplus j \oplus k, s_{j+k} u\alpha^{i(j\oplus k)} + s_k v\alpha^{jk+i(j\oplus k)} + w\alpha^{jk+i(j\oplus k)})(\ell, x), \\
&(i \oplus j \oplus k, s_{k+j} u\alpha^{ij+(i\oplus j)k} + s_k v\alpha^{ij+(i\oplus j)k} + w\alpha^{(i\oplus j)k}) \\
&= (i \oplus j \oplus k, s_{j+k} u\alpha^{ij+(i\oplus j)k-jk} + s_k v\alpha^{jk+i(j\oplus k)} + w\alpha^{jk+i(j\oplus k)})(\ell, x).
\end{aligned}
$$

Here we have used identity (7) in the last step. We obtain

$$
\begin{aligned}
&(i \oplus j \oplus k, s_{k+j} u\alpha^{ij+(i\oplus j)k} + s_k v\alpha^{ij+(i\oplus j)k} + w\alpha^{(i\oplus j)k}) \\
&= (i \oplus j \oplus k \oplus \ell, [s_{j+k+\ell} u\alpha^{ij+(i\oplus j)k-jk} \\
&\quad + s_{k+\ell} v\alpha^{jk+i(j\oplus k)} + s_\ell w\alpha^{jk+i(j\oplus k)} + x]\alpha^{(i\oplus j\oplus k)\ell}).
\end{aligned}
$$

We deduce $\ell = 0$, and can rewrite the above expression as

$$
\begin{aligned}
&s_{k+j} u\alpha^{ij+(i\oplus j)k} + s_k v\alpha^{ij+(i\oplus j)k} + w\alpha^{(i\oplus j)k} \\
&= s_{j+k} u\alpha^{ij+(i\oplus j)k-jk} + s_k v\alpha^{ij+(i\oplus j)k} + w\alpha^{ij+(i\oplus j)k} + x, \\
x &= (s_{j+k} u(1-\alpha^{-jk})\alpha^{ij} + w(1-\alpha^{ij}))\alpha^{(i\oplus j)k}. \qquad \square
\end{aligned}
$$

Proposition 5.4. *Let $Q = \mathrm{Dih}(m, G, \alpha)$ be a dihedral automorphic loop with $\alpha^2 = 1$. Then*

$$A(Q) = \langle [x, y, z] : x, y, z \in Q \rangle = \{[x, y, z] : x, y, z \in Q\} = 0 \times G(1-\alpha).$$

Proof: Here we check all choices of $i, j, k \pmod 2$, using Lemma 5.3.

$$
\begin{aligned}
[(0,u),(0,v),(0,w)] &= (0, u(1-1) + w(1-1)) = (0,0), \\
[(0,u),(1,v),(0,w)] &= (0, -u(1-1) + w(1-1)) = (0,0), \\
[(0,u),(0,v),(1,w)] &= (0, -u(1-1) + w(1-1)) = (0,0), \\
[(0,u),(1,v),(1,w)] &= (0, (u(1-\alpha^{-1}) + w(1-1))\alpha) \\
&= (0, u(1-\alpha^{-1})\alpha) = (0, -u(1-\alpha)), \\
[(1,u),(0,v),(0,w)] &= (0, u(1-1) + w(1-1)) = (0,0), \\
[(1,u),(1,v),(0,w)] &= (0, (-u(1-1)\alpha + w(1-\alpha))) = (0, w(1-\alpha)), \\
[(1,u),(0,v),(1,w)] &= (0, (-u(1-1) + w(1-1))\alpha) = (0,0), \\
[(1,u),(1,v),(1,w)] &= (0, u(1-\alpha)\alpha + w(1-\alpha)), \\
&= (0, u(1-\alpha^{-1})\alpha + w(1-\alpha)), \\
&= (0, (-u+w)(1-\alpha)).
\end{aligned}
$$

We can see that $[(i,u),(j,v),(k,w)] \in 0 \times G(1-\alpha)$. Second, $[(1,u),(1,v),(0,w)] = (0, w(1-\alpha))$. This shows that $\{[x,y,z] : x,y,z \in Q\} = 0 \times G(1-\alpha)$.

Next, we need to show $0 \times G(1-\alpha)$ is subloop of Q. Let $(0, u(1-\alpha))$ and $(0, v(1-\alpha))$ be two elements of $0 \times G(1-\alpha)$. Then

$$\begin{aligned}
(0,u(1-\alpha)) \cdot (0,v(1-\alpha)) &= (0,(u+v)(1-\alpha)),\\
(0,u(1-\alpha)) \backslash (0,v(1-\alpha)) &= (0,(v-u)(1-\alpha)),\\
(0,u(1-\alpha)) / (0,v(1-\alpha)) &= (0,(u-v)(1-\alpha)).
\end{aligned}$$

Finally, to show $0 \times G(1-\alpha)$ is normal in Q we use Lemmas 3.2 and 3.5 to obtain:

$$\begin{aligned}
(0,w(1-\alpha))L_{(j,v),(i,u)} &= (0, s_j u\alpha^{ij} + v\alpha^{ij} + w(1-\alpha)\alpha^{ij} - s_j u\alpha^{ij} - v\alpha^{ij})\\
&= (0, w(1-\alpha)\alpha^{ij}),\\
(0,w(1-\alpha))T_{(i,u)} &= (0, s_i w(1-\alpha) + (1-1)u)\\
&= (0, s_i w(1-\alpha)),\\
(0,w(1-\alpha))R_{(j,v),(i,u)} &= (0, (w(1-\alpha) + s_{-(i+j)}u(1-1))\alpha^{ij})\\
&= (0, w(1-\alpha)\alpha^{ij}). \qquad \square
\end{aligned}$$

Proposition 5.5. *Let* $Q = \mathrm{Dih}(m, G, \alpha)$ *be a dihedral automorphic loop with* $\alpha^2 = 1$. *Then*

$$Q' = 0 \times (G(1-\alpha) + 2G).$$

Proof: The proof is immediate from Propositions 5.2 and 5.4, since $Q' = 0 \times (G(1-\alpha) + 2G)$ is a normal subloop of Q. $\square$

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