Towards a geometric theory for left loops

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Abstract. In [Mwambene E., *Multiples of left loops and vertex-transitive graphs*, Cent. Eur. J. Math. **3** (2005), no. 2, 254–250] it was proved that every vertex-transitive graph is the Cayley graph of a left loop with respect to a quasi-associative Cayley set. We use this result to show that Cayley graphs of left loops with respect to such sets have some properties in common with Cayley graphs of groups which can be used to study a geometric theory for left loops in analogy to that for groups.

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1. Definitions

First we want to introduce the general definition of the Cayley graph of a magma with respect to a Cayley set. Recall that a magma[1] $(M, \cdot)$ is just a set with a closed binary operation. Whenever there is no room for confusion we will say that "M is a magma" without specifying the binary operation.

Definition 1.1. Let M be a magma. A subset $S \subseteq M$ is called a *Cayley set*, if it satisfies the following properties:

(1) $a \notin aS \quad \forall a \in M$,
(2) $a \in (as)S \quad \forall a \in M, \forall s \in S$.

Definition 1.2. Let M be a magma, and let $S \subseteq M$ be a Cayley set. The *Cayley graph*[2] of M with respect to S is $Cay(M,S) = (V,E)$ where $V = M$ and $E = \{\{x, xs\} : x \in M, s \in S\}$.

The reason for using a Cayley set instead of just any set is that it allows the use of undirected graphs without loop-edges.

We will be interested in Cayley graphs whose underlying Cayley set is *quasi-associative*. By that we mean the following:

Definition 1.3. Let M be a magma. A subset $S \subseteq M$ is called *quasi-associative* if $(ab)S = a(bS)$ for all $a, b \in M$.

[1]Sometimes called *groupoid*.

[2]Some authors use the term "Cayley graph" only to talk about Cayley graphs of groups with respect to a generating Cayley set and use the term "groupoid graph" ([6], [7], [8]) or "color group" ([10], [11]) for more general Cayley graphs.

We will be dealing with some important non-associative algebraic structures, so we need the following definitions.

Definition 1.4. Let M be a magma. If for all a, b in M, the equation $ax = b$ has a unique solution, we say that M is a *left quasi-group*. The solution to such equation is denoted by $x = a\backslash b$ and the binary operation $\backslash : (a, b) \mapsto a\backslash b$ is called *left division*. A *left* quasi-group with a *right* neutral element is called a *left loop*.

Analogously, one can define a *right quasi-group* as a magma with *right division* / and define a *right loop*. A magma that is both a left quasi-group and a right quasi-group, is called a *quasi-group* and a quasi-group with a two-sided neutral element is called a *loop* [9].

Recently in [4] Griggs has used Cayley graphs to study Moufang loops.

In any magma M, for each element a in M, we can define the *left translation by a* as the function $L_a : M \to M$ given by $L_a(x) = ax$. In a left quasi-group Q these functions are bijective for all $a \in Q$.

We will be especially interested in Cayley graphs of left loops with respect to quasi-associative Cayley sets.

2. Characterization theorem

In [7], Mwambene gives a characterization of Cayley graphs of left loops with respect to a quasi-associative Cayley set. The method given by Mwambene can be used also to prove Gauyacq's characterization of Cayley graphs of quasi-groups with respect to quasi-associative Cayley sets [3] and Sabidoussi's characterization of Cayley graphs of groups [10].

Recall that a *vertex-transitive* graph is a graph in which its automorphism group acts transitively on the vertices.

Theorem 2.1 (Mwambene)**.** *Let L be a left loop, and let $S \subseteq L$ be a quasi-associative Cayley set. Then the Cayley graph $Cay(L, S)$ is vertex-transitive. In particular the left translations of L are automorphisms of $Cay(L, S)$.*

PROOF: See [8] or [7]. □

One can ask the following question: Given a left loop L and a Cayley set S such that $Cay(L, S)$ is a vertex-transitive graph, is it possible to conclude that the Cayley set is quasi-associative? If the answer to this question were positive that would be very helpful for a future result (see Theorem 2.4), but unfortunately this is not the case, as we can see in the following counterexample.

Counterexample 2.2. Let $L = \{0, 1, 2, 3, 4, 5\}$. Define in L the binary operation $*$ given by Table 1. Note that L is a left loop, moreover, is a loop (the neutral element being 0). One can verify that the set $S = \{1, 5\}$ is a Cayley set, and that the graph $Cay(L, S)$ is the cycle C_6 (Figure 1), which is vertex transitive. Nevertheless, $(2 * 3) * S = \{5, 3\}$ while $2 * (3 * S) = \{0, 5\}$, so S is not quasi-associative.

$*$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	4	5	3	0
2	2	3	5	4	0	1
3	3	4	0	1	5	2
4	4	5	1	0	2	3
5	5	0	3	2	1	4

TABLE 1. Multiplication table of L in Counterexample 2.2

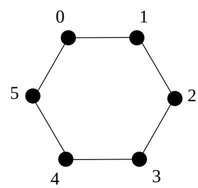

FIGURE 1. Cayley graph of L with respect to S in Counterexample 2.2.

For a given vertex-transitive graph X Mwambene constructs a left loop L and a quasi-associative Cayley set $S \subseteq L$ such that $Cay(L, S)$ is isomorphic to X [7], [8]. We call this construction *Mwambene's method*.

Theorem 2.3 (Mwambene)**.** *Let $X = (V, E)$ be a vertex-transitive graph. Then there exists a left loop L and a quasi-associative Cayley set $S \subseteq L$ such that $Cay(L, S) \cong X$.*

PROOF: We give a short description of the proof following results of Baer in [1].

Given a vertex-transitive graph $X = (V, E)$, let A be its automorphism group and let $u \in V$ be a fixed vertex. Consider the stabilizer subgroup $A_u = \{\sigma \in A : \sigma(u) = u\}$. Choose a left transversal T of A with respect to A_u, that is, a set which contains exactly one element of each left coset of A_u. Now, define a binary operation $*$ in T as follows:

$$(\sigma * \tau)A_u = \sigma\tau A_u \quad \forall \sigma, \tau \in T. \tag{1}$$

In other words, $\sigma * \tau$ is the only element of T that maps the vertex u to the same vertex as $\sigma\tau$ does. In [1] it is shown that $(T, *)$ is a left loop with right neutral element ϵ where $A_u \cap T = \{\epsilon\}$.

Finally, define $S_T := \{\sigma \in T : \{u, \sigma(u)\} \in E\}$. In [8] it is proved that S_T is a quasi-associative Cayley set, and that $Cay(T, S_T)$ is isomorphic to X. □

Theorem 2.4. *Let L be a left loop and let $S \subseteq L$ be a quasi-associative Cayley set. Let $X = Cay(L, S)$. Then, there exists a left loop T isomorphic to L constructed by Mwambene's method starting from the graph X. Moreover, there exists an isomorphism $\Phi : L \to T$, such that $\Phi(S) = S_T$.*

$*$	e	a	a^2	a^2a	aa^2	a^2a^2
e	e	aa^2	a	a^2a^2	a^2	a^2a
a	a	a^2	aa^2	a^2a	e	a^2a^2
a^2	a^2	a^2a	a^2a^2	a	aa^2	e
aa^2	aa^2	a^2a^2	a^2	e	a^2a	a
a^2a	a^2a	a	e	a^2	a^2a^2	aa^2
a^2a^2	a^2a^2	e	a^2a	aa^2	a	a^2

TABLE 2. Multiplication table of L in Counterexample 3.2

PROOF: Let e be the right neutral element in L and let A be the automorphism group of $Cay(L,S)$. Note that the set $T := \{L_a : a \in L\}$ is a left transversal of A with respect to A_e. Now define $\Phi : L \to T$ by $\Phi(a) = L_a$. Since $L_aL_b(e) = ab = L_{ab}(e)$ we conclude that $\Phi(a) * \Phi(b) = L_a * L_b = L_{ab} = \Phi(ab)$ and therefore Φ is an isomorphism.

In addition, the neighbours of e are the elements of eS, but since S is quasi-associative we have that $eS = (ee)S = e(eS)$, and by cancellation we have proved Lemma 2.5[3].

Lemma 2.5. *If L is a left loop with right neutral element e and S is a quasi-associative set, then $S = eS$.*

Then the elements of T that map e into one of its neighbours are precisely $S_T = \{L_s : s \in S\} = \Phi(S)$. $\square$

Note that if the Cayley set S is not quasi-associative then Theorem 2.4 does not ensure the existence of a transversal T isomorphic to the given left loop even if the Cayley graph is vertex-transitive as is the case in Counterexample 2.2.

3. Geometric theory

In this section we will introduce a geometric theory for left loops in analogy to Geometric Group Theory. The problem is that without associativity some results that are very obvious or easy to prove for groups might not even be true for left loops. Here is an example. For groups one has

Proposition 3.1. *Let G be a group and let $S \subseteq G$ be a Cayley set. Then the connected component of the neutral element in the graph $Cay(G,S)$ is the subgroup generated by S. In particular, $Cay(G,S)$ is connected if and only if $G = \langle S \rangle$.*

But for left loops one has the following counterexample.

Counterexample 3.2. Let L be the left loop given by Table 2. One can verify that $S = \{a, a^2a\}$ is a Cayley set, and clearly it generates L. Nevertheless, the Cayley graph is not connected as shown in Figure 2.

[3]In Lemma 2.5 it is not required that S is a Cayley set.

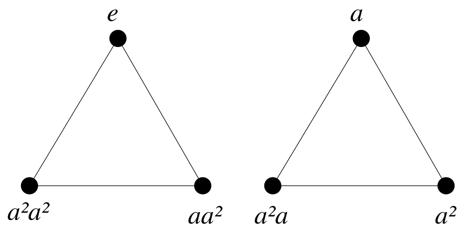

FIGURE 2. Cayley graph for Counterexample 3.2.

In general, if M is a magma, $a \in M$ and $S \subseteq M$ is a Cayley set, the connected component of a in the Cayley graph consists of the elements of the form

$$x = (\dots((as_1)s_2)\dots)s_n, \tag{2}$$

with $s_i \in S$ for every $i = 1, \dots, n$. This could be or not the submagma generated by S, but if M is a left loop and S is quasi-associative we get the following results:

Proposition 3.3. *Let M be a magma and $S \subseteq M$ any quasi-associative set. Then any product of elements of S with any parenthesis arrangement can be expressed also as a left-normed product of the same length of elements of S. More formally, if x is a product of k elements of S (maybe with repetitions), then there exist $s'_1, \dots, s'_k \in S$ such that*

$$x = (\dots((s'_1 s'_2)s'_3)\dots)s'_k. \tag{3}$$

PROOF: Proceed by induction on the length of the product. If the length is 1 or 2 the result is obvious. Now suppose that every product of length $\ell < k$ of elements of S can be expressed also as a left-normed product of the same length. Let x be a product of length k. This means that $x = \alpha\beta$ where α is a product of length $\ell < k$ and β is a product of length $k - \ell$. By induction hypothesis, β can be expressed as a left-normed product of length $k - \ell$ of elements of S, but this means that $\beta = \gamma s$ where γ is a left-normed product of length $k - \ell - 1$ and $s \in S$. Then $x = \alpha(\gamma s)$, but since S is quasi-associative, there exists s' such that $x = (\alpha\gamma)s'$, but by induction hypothesis, $\alpha\gamma$ can be expressed as a left-normed product of length $k - 1$. Therefore x is expressed as a left-normed product of length k. □

Now we can prove a generalization of Proposition 3.1.

Proposition 3.4. *Let L be a left loop and let $S \subseteq L$ be a quasi-associative Cayley set. Then the connected component of the right neutral element in the Cayley graph $Cay(L, S)$ is the left subloop generated by S.*

Proof: The connected component of the right neutral element in $Cay(L,S)$ consists of the elements of the form

$$x = (\dots((es_1)s_2)\dots)s_n, \tag{4}$$

with $s_i \in S$ for all $i = 1 \dots n$. But by Lemma 2.5, $es_1 = s_1' \in S$. So the connected component of the right neutral element consists of all the left-normed products of elements of S, but by Proposition 3.3 these are all the products of elements of S. Now we want to prove that this is in fact the left subloop generated by S.

The set of products of elements of S is obviously closed under product, but it is also closed under left division: Let a, b be products of elements of S and let $x \in L$ be such that $ax = b$. We want to prove that x is a product of elements of S. We proceed by induction on the distance between a and b in $Cay(L,S)$, both of which are in the same connected component as e. If a and b are neighbours, then $b = as$ for some $s \in S$, and then $x = s \in S$. Now suppose that if $ay = c$, and the distance from a to c is less than the distance from a to b, then y is a product of elements of S. One can assume that it is a product of length $d(a,c)$ to get a stronger result. Consider a minimal path from a to b and let c be the neighbour of b in this path. Then $d(a,c) = d(a,b) - 1$ and there exists a product y of $d(a,b) - 1$ elements S such that $ay = c$. There is also some $s \in S$ such that $b = cs$. Then $(ay)s = cs = b$, and by quasi-associativity, there exists $s' \in S$ such that $a(ys') = b$. So $x = ys'$ is a product of length $d(a,b)$ of elements of S such that $ax = b$. □

Corollary 3.5. *Let L be a left loop and let $S \subseteq L$ be a quasi-associative Cayley set. Then $Cay(L,S)$ is connected if and only if $L = \langle S \rangle$.*

In the proof of Proposition 3.4 we have also proved Proposition 3.6, which is also obvious for groups.

Proposition 3.6. *Let L be a left loop and let $S \subseteq L$ be a quasi-associative Cayley set. Let a and b be two different elements in L. Then the distance from a to b in $Cay(L,S)$ is the minimal length of a product of elements of S expressing $a\backslash b$. Also the distance from a to the right neutral element is the minimal length of a product of elements of S expressing a.*

Part of Geometric Group Theory consists in studying groups by the geometric properties of their Cayley graphs. The problem is that the Cayley graph depends on the Cayley set S, so one is constrained to those properties that are invariant under the choice of S within certain class of Cayley sets, as discussed below. What has been done [5] is to study the quasi-isometric invariants of the Cayley graph with respect to generating finite Cayley sets, since it has been shown that Cayley graphs of the same group with respect to different finite generating Cayley sets are quasi-isometric. Here we want to show that it is possible to extend this approach replacing groups by left loops and imposing an additional condition: that the Cayley set be quasi-associative (which is always the case for groups).

First we need the definition of a quasi-isometry.

Definition 3.7. Let (X, d) and (X, d') be metric spaces and let $f : X \to X'$. f is called a *quasi-isometry* if there exist constants $\lambda \geq 1$, $k \geq 0$ and $\delta \geq 0$ such that

$$\frac{1}{\lambda} d(x,y) - k \leq d'(f(x), f(y)) \leq \lambda d(x,y) + k \tag{5}$$

and

$$\forall x' \in X' \; \exists x \in X : \quad d'(f(x), x') \leq \delta. \tag{6}$$

In this case we say that X and X' are *quasi-isometric*.

It is easy to prove that being quasi-isometric is an equivalence relation[2].

Given a group G with two finite generating Cayley sets S and S', the graphs $Cay(G,S)$ and $Cay(G,S')$ are quasi-isometric. In fact we can generalize this result to a non-associative structure as follows:

Theorem 3.8. *Let L be a left loop and let S and S' be two finite quasi-associative Cayley sets that generate L. Then the Cayley graphs $Cay(L,S)$ and $Cay(L,S')$ are quasi-isometric.*

Proof: Denote by $|x|_S$ the minimal length of a product of elements of S expressing x. Let $\lambda_1 := \max_{s \in S} |s|_{S'}$. Let $x, y \in L$. Denote by d the distance in $Cay(L,S)$ and by d' the distance in $Cay(L,S')$. By Proposition 3.6, $x \backslash y$ can be expressed as a product of $d(x,y)$ elements of S. But each of these elements can be expressed as a product of at most λ_1 elements of S', as both graphs are connected according to Corollary 3.5. Then

$$d'(x,y) = |x \backslash y|_{S'} \leq \lambda_1 d(x,y). \tag{7}$$

Analogously, if we define $\lambda_2 := \max_{s' \in S'} |s'|_S$ we get that

$$d(x,y) \leq \lambda_2 d'(x,y). \tag{8}$$

So the identity map is a quasi-isometry between $Cay(L,S)$ and $Cay(L,S')$ with $\lambda = \max\{\lambda_1, \lambda_2\}$, $k = 0$ and $\delta = 0$. □

This result is important because it means that every geometric property of the Cayley graph of a left loop L with respect to a finite generating quasi-associative Cayley set S, are intrinsic properties of the left loop and do not depend on the choice of S. Examples of such properties are hyperbolicity and growth.

Definition 3.9. A geodesic metric space X is *hyperbolic* if there exists a $\delta \geq 0$ such that for every geodesic triangle $\triangle ABC$, and for every point x in the segment AC there exists a point y either in the segment AB or in the segment BC, such that $d(x,y) \leq \delta$.

If X and X' are two quasi-isometric geodesic metric spaces and X is hyperbolic, then X' is hyperbolic too [2]. Since being quasi-isometric is an equivalence relation, one can extend the definition of hyperbolicity to all metric spaces.

Definition 3.10. A metric space X is said to be *hyperbolic* if it is quasi-isometric to a hyperbolic geodesic metric space.

Note that a graph is quasi-isometric to the geodesic metric space in which we identify each edge with a segment of length 1. So to verify if a graph is hyperbolic or not, one can consider the paths of minimal length to be geodesics.

Now it is possible to define a hyperbolic left loop in complete analogy to the definition of a hyperbolic group.

Definition 3.11. Let L be a left loop and let $S \subseteq L$ be a finite quasi-associative Cayley set that generates L. Then L is said to be a *hyperbolic* left loop if $Cay(L, S)$ is a hyperbolic graph.

To get examples of hyperbolic left loops, one can simply apply Mwambene's method to any hyperbolic vertex-transitive graph. By Theorem 2.4 all the hyperbolic left loops with the given Cayley graph can be constructed this way[4].

To define the rate of growth of a left loop, we need first to define the growth function of a vertex-transitive graph of finite degree.

Definition 3.12. Let $X = (V, E)$ be a vertex-transitive graph of finite degree. Let $u \in V$. Then the *growth function of X* is the function $\gamma : \mathbb{N} \to \mathbb{N}$ given by

$$\gamma(n) = |\{x \in V : d(x, u) \leq n\}|. \tag{9}$$

Since this is a definition for vertex-transitive graphs, γ does not depend on the choice of u.

Proposition 3.13. *If X and X' are two quasi-isometric vertex-transitive graphs of finite degree with growth functions γ and γ', then γ and γ' have the same asymptotic behaviour.*

PROOF: See [5]. □

With this information now we can define the rate of growth of a left loop.

Definition 3.14. Let L be a left loop and let $S \subseteq L$ be a finite quasi-associative Cayley set that generates L. The *rate of growth of L* is defined as the asymptotic behaviour of the growth function of the Cayley graph $Cay(L, S)$.

By Proposition 3.13, the rate of growth of a left loop L does not depend on the choice of the finite generating quasi-associative Cayley set, but only on the left loop itself.

In conclusion, we hope that with these definitions, the methods that have been used for groups in Geometric Group Theory can also be applied to left loops and that some of the already known results for groups can be generalized to some non-associative structures.

[4]In the case of a non-hyperbolic graph, we cannot ensure that Mwambene's method constructs all the non-hyperbolic left loops with the given Cayley graph since there might be other left loops with that Cayley graph with respect to a non-quasi-associative Cayley set.

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