An overview of free nilpotent Lie algebras

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Abstract. Any nilpotent Lie algebra is a quotient of a free nilpotent Lie algebra of the same nilindex and type. In this paper we review some nice features of the class of free nilpotent Lie algebras. We will focus on the survey of Lie algebras of derivations and groups of automorphisms of this class of algebras. Three research projects on nilpotent Lie algebras will be mentioned.

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1. Introduction

According to Levi's theorem, any finite-dimensional Lie algebra of characteristic zero decomposes as a direct sum of a semisimple Lie algebra and its unique maximal solvable ideal. The classification of semisimple Lie algebras over the complex field was settled at the beginning of the last century. Around 1945, A.I. Malcev [22] reduced the classification of complex solvable Lie algebras to the classification of nilpotent Lie algebras, their derivation algebras, groups of automorphisms and several invariants. But the classification of nilpotent algebras is a wild problem. Most of the results achieved in this direction are partial classifications of algebras satisfying particular properties (2-step nilpotent, maximal rank) or classifications in low (modest) dimension (see [12] for a historical survey).

In 1971, T. Sato [30] (see also [11]) showed that any nilpotent Lie algebra is isomorphic to a quotient, by a suitable ideal, of a free nilpotent Lie algebra of the same nilindex and type. Among the results in [30] we point out the study of the derivations and automorphisms of free nilpotent Lie algebras. It is proved that the Levi subalgebra of the Lie algebra of derivations of any free nilpotent Lie algebra of type d is the special linear algebra $\mathfrak{sl}_d(k)$ of $d \times d$ traceless matrices [30, Proposition 2]. Basic facts on nilpotent Lie algebras are encoded in this simple Lie algebra. On the other hand, the general linear group $GL_d(k)$ of $d \times d$ matrices plays an important role in the construction of the group of automorphisms [30, Proposition 3]. Some general problems on nilpotent Lie algebras can be illuminated by their previous solutions on free nilpotent Lie algebras (see [4], [5], [28]).

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In this paper we will survey the main features of free nilpotent Lie algebras and some recent research on nilpotent Lie algebras. The paper splits into the introductory section, and two main sections. In Section 2, we present the basic terminology on general Lie algebras and some well-known results on the structure of a free nilpotent Lie algebra $\mathfrak{n}_{d,t}$ of type d and nilindex t. The results on structure are collected from different papers. The original results in the paper are included in subsections 2.3 and 2.4. By using irreducible representations of the simple split 3-dimensional Lie algebra $\mathfrak{sl}_2(k)$ we will obtain *nested bases* of free nilpotent Lie algebras in subsection 2.3. These bases let us give explicit matrix representations of derivations and automorphisms of $\mathfrak{n}_{2,4}$ and $\mathfrak{n}_{3,3}$ in subsection 2.4. Section 3 is devoted to explaining and reviewing some results on three different research projects: quasiclassical nilpotent Lie algebras, Lie algebras with a given nilradical, and Anosov Lie algebras. Apart from theoretical considerations, the interest in these projects comes from their physical applications (see [4], [21], [32] and references therein). We also include a series of tables with information on different bases and matrix representations of derivations and automorphisms of free nilpotent Lie algebras in low dimension.

Throughout the paper, vector spaces are considered to be finite-dimensional over a field k of characteristic 0.

2. Free nilpotent Lie algebras

We introduce the basic terminology on Lie algebras, and the usual definition of a free nilpotent Lie algebra. We also present some nice features of derivations and automorphisms of this class of Lie algebras.

2.1 Notation and terminology on Lie algebras. A *Lie algebra* $\mathfrak{g}$ is a vector space endowed with a skewsymmetric binary product, $[a, b]$ (we shall refer to this product as the Lie bracket) that satisfies the Jacobi identity $J(a, b, c) = 0$, where $J(a, b, c) = [[a, b], c] + [[c, a], b] + [[b, c], a]$.

Any associative algebra $\mathfrak{a}$ with product ab becomes a Lie algebra $\mathfrak{a}^-$ by defining the Lie bracket $[a, b] := ab - ba$. In this way, we get the general linear Lie algebra $\mathfrak{gl}(V)$ as the Lie algebra $\mathrm{End}\,(V)^-$ of endomorphisms of a vector space V.

For a given Lie algebra $\mathfrak{g}$, the Lie bracket of two subspaces U and V is the linear span $[U, V] = \mathrm{span}\,\langle [u, v] : u \in U, v \in V\rangle$.

Definition 1. The *derived series* of $\mathfrak{g}$ is defined recursively as $\mathfrak{g} = \mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}]$. The *lower central series* (l.c.s. for short) is also defined recursively as: $\mathfrak{g} = \mathfrak{g}^1$ and $\mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}]$. The Lie algebra $\mathfrak{g}$ is called *solvable* if the derived series vanishes, i.e. there exists $t \in \mathbb{N}$ such that $\mathfrak{g}^{(t)} = 0$. If the lower central series terminates, then $\mathfrak{g}$ is called *nilpotent*. The smallest value of t for which $\mathfrak{g}^{t+1} = 0$ is called the *degree of nilpotency* or *nilindex* of $\mathfrak{g}$.

Definition 2. The *solvable radical* of $\mathfrak{g}$, denoted $\mathfrak{r}$, is the maximal solvable ideal of $\mathfrak{g}$. We also denote by $\mathfrak{n}$ the *nilpotent radical* or *nilradical* of $\mathfrak{g}$, which is the biggest nilpotent ideal.

Definition 3. In case $\mathfrak{g}$ has no proper ideals and $\mathfrak{g}^2 \neq 0$, $\mathfrak{g}$ is a *simple* Lie algebra. The Lie algebras which are direct sums of ideals that are simple as Lie algebras are called *semisimple*.

Levi's Theorem asserts that any Lie algebra can be built from solvable and semisimple Lie algebras:

Theorem 2.1 (Eugenio E. Levi, 1905)**.** *For a given finite-dimensional Lie algebra* $\mathfrak{g}$ *of characteristic* 0 *with solvable radical* $\mathfrak{r}$*, there exists a semisimple subalgebra* $\mathfrak{s}$ *of* $\mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r}$. □

The subalgebra $\mathfrak{s}$ in the previous theorem is called the *Levi subalgebra* of the Lie algebra $\mathfrak{g}$. In 1942, A.I. Malcev proved that any two Levi subalgebras of a fixed Lie algebra $\mathfrak{g}$ are conjugate by an (inner) automorphism of the form $\exp(\operatorname{ad} z)$ for some element z in the nilradical of $\mathfrak{g}$.

Definition 4. A *derivation* of $\mathfrak{g}$ is a linear map satisfying the Leibniz rule $d([x,y]) = [d(x),y] + [x,d(y)]$. For $x \in \mathfrak{g}$, the map $\operatorname{ad} x(a) = [x,a]$ is a derivation which is called an *inner derivation*. An *automorphism* Φ of $\mathfrak{g}$ is a bijective map such that $\Phi([x,y]) = [\Phi(x),\Phi(y)]$.

The Lie bracket $[d_1,d_2] = d_1d_2 - d_2d_1$ of two given derivations is a derivation; so, the whole set $\operatorname{Der}\mathfrak{g}$ of derivations of $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$. The group of automorphisms of $\mathfrak{g}$ will be denoted as $\operatorname{Aut}\mathfrak{g}$. The set of inner derivations $\operatorname{Inner}\mathfrak{g}$ is an ideal of $\operatorname{Der}\mathfrak{g}$.

From inner derivations, we can define the (restricted) adjoint map $ad_{\mathfrak{s}} : \mathfrak{s} \to \mathfrak{gl}(\mathfrak{r})$ given by $x \to \operatorname{ad} x$. This map is a homomorphism of Lie algebras, so the radical of a Lie algebra $\mathfrak{g}$ is an $\mathfrak{s}$-module. In general:

Definition 5. A representation of $\mathfrak{g}$ is a homomorphism of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ where V is a vector space. The vector space V is called a $\mathfrak{g}$-*module* and $x \cdot v = \rho(x)(v)$ is used to denote the way the algebra $\mathfrak{g}$ acts on V via ρ. The module V is *irreducible* if it is non-trivial and does not contain proper submodules.

2.2 Free nilpotent Lie algebras: examples and features. From now on, $\mathfrak{n}$ will be an arbitrary nilpotent Lie algebra. The *type of* $\mathfrak{n}$ is the codimension of $\mathfrak{n}^2$ in $\mathfrak{n}$. Following [30] and [11], any t-nilpotent Lie algebra of type d can be viewed as a quotient of a certain "universal" nilpotent Lie algebra which can be defined through the *free Lie algebra* (see [16, Section 4, Chapter V]) on d generators in the following way:

Definition 6. Let $\mathfrak{FL}(\mathfrak{m})$ be the free Lie algebra on the set of generators $\mathfrak{m} = \{x_1, \dots, x_d\}$, $d \geq 2$. For any $t \geq 1$, the quotient

$$\mathfrak{n}_{d,t} = \frac{\mathfrak{FL}(\mathfrak{m})}{\mathfrak{FL}(\mathfrak{m})^{t+1}}$$

is called the *free t-nilpotent Lie algebra on d generators*.

The free Lie algebra $\mathfrak{FL}(\mathfrak{m})$ is spanned as a vector space by the linear combinations of monomials $[x_{i_1}, \dots, x_{i_s}] = [\dots[[x_{i_1}, x_{i_2}]x_{i_3}]\dots x_{i_s}]$, $s \geq 1$, where $x_{i_j} \in \mathfrak{m}$. The ideal $\mathfrak{FL}(\mathfrak{m})^{t+1}$ is the $(t+1)$-st term of the l.c.s. of $\mathfrak{FL}(\mathfrak{m})$, so it is spanned as a vector space by monomials of length $s \geq t+1$. In low nilindex, we can get simple models of free nilpotent Lie algebras by using multilinear algebra, as the next example shows:

Example 1. The abelian Lie algebra $\mathfrak{n}_{d,1}$ is just a d-dimensional vector space. From [11] and [5] we have:

- Any *free 2-nilpotent algebra of type d* is given by the direct sum $n_{d,2} = k^n \oplus \Lambda^2 k^n$ and the natural Lie bracket: $[u, v] = u \wedge v$, for $u, v \in k^n$ and $[k^n, \Lambda^2 k^n] = 0$. The smaller case $\mathfrak{n}_{2,2} = kx \oplus ky \oplus kz$, with nonzero product $[x, y] = z$ is the Heisenberg 3-dimensional Lie algebra.
- Any *free 3-nilpotent algebra of type d* can be built as $n_{d,3} = k^n \oplus \Lambda^2 k^n \oplus \mathfrak{t}$, where $\mathfrak{t} = \operatorname{span}\langle 2u \otimes (v \wedge w) + v \otimes (u \wedge w) + w \otimes (v \wedge u) : u, v, w \in k^n \rangle$. In this case, the Lie bracket is given by declaring $[u, v] = u \wedge v$ and

$$[u, v \wedge w] = \frac{2}{3} u \otimes (v \wedge w) + \frac{1}{3} v \otimes (u \wedge w) + \frac{1}{3} w \otimes (v \wedge u),$$

for $u, v, w \in k^n$ (other bracket products are trivial). $\square$

The next features of the structure of free nilpotent Lie algebras, their derivations and automorphisms have been collected from [4], [11] and [30]:

Proposition 2.2. *The free nilpotent Lie algebra $\mathfrak{n}_{d,t}$ satisfies:*

a) *$\mathfrak{n}_{d,t}$ is t-nilpotent and of type d.*
b) *$\mathfrak{n}_{d,t} = \mathfrak{m} \oplus \mathfrak{m}^2 \oplus \cdots \oplus \mathfrak{m}^t$ is a quasi-cyclic Lie algebra and* $\dim \mathfrak{m}^s = \frac{1}{s} \sum_{a|s} \mu(a) d^{s/a}$ *where μ is the Möbius function.*
c) *The terms in the l.c.s. of $\mathfrak{n}_{d,t}$ are $\mathfrak{n}_{d,t}^j = \oplus_{s=j}^t m^s$, for $1 \leq j \leq t$.*
d) *The center of $\mathfrak{n}_{d,t}$ is $Z(\mathfrak{n}_{d,t}) = \mathfrak{m}^t$.*
e) *Any t-nilpotent Lie algebra of type d is a quotient of $\mathfrak{n}_{d,t}$.*
f) *Given an ideal J and the corresponding nilpotent Lie algebra quotient $\mathfrak{n} = \frac{\mathfrak{n}_{d,t}}{J}$, the Lie algebra of derivations of $\mathfrak{n}$ is* $\operatorname{Der} \mathfrak{n} = \frac{\mathcal{D}_J}{\mathcal{D}_0}$, *where $\mathcal{D}_J$ is the subalgebra of derivations of $\mathfrak{n}_{d,t}$ that satisfy $d(J) \subseteq J$ and $\mathcal{D}_0$ is the set of derivations of $\mathfrak{n}_{d,t}$ such that $d(\mathfrak{n}_{d,t}) \subseteq J$.*
g) *Up to isomorphism, the Levi subalgebra of* $\operatorname{Der} \mathfrak{n}_{d,t}$ *is $\mathfrak{sl}_d(k)$. Then,* $\operatorname{Der} \mathfrak{n}_{d,t} = \mathfrak{sl}_d(k) \oplus \mathfrak{r}$, *where $\mathfrak{r}$ is the solvable radical of* $\operatorname{Der} \mathfrak{n}_{d,t}$.

h) *The group of automorphisms is a semidirect product of the general linear group $GL_d(k)$.*

PROOF: Assertions a), b), c) and e) follows from the definitions and the results in [11] and [30]. For d) see [4]. The final statements are consequence of the results and proofs in [30, Section 2]. □

Starting with the (ordered) set of generators $\mathfrak{m}$ of $\mathfrak{n}_{d,t}$, and following the recursive procedure given in [14], we get the so called *Hall basis* of $\mathfrak{n}_{d,t}$, the most natural basis for a free nilpotent Lie algebra. In [13], the authors present an algorithm that, using polynomial functions, determines a set of d generators for the free nilpotent Lie algebra $\mathfrak{n}_{d,t}$. The Hall basis associated to this set of generators of polynomial functions has nice properties that, according to [13, Section 3], can be used to derive some results in control theory, and to compute the coefficients in the Baker-Campbell-Hausdorff formula and the universal enveloping algebra of a free Lie algebra.

From the structural properties of $\mathfrak{n}_{d,t}$ given in Proposition 2.2, and using the representation theory of Lie algebras, we will present an alternative method to get different bases of a free nilpotent Lie algebra with rational structure constants (rescaling we can assume that the constants are integers). These bases will be used to get matrix representations of derivations and automorphisms of $\mathfrak{n}_{d,t}$.

2.3 Derivations and bases. The following proposition puts together two statements in [30, Section 2, Propositions 2 and 3] that are essential to get the whole set of derivations and automorphisms of a free nilpotent Lie algebra:

Proposition 2.3. *Any linear map from $\mathfrak{m}$ to $\mathfrak{n}_{d,t}$ can be extended to a derivation of $\mathfrak{n}_{d,t}$ in a unique way, as well as to an endomorphism of $\mathfrak{n}_{d,t}$ as Lie algebra. In particular, the set of derivations of $\mathfrak{n}_{d,t}$ is completely determined by the set of linear maps $\operatorname{Hom}(\mathfrak{m},\mathfrak{n}_{d,t})$, and the set of automorphisms of $\mathfrak{n}_{d,t}$ is given by the set of linear maps $\{\varphi : \mathfrak{m} \to \mathfrak{n}_{d,t} : \operatorname{proj}_{\mathfrak{m}} \circ \varphi \in \operatorname{GL}(\mathfrak{m},\mathfrak{m})\}$, where $\operatorname{proj}_{\mathfrak{m}}$ denotes the projection map on $\mathfrak{m}$.* □

The procedure to extend any linear map $\varphi : \mathfrak{m} \to \mathfrak{n}_{d,t}$ to a derivation is given by applying the Leibniz rule to $\mathfrak{m}^2$, $d_\varphi([x_i, x_j]) = [\varphi(x_i), x_j] + [x_i, \varphi(x_j)]$, and extending to the monomials $[x_{i_1}, \dots x_{i_j}]$ by induction. To obtain an automorphism we start with any linear map $\varphi : \mathfrak{m} \to \mathfrak{n}_{d,t}$ such that $\pi_{\mathfrak{m}} \circ \varphi \in \operatorname{GL}(\mathfrak{m},\mathfrak{m})$ where $\pi_{\mathfrak{m}} : \mathfrak{n}_{d,t} \to \mathfrak{m}$ is the canonical projection. In this case, instead of the Leibniz rule, we make use of $\Phi_\varphi[x_i, x_j] = [\varphi(x_i), \varphi(x_j)]$ and extend to all monomials by induction. Following these two ideas, we found several patterns in the set of derivations of free nilpotent Lie algebras (see [5, Section 2]):

Proposition 2.4. *Let $\mathfrak{n}_{d,t}$ be the free nilpotent Lie algebra given by the set of generators $\mathfrak{m}$. Then, the Lie algebra of derivations of $\mathfrak{n}_{d,t}$ decomposes as:*

$$\operatorname{Der} \mathfrak{n}_{d,t} = \bigoplus_{j=1}^{t} \operatorname{Der}_j \mathfrak{n}_{d,t},$$

where $\operatorname{Der}_j \mathfrak{n}_{d,t} = \{d \in \operatorname{Der} \mathfrak{n}_{d,t} : d(\mathfrak{m}) \subseteq \mathfrak{m}^j\}$*. Moreover, the map* $id_{d,t}$ *defined by* $id_{d,t}|_{\mathfrak{m}^s} = s \cdot Id$ *for* $s \geq 1$ *is a derivation and:*

a) $\operatorname{Der}_1 \mathfrak{n}_{d,t} = \operatorname{Der}_1^0 \mathfrak{n}_{d,t} \oplus k \cdot id_{d,t}$ *is a Lie subalgebra of* $\operatorname{Der} \mathfrak{n}_{d,t}$ *isomorphic to the general linear Lie algebra* $\mathfrak{gl}_d(k)$*. The derived subalgebra of* $\operatorname{Der}_1 \mathfrak{n}_{d,t}$, $\operatorname{Der}_1^0 \mathfrak{n}_{d,t} = [\operatorname{Der}_1 \mathfrak{n}_{d,t}, \operatorname{Der}_1 \mathfrak{n}_{d,t}]$*, is isomorphic to the special linear algebra* $\mathfrak{sl}_d(k)$*, a simple Lie algebra of Cartan type* A_{d-1}*.*

b) *The solvable radical of* $\operatorname{Der} \mathfrak{n}_{d,t}$ *is* $\mathfrak{R}_{d,t} = k \cdot id_{d,t} \oplus \mathfrak{N}_{d,t}$*, where* $\mathfrak{N}_{d,t} = \bigoplus_{j \geq 2}^{t} \operatorname{Der}_j \mathfrak{n}_{d,t}$ *is the nilradical of* $\operatorname{Der} \mathfrak{n}_{d,t}$*.*

In particular, $\operatorname{Der}_1^0 \mathfrak{n}_{d,t}$ *is a Levi subalgebra of* $\operatorname{Der} \mathfrak{n}_{d,t}$*.*

Proof: This follows from Proposition 2.3, and the comments in [30] and [5]. □

In [30], T. Sato described exactly $\operatorname{Der}_1^0 \mathfrak{n}_{d,t}$ as: *the collection of extensions of linear endomorphisms of* $\mathfrak{m}$ *whose traces are zero.*

This remark leads to the following result inspired by [32, Theorem 2]:

Lemma 2.5. *Let* $\mathfrak{s}$ *be a simple Lie algebra with a faithful representation on a vector space of dimension* d*. Then, there exists at least one homomorphism of Lie algebras* $\rho : \mathfrak{s} \to \operatorname{Der}_1^0 \mathfrak{n}_{d,t} \subseteq \mathfrak{gl}(\mathfrak{n}_{d,t})$ *satisfying* $\rho(\mathfrak{m}) \subseteq \mathfrak{m}$*. Moreover, such a* ρ *is a representation of* $\mathfrak{s}$ *on* $\mathfrak{n}_{d,t}$ *and* $\mathfrak{n}_{d,t} = \oplus_{i=1}^{t} \mathfrak{m}^s$ *is an* $\mathfrak{s}$*-module decomposition. In particular, the irreducible components of* $\mathfrak{m}^s$ *for* $s \geq 2$ *are among the irreducible components of the tensor product representation* $\mathfrak{m} \otimes \mathfrak{m}^{s-1}$ *induced by* ρ*.*

Proof: Without loss of generality, we can consider the d-dimensional representation on the set $\mathfrak{m}$ of generators of $\mathfrak{n}_{d,t}$. So, we have a homomorphism $\rho_1 : \mathfrak{s} \to \mathfrak{gl}(\mathfrak{m})$ and, since $\mathfrak{s}$ is simple, $\mathfrak{s} \cong \rho_1(\mathfrak{s}) \subseteq \mathfrak{sl}(\mathfrak{m})$. Now, from Proposition 2.3, $\rho_1(\mathfrak{s})$ can be embedded into $\operatorname{Der}_1^0 \mathfrak{n}_{d,t}$; the embedding is in fact a homomorphism of Lie algebras, so we have a representation $\rho : \mathfrak{s} \to \operatorname{Der}_1^0 \mathfrak{n}_{d,t} \subseteq \mathfrak{gl}(\mathfrak{n}_{d,t})$ and $\mathfrak{m}$ is a submodule. Since ρ is a representation given by derivations, for every $d \in \rho(\mathfrak{s})$, we have $d(\mathfrak{m}) \subseteq \mathfrak{m}$ and $d(\mathfrak{m}^2) = d([\mathfrak{m}, \mathfrak{m}]) = [d(\mathfrak{m}), \mathfrak{m}] \subseteq \mathfrak{m}^2$. Hence $\mathfrak{m}^2$ also is a submodule. Applying this argument recursively, we get that the direct sum $n_{d,t} = \oplus_{i=1}^{t} \mathfrak{m}^s$ is an $\mathfrak{s}$-module decomposition. Note that the linear map $[\cdot, \cdot] : \mathfrak{m} \otimes \mathfrak{m}^{s-1} \to \mathfrak{m}^s$ is a homomorphism of $\mathfrak{s}$-modules which is onto; this proves the last assertion. □

The 3-dimensional set $\mathfrak{sl}_2(k)$ of 2×2 matrices of trace 0 is a simple Lie algebra. Any special linear Lie algebra $\mathfrak{sl}_d(k)$ of $d \times d$ traceless matrices contains different copies of $\mathfrak{sl}_2(k)$. One of the most interesting tools provided by $\mathfrak{sl}_2(k)$ is its representation theory. For every $n \geq 1$, there is a unique faithful $\mathfrak{sl}_2(k)$-irreducible representation $V(n)$ of dimension $n + 1$ (see [15] for a complete description). Moreover, applying the Clebsch-Gordan formula we get the tensor product decomposition:

$$V(n) \otimes_k V(m) \cong V(n+m) \oplus V(n+m-2) \oplus \ldots \oplus V(n-m),$$

(here $n \geq m$ is assumed). From Lemma 2.5 and using the irreducible $\mathfrak{sl}_2(k)$-modules $V(n)$ we can find bases for $\mathfrak{n}_{d,t}$ in a recursive way. Our next example explains this technique (we follow the proof of [5, Proposition 2.3] and terminology and results on representation theory of $\mathfrak{sl}_2(k)$ from [15]):

Example 2. *Nested bases for* $\mathfrak{n}_{2,t}$. In this case, $\mathrm{Der}_1^0\,\mathfrak{n}_{2,t} \cong \mathfrak{sl}_2(k)$ for which $\mathfrak{m}$ is the natural module $V(1)$. Then using the formula in Proposition 2.2 for computing the dimension of each component $\mathfrak{m}^s$ we have:

- $\mathfrak{m}^2$ is 1-dimensional and $\mathfrak{m}^2 \subseteq \mathfrak{m} \otimes \mathfrak{m} = V(1) \otimes V(1) = V(2) \oplus V(0)$. So $\mathfrak{m}^2 = V(0)$, is a trivial module. Thus, considering the standard basis v_0, v_1 of $\mathfrak{m} = V(1)$ and $\mathfrak{m}^2 = \mathrm{span}\langle w_0 = [v_0, v_1]\rangle$ we get the basis of $\mathfrak{n}_{2,2}$ $\{v_0, v_1, w_0\}$.
- $\mathfrak{m}^3$ is 2-dimensional and $\mathfrak{m}^3 \subseteq \mathfrak{m} \otimes \mathfrak{m}^2 = V(1) \otimes V(0) = V(1)$. So $\mathfrak{m}^3 = V(1)$, is a 2-irreducible module. Since $\mathfrak{m}^3 = [\mathfrak{m}, \mathfrak{m}^2] = \mathrm{span}\langle z_0 = [v_0, w_0], z_1 = [v_1, w_0]\rangle$, by adding the set $\{z_0, z_1\}$, which forms a standard basis of $V(1)$, to the basis of $\mathfrak{n}_{2,2}$ previously computed, we get the basis of $\mathfrak{n}_{2,3}$.
- $\mathfrak{m}^4$ is 3-dimensional and $\mathfrak{m}^4 \subseteq \mathfrak{m} \otimes \mathfrak{m}^3 = V(1) \otimes V(1) = V(2) \oplus V(0)$. So $\mathfrak{m}^4 = V(2)$, is a 3-irreducible module. From $\mathfrak{m}^4 = [\mathfrak{m}, \mathfrak{m}^3]$ and $[v_1, z_0] = [v_0, z_1]$, we arrive at the set $\{x_0 = [v_0, z_0], x_1 = 2[v_1, z_0], x_2 = [v_1, z_0]\}$, a standard basis of $V(2)$ inside $\mathfrak{m}^4$. Now $\{x_0, x_1, x_2\}$ along with the previous basis of $\mathfrak{n}_{2,3}$, provides a basis of $\mathfrak{n}_{2,4}$.
- $\mathfrak{m}^5$ is 6-dimensional and $\mathfrak{m}^5 \subseteq \mathfrak{m} \otimes \mathfrak{m}^4 = V(1) \otimes V(2) = V(3) \oplus V(1)$, so $\mathfrak{m}^5 = V(3) \oplus V(1)$. In this case, the set $\{y_0 = [v_0, x_0], y_1 = [v_0, x_1] + [v_1, x_0], y_2 = [v_1, x_1] + [v_0, x_2], y_3 = [v_1, x_2]\}$ spans a module of type $V(3)$ (in fact it is a standard basis) and the set $\{u_0 = [v_0, x_1] - 2[v_1, x_0], u_1 = -[v_1, x_1] + 2[v_0, x_2]\}$ spans a $V(1)$ module (standard basis). In this case, $[\mathfrak{m}^i, \mathfrak{m}^j] = 0$ for $i, j \geq 3$ or $i = 2$ and $j \geq 4$ and the product relation $[w_0, z_i] = \frac{1}{2}u_i$ follows easily using the Jacobi identity. Then, a basis of $\mathfrak{n}_{2,5}$ is given by that of $\mathfrak{n}_{2,4}$ and $\{y_0, y_1, y_2, y_3, u_0, u_1\}$.

$\mathfrak{m}$	$\mathfrak{m}^2$	$\mathfrak{m}^3$	$\mathfrak{m}^4$	$\mathfrak{m}^5$
$V(1)$	$V(0)$	$V(1)$	$V(2)$	$V(3) \oplus V(1)$
v_0, v_1	$[v_0, v_1] = w_0$	$[v_0, w_0] = z_0$	$[v_0, z_0] = x_0$	$\dots$
		$[v_1, w_0] = z_1$	$[v_0, z_1] = [v_1, z_0] = \frac{1}{2}x_1$	$\dots$
			$[v_1, z_1] = x_2$	

TABLE 1. Nested bases for $\mathfrak{n}_{2,t}$

Example 3. *Nested bases for* $\mathfrak{n}_{3,t}$. In this case, $\mathrm{Der}_1^0\,\mathfrak{n}_{2,t} \cong \mathfrak{sl}_3(k)$. From the 3-dimensional representation $\mathfrak{m} = V(2)$ of $\mathfrak{sl}_2(k)$ with standard basis $\{v_0, v_1, v_2\}$ and using arguments analogous to that given in Example 2, we get the bases of

$\mathfrak{m}$	$\mathfrak{m}^2$	$\mathfrak{m}^3$		$\mathfrak{m}^4$
$V(2)$	$V(2)$	$V(4) \oplus V(2)$		$V(6) \oplus V(4) \oplus 2V(2)$
v_0, v_1, v_2	$[v_0, v_1] = w_0$	$[v_0, w_0] = z_0$	$[v_1, w_1] = \frac{1}{2} z_2$	$\ldots$
	$[v_0, v_2] = w_1$	$[v_0, w_1] = \frac{1}{2}(z_1 + x_0)$	$[v_1, w_2] = \frac{1}{2}(z_3 + x_2)$	$\ldots$
	$[v_1, v_2] = w_2$	$[v_1, w_0] = \frac{1}{2}(z_1 - x_0)$	$[v_2, w_1] = \frac{1}{2}(z_3 - x_2)$	
		$[v_0, w_2] = \frac{1}{4}(z_2 + x_1)$	$[v_2, w_2] = z_4$	
		$[v_2, w_0] = \frac{1}{4}(z_2 - x_1)$		
$V(1) \oplus V(0)$	$V(1) \oplus V(0)$	$V(2) \oplus 2V(1) \oplus V(0)$		$\ldots$
v_0, v_1, v	$[v_0, v_1] = w$	$[v_0, w_0] = u_0$	$[v_0, w] = z_0$	$\ldots$
	$[v_0, v] = w_0$	$[v_0, w_1] = \frac{1}{2}(u_1 + y_0)$	$[v_1, w] = z_1$	$\ldots$
	$[v_1, v] = w_1$	$[v_1, w_0] = \frac{1}{2}(u_1 - y_0)$	$[v, w_0] = x_0$	
		$[v_1, w_1] = u_2$	$[v, w_1] = x_1$	
		$[v, w] = y_0$		

TABLE 2. Nested bases for $\mathfrak{n}_{3,t}$

$\mathfrak{n}_{3,2}$ and $\mathfrak{n}_{3,3}$ given in Table 2. In this case, we can also start from the reducible module decomposition $\mathfrak{m} = V(1) \oplus V(0)$ and then we get $\mathfrak{m}^2 = V(1) \oplus V(0)$ and $\mathfrak{m}^3 = V(2) \oplus 2V(1) \oplus V(0)$ as unique possibilities. The basis and the multiplication table starting from this non irreducible decomposition are also included in Table 2.

Remark 1. The *Hall bases* of $\mathfrak{n}_{2,3}$ and $\mathfrak{n}_{3,2}$ given in [4] agree with the bases obtained from representation theory in our Examples 2 and 3. For $d \geq 3$, we can use modules of other simple algebras of $\mathfrak{sl}_d(k)$ to get many different bases.

2.4 Derivations and automorphisms of $\mathfrak{n}_{2,4}$ and $\mathfrak{n}_{3,3}$. From the basis of $\mathfrak{n}_{2,4}$ given in Example 2 (see also Table 1) and canonical computations ($D([x,y]) = [D(x), y] + [x, D(y)]$ and $\phi([x,y]) = [\phi(x), \phi(y)]$ for D a derivation and ϕ an automorphism), we can describe $\operatorname{Der} \mathfrak{n}_{2,4}$ and $\operatorname{Aut} \mathfrak{n}_{2,4}$ using 8×8 matrices. A general derivation of $\mathfrak{n}_{2,4}$ has a matrix of the form ($\beta, \alpha_i \in k$):

$$\begin{pmatrix} \alpha_1 + \beta & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_3 & -\alpha_1 + \beta & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_4 & \alpha_5 & 2\beta & 0 & 0 & 0 & 0 & 0 \\ \alpha_6 & \alpha_7 & \alpha_5 & \alpha_1 + 3\beta & \alpha_2 & 0 & 0 & 0 \\ \alpha_8 & \alpha_9 & -\alpha_4 & \alpha_3 & -\alpha_1 + 3\beta & 0 & 0 & 0 \\ \alpha_{10} & \alpha_{11} & \alpha_7 & \alpha_5 & 0 & 2\alpha_1 + 4\beta & 2\alpha_2 & 0 \\ \alpha_{12} & \alpha_{13} & \frac{\alpha_9 - \alpha_6}{2} & -\frac{\alpha_4}{2} & \frac{\alpha_5}{2} & \alpha_3 & 4\beta & \alpha_2 \\ \alpha_{14} & \alpha_{15} & -\alpha_8 & 0 & -\alpha_4 & 0 & 2\alpha_3 & -2\alpha_1 + 4\beta \end{pmatrix}$$

Any element in the subalgebra $\mathrm{Der}_1\,\mathfrak{n}_{2,4} = \mathrm{Der}_1^0\,\mathfrak{n}_{2,4} \oplus k \cdot id_{2,t} \cong \mathfrak{gl}_2(k)$ has the matrix representation:

$$\begin{pmatrix} \alpha_1+\beta & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_3 & -\alpha_1+\beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1+3\beta & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_3 & -\alpha_1+3\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\alpha_1+4\beta & 2\alpha_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_3 & 4\beta & \alpha_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\alpha_3 & -2\alpha_1+4\beta \end{pmatrix}$$

The Levi subalgebra $\mathrm{Der}_1^0\,\mathfrak{n}_{2,4} \cong \mathfrak{sl}_2(k)$ is the set of traceless matrices of $\mathrm{Der}_1\,\mathfrak{n}_{2,4}$ ($\beta = 0$). The derivation $id_{2,4}$ is given by taking $\beta = 1$ and $\alpha_i = 0$. A general automorphism of $\mathfrak{n}_{2,4}$ is represented by a matrix of the form ($\beta, \alpha_i \in k$):

$$\begin{pmatrix} \alpha_1 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_3 & \alpha_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha_5 & \alpha_6 & \Delta_{14}^{23} & 0 & 0 & 0 & 0 & 0 \\ \alpha_7 & \alpha_8 & \Delta_{16}^{25} & \alpha_1\Delta_{14}^{23} & \alpha_2\Delta_{14}^{23} & 0 & 0 & 0 \\ \alpha_9 & \alpha_{10} & \Delta_{36}^{45} & \alpha_3\Delta_{14}^{23} & \alpha_4\Delta_{14}^{23} & 0 & 0 & 0 \\ \alpha_{11} & \alpha_{12} & \Delta_{18}^{27} & \alpha_1\Delta_{16}^{25} & \alpha_2\Delta_{16}^{25} & \alpha_1^2\Delta_{14}^{23} & 2\alpha_1\alpha_2\Delta_{14}^{23} & \alpha_2^2\Delta_{14}^{23} \\ \alpha_{13} & \alpha_{14} & \frac{\Delta_{1\overline{10}}^{29}-\Delta_{38}^{47}}{2} & \frac{\alpha_3\Delta_{16}^{25}+\alpha_1\Delta_{36}^{45}}{2} & \frac{\alpha_4\Delta_{16}^{25}+\alpha_2\Delta_{36}^{45}}{2} & \alpha_1\alpha_3\Delta_{14}^{23} & \alpha_1\alpha_4\Delta_{14}^{23} & 2\alpha_2\alpha_4\Delta_{14}^{23} \\ \alpha_{15} & \alpha_{16} & \Delta_{3\overline{10}}^{49} & \alpha_3\Delta_{36}^{45} & \alpha_4\Delta_{36}^{45} & \alpha_3^2\Delta_{14}^{23} & 2\alpha_3\alpha_4\Delta_{14}^{23} & \alpha_4^2\Delta_{14}^{23} \end{pmatrix}$$

where $\Delta_{ij}^{kl} = \alpha_i\alpha_j - \alpha_k\alpha_l$ and $\Delta_{14}^{23} \neq 0$. The derivation algebra and the group of automorphisms of the free nilpotent Lie algebras $\mathfrak{n}_{2,t}$ for $t = 1, 2, 3$ can be represented by matrices (α_{ij}) relative to the bases given in Table 1. These matrices are displayed in Table 3.

t	Der $\mathfrak{n}_{2,t}$	Aut $\mathfrak{n}_{2,t}$
1	$\begin{pmatrix} \alpha_1+\beta & \alpha_2 \\ \alpha_3 & -\alpha_1+\beta \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}$, $\epsilon = \alpha_1\alpha_4 - \alpha_2\alpha_3 \neq 0$
2	$\begin{pmatrix} \alpha_1+\beta & \alpha_2 & 0 \\ \alpha_3 & -\alpha_1+\beta & 0 \\ \alpha_4 & \alpha_5 & 2\beta \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & \alpha_4 & 0 \\ \alpha_5 & \alpha_6 & \epsilon \end{pmatrix}$
3	$\begin{pmatrix} \alpha_1+\beta & \alpha_2 & 0 & 0 & 0 \\ \alpha_3 & -\alpha_1+\beta & 0 & 0 & 0 \\ \alpha_4 & \alpha_5 & 2\beta & 0 & 0 \\ \alpha_6 & \alpha_7 & \alpha_5 & \alpha_1+3\beta & \alpha_2 \\ \alpha_8 & \alpha_9 & -\alpha_4 & \alpha_3 & -\alpha_1+3\beta \end{pmatrix}$	$\begin{pmatrix} \alpha_1 & \alpha_2 & 0 & 0 & 0 \\ \alpha_3 & \alpha_4 & 0 & 0 & 0 \\ \alpha_5 & \alpha_6 & \epsilon & 0 & 0 \\ \alpha_7 & \alpha_8 & \alpha_1\alpha_6-\alpha_2\alpha_5 & \epsilon\alpha_1 & \epsilon\alpha_2 \\ \alpha_9 & \alpha_{10} & \alpha_3\alpha_6-\alpha_4\alpha_5 & \epsilon\alpha_3 & \epsilon\alpha_4 \end{pmatrix}$

TABLE 3. Derivations and automorphisms of $\mathfrak{n}_{2,t}$

In the cases $\mathfrak{n}_{2,3}$ and $\mathfrak{n}_{3,2}$, we get descriptions analogous to those given in [4]. Derivations of $\mathfrak{n}_{3,t}$ for $t = 1, 2$ are given in Table 5.

t	$\mathrm{Der}_1^0\,\mathfrak{n}_{2,t}$	$\mathrm{Inner}\,\mathfrak{n}_{2,t}$
1	$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & -\alpha_1 \end{pmatrix}$	0
2	$\begin{pmatrix} \alpha_1 & \alpha_2 & 0 \\ \alpha_3 & -\alpha_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha_4 & \alpha_5 & 0 \end{pmatrix}$
3	$\begin{pmatrix} \alpha_1 & \alpha_2 & 0 & 0 & 0 \\ \alpha_3 & -\alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 & \alpha_2 \\ 0 & 0 & 0 & \alpha_3 & -\alpha_1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha_4 & \alpha_5 & 0 & 0 & 0 \\ \alpha_6 & 0 & \alpha_5 & 0 & 0 \\ 0 & \alpha_6 & -\alpha_4 & 0 & 0 \end{pmatrix}$

TABLE 4. Levi subalgebra and inner derivation algebra of $\mathrm{Der}\,\mathfrak{n}_{2,t}$.

t	$\mathrm{Der}\,\mathfrak{n}_{3,t}$
1	$\begin{pmatrix} \alpha_1+\beta & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5+\beta & \alpha_6 \\ \alpha_7 & \alpha_8 & -(\alpha_1+\alpha_5)+\beta \end{pmatrix}$
2	$\begin{pmatrix} \alpha_1+\beta & \alpha_2 & \alpha_6 & 0 & 0 & 0 \\ \alpha_4 & \alpha_5+\beta & \alpha_7 & 0 & 0 & 0 \\ \alpha_6 & \alpha_7 & -(\alpha_1+\alpha_5)+\beta & 0 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & \alpha_1+\alpha_5+2\beta & \alpha_6 & -\alpha_3 \\ \beta_4 & \beta_5 & \beta_6 & \alpha_8 & -\alpha_5+2\beta & \alpha_2 \\ \beta_7 & \beta_8 & \beta_9 & -\alpha_7 & \alpha_4 & -\alpha_1+2\beta \end{pmatrix}$

TABLE 5. Derivations of $\mathfrak{n}_{3,t}$

t	$\mathrm{Aut}\,\mathfrak{n}_{3,t}$	
1	$\mathbf{A} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}$	$\det \mathbf{A} \neq 0$
2	$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 & 0 \\ \alpha_4 & \alpha_5 & \alpha_6 & 0 & 0 & 0 \\ \alpha_7 & \alpha_8 & \alpha_9 & 0 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 & \Delta_{15}^{24} & \Delta_{16}^{34} & \Delta_{26}^{35} \\ \beta_4 & \beta_5 & \beta_6 & \Delta_{18}^{27} & \Delta_{19}^{37} & \Delta_{29}^{38} \\ \beta_7 & \beta_8 & \beta_9 & \Delta_{48}^{57} & \Delta_{49}^{67} & \Delta_{59}^{68} \end{pmatrix}$	$\Delta_{ij}^{kl} = \alpha_i\alpha_j - \alpha_k\alpha_l$

TABLE 6. Automorphisms of $\mathfrak{n}_{3,t}$

The general shape of a derivation of $\mathfrak{n}_{3,3}$ is:

$$\left(\begin{array}{cccccccccccccc}
\alpha_1+\beta & \alpha_2 & \alpha_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_4 & \alpha_5+\beta & \alpha_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\alpha_7 & \alpha_8 & -\sigma_{15}+\beta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\beta_1 & \beta_2 & \beta_3 & a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\beta_4 & \beta_5 & \beta_6 & a_{21} & a_{22} & a_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\beta_7 & \beta_8 & \beta_9 & a_{31} & a_{32} & a_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\mu_1 & \mu_2 & \mu_3 & b_{11} & b_{12} & b_{13} & c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & c_{17} & c_{18} \\
\mu_4 & \mu_5 & \mu_6 & b_{21} & b_{22} & b_{23} & c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} & c_{27} & c_{28} \\
\mu_7 & \mu_8 & \mu_9 & b_{31} & b_{32} & b_{33} & c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} & c_{37} & c_{38} \\
\mu_{10} & \mu_{11} & \mu_{12} & b_{41} & b_{42} & b_{43} & c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} & c_{47} & c_{48} \\
\mu_{13} & \mu_{14} & \mu_{15} & b_{51} & b_{52} & b_{53} & c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} & c_{57} & c_{58} \\
\mu_{16} & \mu_{17} & \mu_{18} & b_{61} & b_{62} & b_{63} & c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} & c_{67} & c_{68} \\
\mu_{19} & \mu_{20} & \mu_{21} & b_{71} & b_{72} & b_{73} & c_{71} & c_{72} & c_{73} & c_{74} & c_{75} & c_{76} & c_{77} & c_{78} \\
\mu_{22} & \mu_{23} & \mu_{24} & b_{81} & b_{82} & b_{83} & c_{81} & c_{82} & c_{83} & c_{84} & c_{85} & c_{86} & c_{87} & c_{88}
\end{array}\right)$$

where $(a_{ij}),(b_{ij})$ and (c_{ij}) are given by:

$$(a_{ij}) = \begin{pmatrix} \alpha_1+\alpha_5+2\beta & \alpha_6 & -\alpha_3 \\ \alpha_8 & -\alpha_5+2\beta & \alpha_2 \\ -\alpha_7 & \alpha_4 & -\alpha_1+2\beta \end{pmatrix}$$

$$(b_{ij}) = \begin{pmatrix} \beta_2 & \beta_3 & 0 \\ \frac{\beta_5-\beta_1}{2} & \frac{\beta_6}{2} & \frac{\beta_3}{2} \\ \frac{\beta_8-2\beta_4}{4} & \frac{\beta_9-\beta_1}{4} & \frac{2\beta_6-\beta_2}{4} \\ -\frac{\beta_7}{2} & -\frac{\beta_4}{2} & \frac{\beta_9-\beta_5}{2} \\ 0 & -\beta_7 & -\beta_8 \\ \frac{\beta_1+\beta_5}{2} & \frac{\beta_6}{2} & -\frac{\beta_3}{2} \\ \frac{\beta_8}{4} & \frac{\beta_1+\beta_9}{4} & \frac{\beta_2}{4} \\ -\frac{\beta_7}{2} & \frac{\beta_4}{2} & \frac{\beta_5+\beta_9}{2} \end{pmatrix}$$

$$(c_{ij}) = \begin{pmatrix} \Delta+3\beta & \sigma_{26} & 0 & 0 & 0 & -\tau_{26} & -4\alpha_3 & 0 \\ \frac{\sigma_{48}}{2} & \frac{\Delta+6\beta}{2} & \sigma_{26} & 0 & 0 & -\frac{3\alpha_5}{2} & \tau_{26} & -\alpha_3 \\ 0 & \frac{3\sigma_{48}}{4} & 3\beta & \frac{3\sigma_{26}}{4} & 0 & \frac{3\tau_{48}}{4} & 0 & \frac{3\tau_{26}}{4} \\ 0 & 0 & \sigma_{48} & \frac{-\Delta+6\beta}{2} & \frac{\sigma_{26}}{2} & \alpha_7 & \tau_{48} & \frac{2\alpha_5-\beta}{2} \\ 0 & 0 & 0 & \sigma_{48} & -\Delta+3\beta & 0 & 4\alpha_7 & -\tau_{48} \\ -\frac{\tau_{48}}{2} & \frac{-2\alpha_5+\beta}{2} & \tau_{26} & \alpha_3 & 0 & \frac{\Delta+6\beta}{2} & \sigma_{26} & 0 \\ -\frac{\alpha_7}{2} & \frac{\sigma_{48}}{2} & 0 & \frac{\tau_{26}}{4} & \frac{\alpha_3}{2} & \frac{\tau_{48}}{2} & 3\beta & \frac{\sigma_{26}}{4} \\ 0 & -\alpha_7 & \tau_{48} & \frac{2\alpha_5-\beta}{2} & -\frac{\tau_{26}}{2} & 0 & \sigma_{48} & \frac{-\Delta+6\beta}{2} \end{pmatrix}$$

with $\Delta = 2\alpha_1 + \alpha_5$, $\sigma_{ij} = \alpha_i + \alpha_j$ and $\tau_{ij} = \alpha_i - \alpha_j$. Similar computations can be done to get the general matrix of any element in $\operatorname{Aut} \mathfrak{n}_{3,3}$.

3. Some research projects on nilpotent Lie algebras

The general knowledge of Lie algebras and their classification can be useful for both theoretical considerations and practical purposes. The representations of some simple Lie algebras as $\mathfrak{su}(2, \mathbb{R})$ and $\mathfrak{sl}(3, \mathbb{C})$, appear in problems of particle physics; Heisenberg algebras play a fundamental role in quantum mechanics [29], and Yang-Mills gauge theories are related to quasiclassical Lie algebras [24]. In this section we discuss three theoretical research projects on nilpotent Lie algebras with potential applications. We are currently working on project #2; the other two projects will be considered for future work.

3.1 Research project #1: Regular quadratic nilpotent Lie algebras. A Lie algebra endowed with a nondegenerate, symmetric, invariant bilinear form is called *regular quadratic Lie algebra* or *quasiclassical algebra*; such Lie algebras are also known as *metric Lie algebras*. Semisimple Lie algebras with the Killing form are quasiclassical algebras. This class of algebras is useful in conformal field theory and string theory [10], constitutes the basis for the construction of bialgebras, and gives rise to pseudo-Riemannian geometry. The first structure results on general quadratic Lie algebras appear in [9] and [23]; paper [9] focuses on quasiclassical Lie algebras with nontrivial center and includes a complete classification of quadratic nilpotent Lie algebras of dimension ≤ 7. New general classifications are included in [17] and [20]. The results in [17] lead to the classification of indecomposable quasiclassical nilpotent Lie algebras of dimension ≤ 10 in [18]. Recently, in [4] the authors prove that $\mathfrak{n}_{2,3}$ and $\mathfrak{n}_{3,2}$ are the unique free nilpotent Lie algebras that are regular quadratic.

3.2 Research project #2: Lie algebras with a given nilradical. This project is a reformulation of a problem related to the Levi decomposition of a Lie algebra: For a given solvable algebra $\mathfrak{r}$, classify all Lie algebras without semisimple ideals such that $\mathfrak{r}$ is their solvable radical. A Lie algebra without semisimple ideals is called faithful. In 1944, I.A. Malcev [26, Theorem 4.4, Section 4] gives a general answer to this classical problem in terms of derivations and automorphisms of the solvable Lie algebra $\mathfrak{r}$ (see [25] for a complete explanation). According to Malcev, the problem has a positive answer only in case $\operatorname{Der} \mathfrak{r}$ has nonzero Levi subalgebras. This is the main argument showing that there are no faithful nonsolvable Lie algebras with radical a filiform Lie algebra of dimension ≥ 4 (see [2], [5, Corollary 2.6]). For a given nilpotent Lie algebra $\mathfrak{n}$, we can study two questions (the second one depends on the first):

- *Question #2.1: Classify solvable Lie algebras with nilradical* $\mathfrak{n}$.
- *Question #2.2: Classify nonsolvable Lie algebras with nilradical* $\mathfrak{n}$.

Following Malcev's ideas, a general technique to solve both questions is based on extending nilpotent Lie algebras by convenient subalgebras of their Lie algebras of derivations; the isomorphisms among different extensions are determined by using the group of automorphisms. In *Question #2.1*, the solvable algebras arise by means of subalgebras of Der $\mathfrak{n}$ without nilpotent elements for which the corresponding derived subalgebra is contained in the ideal Inner $\mathfrak{n}$. In *Question #2.2* and according to [33, Section 2], the nonsolvable algebras arise from subalgebras of Der $\mathfrak{n}$ that satisfy the previous conditions and the additional feature of being centralized by any Levi subalgebra of Der $\mathfrak{n}$. Explicit classifications that follow these ideas are given in [29], [6], [7] and [1]. Some general structural results and methods on this research project can be found in [27] and [5]. In the last paper the results involve free nilpotent algebras and their quotients.

3.3 Research project #3: Anosov Lie algebras. Anosov diffeomorphisms give examples of structurally stable dynamical systems (see [19, Section 2] for a precise definition). In 1967, S. Smale [31] raised the problem of classifying the nilmanifolds admitting Anosov diffeomorphisms; at the level of Lie algebras, this problem corresponds to the classification of Anosov Lie algebras.

Following [19], a rational Lie algebra $\mathfrak{n}$ of dimension d is said to be Anosov, if it admits an hyperbolic automorphism τ (i.e. all eigenvalues of τ have absolute value different from 1). The map τ is called an Anosov automorphism of the Lie algebra. It is well known that any Anosov Lie algebra is necessarily nilpotent. The free nilpotent Lie algebra $\mathfrak{n}_{d,t}$ is Anosov in case $t < d$.

In 1970, L. Auslander and J. Scheuneman [3] established the correspondence between Anosov automorphisms of nilpotent Lie algebras, and semisimple hyperbolic automorphisms of free nilpotent Lie algebras preserving ideals that satisfy four special conditions called the Auslander-Scheuneman conditions. Following this approach, the study of ideals of free nilpotent Lie algebras yields in [28] general properties of Anosov Lie algebras. The results therein extend the classification of Anosov Lie algebras to some new classes of two-step Lie algebras.

Some background on the present state of knowledge regarding Anosov Lie algebras can be found in [21]. Several natural questions on this class of Lie algebras are included in [19, Section 1].

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