Free associative algebras, noncommutative Gröbner bases, and universal associative envelopes for nonassociative structures

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Abstract. First, we provide an introduction to the theory and algorithms for noncommutative Gröbner bases for ideals in free associative algebras. Second, we explain how to construct universal associative envelopes for nonassociative structures defined by multilinear operations. Third, we extend the work of Elgendy (2012) for nonassociative structures on the 2-dimensional simple associative triple system to the 4- and 6-dimensional systems.

Keywords: free associative algebras; Gröbner bases; composition (diamond) lemma; universal associative envelopes; Lie algebras and triple systems; PBW theorem; Jordan algebras and triple systems; trilinear operations; computer algebra

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1. Introduction

Our primary goal is to apply the theory of noncommutative Gröbner bases in free associative algebras to construct universal associative envelopes for nonassociative systems defined by multilinear operations. We take an algorithmic approach, developing just enough theory to motivate the computational methods. We leave some of the easier proofs as exercises for the reader, and mention a number of open research problems. We begin by recalling the definitions of the most familiar nonassociative structures, finite dimensional Lie and Jordan algebras, and their universal associative enveloping algebras. We work over a field F of characteristic not 2.

1.1 Lie algebras. Lie algebras are defined by the polynomial identities of degree ≤ 3 satisfied by the Lie bracket $[x,y] = xy - yx$ in every associative algebra, namely anticommutativity and the Jacobi identity:

$$[x,x] \equiv 0, \qquad [[x,y],z] + [[y,z],x] + [[z,x],y] \equiv 0.$$

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Every polynomial identity satisfied by the Lie bracket in every associative algebra is a consequence of these two identities; see Corollary 7.2.

Definition 1.1. Let A be an associative algebra with product xy. We write A^- for the Lie algebra with the same underlying vector space as A, but the associative operation is replaced by the Lie bracket $[x, y] = xy - yx$. If a Lie algebra L is isomorphic to a subalgebra of A^- then A is an **associative envelope** for L.

Example 1.2. If $L = \mathfrak{sl}_n(F)$ is the Lie algebra of $n \times n$ matrices of trace 0, then L is a subalgebra of A^- where $A = M_n(F)$ is the associative algebra of $n \times n$ matrices.

Definition 1.3. The **universal associative envelope** $U(L)$ of a Lie algebra L is the unital associative algebra satisfying the following universal property, which implies that $U(L)$ is unique up to isomorphism: There is a morphism of Lie algebras $\alpha\colon L \to U(L)^-$ such that for any unital associative algebra A and any morphism of Lie algebras $\beta\colon L \to A^-$, there is a unique morphism of associative algebras $\gamma\colon U \to A$ satisfying $\beta = \gamma \circ \alpha$. In categorical terminology, the functor sending a Lie algebra L to its universal associative envelope $U(L)$ is left adjoint to the functor sending an associative algebra A to the Lie algebra A^-.

Lemma 1.4. *The subset $\alpha(L)$ generates $U(L)$. If A is another associative envelope for L, and A is generated by the subset L, then A is isomorphic to a quotient of $U(L)$; that is, $A \approx U(L)/I$ for some ideal I.*

We will see that $U(L)$ is always infinite dimensional, and that α is always injective, so that L is isomorphic to a subalgebra of $U(L)^-$. This follows from the PBW theorem (Theorem 7.1) that we will prove using Gröbner bases.

Example 1.5. If L is the n-dimensional Lie algebra with basis $\{x_1, \ldots, x_n\}$ and trivial commutation relations $[x_i, x_j] = 0$ for all i, j then $U(L) \approx F[x_1, \ldots, x_n]$, the commutative associative polynomial algebra in n variables.

1.2 Jordan algebras. Jordan algebras are defined by the polynomial identities of degree ≤ 4 satisfied by the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$ in every associative algebra, namely commutativity and the Jordan identity:

$$x \circ y \equiv y \circ x, \qquad ((x \circ x) \circ y) \circ x \equiv (x \circ x) \circ (y \circ x).$$

(We sometimes omit the scalar $\frac{1}{2}$ in the definition of the Jordan product.) In contrast to Lie algebras, there exist further identities satisfied by the Jordan product in every associative algebra which are not consequences of these two identities. The simplest such identities have degree 8 and were discovered 50 years ago; see [65].

Definition 1.6. Let A be an associative algebra with product xy. We write A^+ for the Jordan algebra with the same underlying vector space as A, but the associative operation is replaced by the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$. If a Jordan algebra J is isomorphic to a subalgebra of A^+ then A is an **associative envelope** for J.

Example 1.7. If $J = S_n(F)$ is the Jordan algebra of symmetric $n \times n$ matrices, then J is a subalgebra of A^+ where $A = M_n(F)$.

Definition 1.3 and Lemma 1.4 have obvious analogues for Jordan algebras. If J is finite dimensional, then so is its universal associative envelope $U(J)$. However, the natural map from J to $U(J)$ may not be injective; hence, strictly speaking, $U(J)$ may not be an associative envelope for J in the sense of Definition 1.6.

Example 1.8. If J is the n-dimensional Jordan algebra with basis $\{x_1, \dots, x_n\}$ and trivial Jordan products $x_i \circ x_j = 0$ for all i, j then $U(J) \approx \Lambda(x_1, \dots, x_n)$, the exterior (Grassmann) algebra on n generators, and hence $\dim U(J) = 2^n$.

Definition 1.9. This definition has no analogue for Lie algebras. If a Jordan algebra J has an associative envelope then J is a **special** Jordan algebra; otherwise, J is an **exceptional** Jordan algebra.

Example 1.10. The real vector space $H_3(\mathbb{O})$ of 3×3 Hermitian matrices over the 8-dimensional division algebra $\mathbb{O}$ of octonions is closed under the Jordan product and is a 27-dimensional exceptional Jordan algebra.

2. Free associative algebras

The following exposition of the theory of noncommutative Gröbner bases is based on de Graaf [48, §§6.1-6.2], but we fill in some details. The best-known original paper is Bergman [10]; similar results were published a little earlier in Bokut [13]. The work of Bokut was motivated by Shirshov's work on Lie algebras [114]; Shirshov's papers have appeared recently in English translation [115].

Definition 2.1. Let $X = \{x_1, \dots, x_n\}$ be an **alphabet**: a finite set of indeterminates (or **letters**). We totally order X by $x_i \prec x_j$ if and only if $i < j$. We write X^* for the set of **monomials** (or **words**) $w = x_{i_1} \cdots x_{i_k}$ where $x_{i_1}, \dots, x_{i_k} \in X$ and $k \geq 0$. (If $k = 0$ then we have the **empty word** $w = 1$.) The **degree** of $w = x_{i_1} \cdots x_{i_k}$ is the number of letters it contains, counting repetitions: $\deg(w) = k$. We define **concatenation** on X^* by $(u, v) \mapsto uv$ for any $u, v \in X^*$; this associative operation makes X^* into the **free monoid** generated by X.

Example 2.2. If $X = \{a\}$ then $X^* = \{a^k \mid k \geq 0\}$ consists of the non-negative powers of a; we have $a^i a^j = a^{i+j}$ so X^* is commutative. If $|X| \geq 2$ then X^* is noncommutative. If $X = \{a, b\}$ then X^* has 2^k distinct words of degree k for $k \geq 0$.

Definition 2.3. A nonempty word $u \in X^*$ is a **subword** (or **factor** or **divisor**) of $w \in X^*$ if $w = v_1 u v_2$ for some $v_1, v_2 \in X^*$. If $v_1 = 1$ then u is a **left** subword; if $v_2 = 1$ then u is a **right** subword. If $u \neq w$ then u is a **proper** subword of w.

Definition 2.4. The total order on X induces one on X^*, the **deglex** (degree lexicographical) order: If $u, w \in X^*$ then $u \prec w$ (u **precedes** w) if and only if

(i) either $\deg(u) < \deg(w)$,

(ii) or $\deg(u) = \deg(w)$ where $u = vx_iu'$ and $w = vx_jw'$ for some $v, u', w' \in X^*$ and $x_i, x_j \in X$ with $x_i \prec x_j$.

Condition (ii) says that we find the common left subword v of highest degree, and compare the next letters using the total order on X.

Example 2.5. For $X = \{a, b\}$ with $a \prec b$, the words in X^* of degree ≤ 3 are:

$$1 \prec a \prec b \prec a^2 \prec ab \prec ba \prec b^2 \prec a^3 \prec a^2b \prec aba \prec ab^2 \prec ba^2 \prec bab \prec b^2a \prec b^3.$$

Definition 2.6. A total order on X^* is **multiplicative** if $u \prec v$ implies $uw \prec vw$ and $wu \prec wv$ for all $w \in X^*$ (equivalently, $w_1uw_2 \prec w_1vw_2$ for all $w_1, w_2 \in X^*$).

Definition 2.7. A total order on X^* satisfies the **descending chain condition** (**DCC**) if $w_1 \succeq w_2 \succeq \cdots \succeq w_n \succeq \cdots$ implies $w_n = w_{n+1} = \cdots$ for some n: there are no infinite strictly decreasing sequences; the set $\{v \in X^* \mid v \prec w\}$ is finite for every $w \in X^*$. This allows us to use induction with respect to the total order.

Lemma 2.8. *The total order of Definition* 2.4 *is multiplicative and satisfies DCC.*

Definition 2.9. We write $F\langle X \rangle$ for the vector space with basis X^*. Concatenation in X^* extends bilinearly to $F\langle X \rangle$: for $a_i, b_j \in F$ and $u_i, v_j \in X^*$ we set

$$\Big(\sum_i a_i u_i \Big) \Big(\sum_j b_j v_j \Big) = \sum_{i,j} a_i b_j u_i v_j.$$

This makes $F\langle X \rangle$ into the **free associative algebra** generated by X; the empty word acts as the unit element. Elements of $F\langle X \rangle$ are linear combinations of monomials in X^*; we call them **noncommutative polynomials**.

Example 2.10. If $X = \{a\}$ then $F\langle X \rangle = F[a]$, the algebra of commutative associative polynomials in one variable. If X has two or more elements, then $F\langle X \rangle$ and $F[X]$ do not coincide: $F[X]$ is commutative but $F\langle X \rangle$ is noncommutative.

Definition 2.11. Consider a nonzero element $f \in F\langle X \rangle$:

$$f = \sum_{i \in \mathcal{I}} a_i u_i,$$

where $\mathcal{I}$ is a nonempty finite index set and $a_i \neq 0$ for all $i \in \mathcal{I}$. The **support** of f is the set of monomials occurring in f: $\mathrm{support}(f) = \{u_i \mid i \in \mathcal{I}\}$. By convention $\mathrm{support}(0) = \emptyset$. If $f \neq 0$ then $\mathrm{support}(f)$ is a nonempty finite set; the greatest element with respect to the total order $\prec$ is the **leading monomial** $LM(f)$. The coefficient of $LM(f)$ is the **leading coefficient** $lc(f)$. If $lc(f) = 1$ then f is **monic**. For a subset $S \subseteq F\langle X \rangle$, we write $LM(S) = \{LM(f) \mid f \in S\}$.

Definition 2.12. The **standard form** of a nonzero element $f \in F\langle X \rangle$ consists of f divided by $lc(f)$ with the monomials in reverse deglex order. Thus the standard form is monic and the leading monomial occurs in the first (leftmost) position.

3. Universal associative envelopes of Lie and Jordan algebras

Definition 3.1. Every associative algebra A is isomorphic to a quotient $F\langle X\rangle/I$ for some set X and some ideal $I \subseteq F\langle X\rangle$. If I is generated by the subset $G \subset I$ then the pair (X, G) is a **presentation** of A by **generators** and **relations**.

3.1 Lie algebras. Let L be a Lie algebra with basis $X = \{x_1, \dots, x_d\}$ and structure constants $c_{ij}^k \in F$ for $1 \le i, j, k \le d$:

$$[x_i, x_j] = \sum_{k=1}^{d} c_{ij}^k x_k.$$

Let $F\langle X\rangle$ be the free associative algebra generated by X. (We regard the basis elements as formal variables, but this should not cause confusion.) Let I be the ideal in $F\langle X\rangle$ generated by these $d(d-1)/2$ elements for $1 \le j < i \le d$:

$$x_i x_j - x_j x_i - \sum_{k=1}^{d} c_{ij}^k x_k.$$

The quotient algebra $U(L) = F\langle X\rangle/I$ is the universal associative envelope of L.

Example 3.2. The Lie algebra $\mathfrak{sl}_2(F)$ of 2×2 matrices of trace 0 has this basis:

$$h = E_{11} - E_{22} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e = E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad f = E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

These basis elements satisfy the equations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$ from which we obtain the generators $he - eh - 2e$, $hf - fh + 2f$, $ef - fe - h$ for I. The universal associative envelope is the quotient $U(\mathfrak{sl}_2(F)) = F\langle h, e, f\rangle/I$.

3.2 Jordan algebras. Let J be a Jordan algebra with basis $X = \{x_1, \dots, x_d\}$ and structure constants $c_{ij}^k \in F$ for $1 \le i, j, k \le d$:

$$x_i \circ x_j = \sum_{k=1}^{d} c_{ij}^k x_k.$$

Let $I \subseteq F\langle X\rangle$ be the ideal generated by these $d(d+1)/2$ elements for $1 \le j \le i \le d$:

$$\tfrac{1}{2}(x_i x_j + x_j x_i) - \sum_{k=1}^{d} c_{ij}^k x_k.$$

The quotient algebra $U(J) = F\langle X\rangle/I$ is the universal associative envelope of J.

Example 3.3. The Jordan algebra $S_2(F)$ of symmetric 2×2 matrices has basis

$$a = E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad b = E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad c = E_{12} + E_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

These basis elements satisfy the equations $a \circ a = 2a$, $a \circ b = 0$, $a \circ c = c$, $b \circ b = 2b$, $b \circ c = c$, $c \circ c = 2a + 2b$ from which we obtain the generators $a^2 - a$, $ba + ab$, $ca + ac - c$, $b^2 - b$, $cb + bc - c$, $c^2 - b - a$ for I. The universal associative envelope is the quotient $U(S_2(F)) = F\langle a, b, c\rangle / I$.

4. Normal forms of noncommutative polynomials

To understand the structure of the quotient algebra $F\langle X\rangle/I$, we need to find a basis and express the product of any two basis elements as a linear combination of basis elements. It suffices to construct a Gröbner basis for I: a set of generators with special properties which will be explained in this section and the next.

4.1 Normal forms modulo an ideal. A basis for $F\langle X\rangle/I$ is a subset $B \subset F\langle X\rangle$ consisting of coset representatives: the elements $b + I$ for $b \in B$ are linearly independent in $F\langle X\rangle/I$ and span $F\langle X\rangle/I$. Equivalently, B is a basis for a complement $C(I)$ to I in $F\langle X\rangle$, meaning that $F\langle X\rangle = I \oplus C(I)$, the direct sum of subspaces.

Lemma 4.1. *Assume that I is an ideal in $F\langle X\rangle$ and that B is a subset of $F\langle X\rangle$. The set $\{b + I \mid b \in B\}$ is a basis of $F\langle X\rangle/I$ if and only if the elements of B are linearly independent and $F\langle X\rangle = I \oplus \operatorname{span}(B)$.*

Definition 4.2. Let I be an ideal in $F\langle X\rangle$. The set $N(I) \subseteq X^*$ of **normal words** modulo I consists of all monomials which are *not* leading monomials of elements of I: that is, $N(I) = X^* \setminus LM(I) = \{w \in X^* \mid w \notin LM(I)\}$. The **complement** to I in $F\langle X\rangle$ is the subspace $C(I) \subseteq F\langle X\rangle$ with basis $N(I)$.

Proposition 4.3. *We have $F\langle X\rangle = I \oplus C(I)$.*

Proof: We follow de Graaf [48, Proposition 6.1.1] but provide more details. The basic idea is the division algorithm for noncommutative polynomials.

First, we prove that $I \cap C(I) = \{0\}$. Assume that $f \in I$ and $f \in C(I)$. If $f \neq 0$ then $f \in I$ implies $LM(f) \in LM(I)$, but $f \in C(I)$ implies $LM(f) \in N(I)$ so $LM(f) \notin LM(I)$. This contradiction implies $f = 0$.

Second, we prove that if $f \in F\langle X\rangle$ then $f = g + h$ with $g \in I$ and $h \in C(I)$. If $f = 0$ then $g = h = 0$, so we now assume $f \neq 0$. We use induction on leading monomials with respect to the total order $\prec$ on X^* which satisfies the DCC.

For the basis of the induction, if $LM(f) = 1$ (empty word) then $f = \alpha \in F\setminus\{0\}$. If $I = F\langle X\rangle$ then $N(I) = \emptyset$, $C(I) = \{0\}$, and $f = \alpha + 0$ with $\alpha \in I$, $0 \in C(I)$. If $I \neq F\langle X\rangle$ then $1 \notin LM(I)$ so $1 \in N(I)$; we have $f = 0 + \alpha$ with $0 \in I$, $\alpha \in C(I)$.

Since X^* satisfies the DCC, we may now assume the claim for all $f_0 \in F\langle X\rangle$ with $LM(f_0) \prec LM(f)$. This inductive hypothesis depends on the fact that only finitely many elements of X^* precede $LM(f)$. We have $f = \alpha LM(f) + f_0$ where $\alpha = lc(f) \in F \setminus \{0\}$, and either $f_0 = 0$ or $LM(f_0) \prec LM(f)$.

If $f_0 = 0$ then $f = \alpha LM(f)$; if $LM(f) \in I$ then $f = \alpha LM(f) + 0 \in I + C(I)$, and if $LM(f) \notin I$ then $LM(f) \in N(I)$ and $f = 0 + \alpha LM(f) \in I + C(I)$.

If $f_0 \neq 0$ then $LM(f_0) \prec LM(f)$; by induction $f_0 = g_0 + h_0$ with $g_0 \in I$, $h_0 \in C(I)$. We have two cases: $LM(f) \in N(I)$, $LM(f) \notin N(I)$. If $LM(f) \in N(I)$

then $f = \alpha LM(f) + (g_0 + h_0) = g_0 + (\alpha LM(f) + h_0) \in I + C(I)$. If $LM(f) \notin N(I)$ then by definition of $N(I)$ we have $LM(f) = LM(k)$ for some $k \in I \setminus \{0\}$. (We cannot assume $LM(f) \in I$. We are choosing $k \in I$ which has the same leading monomial as f. Finding an algorithm to construct such an element k is one of the main goals of the theory of noncommutative Gröbner bases.)

Write $k = \beta LM(k) + k_0$ where $\beta = lc(k) \in F \setminus \{0\}$, and either $k_0 = 0$ or $LM(k_0) \prec LM(k) = LM(f)$. Then

$$\begin{aligned} f - \frac{\alpha}{\beta} k &= \Big(\alpha LM(f) + (g_0 + h_0) \Big) - \frac{\alpha}{\beta} \Big(\beta LM(k) + k_0 \Big) \\ &= \alpha LM(f) + g_0 + h_0 - \alpha LM(k) - \frac{\alpha}{\beta} k_0 = g_0 + h_0 - \frac{\alpha}{\beta} k_0. \end{aligned}$$

If $k_0 = 0$ then

$$f = \Big(\frac{\alpha}{\beta} k + g_0 \Big) + h_0 \in I + C(I).$$

If $k_0 \neq 0$ then by induction $k_0 = \ell_0 + m_0$ where $\ell_0 \in I$ and $m_0 \in C(I)$. We have

$$\begin{aligned} f &= \frac{\alpha}{\beta} k + g_0 + h_0 - \frac{\alpha}{\beta} k_0 = \frac{\alpha}{\beta} k + g_0 + h_0 - \frac{\alpha}{\beta} \big(\ell_0 + m_0 \big) \\ &= \Big(\frac{\alpha}{\beta} k + g_0 - \frac{\alpha}{\beta} \ell_0 \Big) + \Big(h_0 - \frac{\alpha}{\beta} m_0 \Big). \end{aligned}$$

The first three terms belong to I, and the last two terms belong to $C(I)$. $\square$

Corollary 4.4. *Let I be an ideal in $F\langle X \rangle$. Then every element $f \in F\langle X \rangle$ has a unique decomposition $f = g + h$ where $g \in I$ and $h \in C(I)$.*

Definition 4.5. For $f \in F\langle X \rangle$ and an ideal $I \subseteq F\langle X \rangle$, the element $h \in C(I)$ uniquely determined by Corollary 4.4 is the **normal form** of f modulo I; we write $h = NF_I(f)$ or simply $NF(f)$ if I is understood.

Lemma 4.6. *Let $I \subseteq F\langle X \rangle$ be an ideal. Define a product $f \cdot g$ on $C(I)$ as follows: For any $f, g \in C(I)$ set $f \cdot g = NF_I(fg)$. Then the algebra consisting of the vector space $C(I)$ with the product $f \cdot g$ is isomorphic to the quotient algebra $F\langle X \rangle / I$.*

This shows how to find a basis and structure constants for the quotient $F\langle X \rangle / I$. But we must be able to determine the basis $N(I)$ of the complement $C(I)$, and to calculate the normal form $NF_I(f)$ for every element $f \in F\langle X \rangle$.

4.2 Computing normal forms. Our next task is to find an algorithm whose input is an element $f \in F\langle X \rangle$ and an ideal $I \subseteq F\langle X \rangle$ given by a set G of generators, and whose output is the normal form $NF_I(f)$. We first give an algorithm for computing the normal form $NF(f, G)$ of f with respect to G. Unfortunately, the output depends on G: if G_1 and G_2 are two generating sets for the ideal I, then we may have $NF(f, G_1) \neq NF(f, G_2)$. Even for one set G, the output depends on the choice of reductions performed at each step; see Example 4.9. Therefore in general the output is not the normal form of f modulo I. On the other hand, if G is a Gröbner basis for I then we always have $NF(f, G) = NF_I(f)$.

Definition 4.7. If $f \in F\langle X\rangle$ and G is a finite subset of $F\langle X\rangle$, then f is in **normal form** with respect to G if this condition holds: for every $g \in G$ and $w \in \text{support}(f)$, the leading monomial $LM(g)$ is not a subword of w.

The algorithm for computing $NF(f, G)$ is similar to the proof of Proposition 4.3: the division algorithm for noncommutative polynomials. We may assume without loss of generality that the elements of G are monic. Consider the set $LM(G)$ of leading monomials of elements of G. For $v \in LM(G)$ and $w \in \text{support}(f)$ we can easily determine if v is a subword of w. If this never occurs, then f is in normal form with respect to G, and we are done. Otherwise, $w = u_1 v u_2$ for some $u_1, u_2 \in X^*$, and f contains the term αw for some $\alpha \in F \setminus \{0\}$. There exists $g \in G$ with $LM(g) = v$; we replace f by $f_2 = f - \alpha u_1 g u_2$ to eliminate the term αw. Repeating this procedure, we obtain a sequence $f_1 = f, f_2, f_3, \ldots, f_n, \cdots \in F\langle X\rangle$ which converges since X^* satisfies the DCC; see Figure 1.

NormalForm(f, G)

Input: An element $f \in F\langle X\rangle$ and a finite monic subset $G \subset F\langle X\rangle$.

Output: A normal form of f with respect to G.

(1) Set $n \leftarrow 0$, $f_0 \leftarrow 0$, $f_1 \leftarrow f$.
(2) While $f_n \neq f_{n+1}$ do:
  (a) Set $n \leftarrow n + 1$.
  (b) If $w = u_1 v u_2$ for some $w \in \text{support}(f_n)$ and $v = LM(g) \in LM(G)$ then set $f_{n+1} \leftarrow f_n - \alpha u_1 g u_2$ else set $f_{n+1} \leftarrow f_n$.
(3) Return f_n.

FIGURE 1. Algorithm for a normal form of f with respect to G

Lemma 4.8. *In Figure 1 we have $LM(f_1) \succeq LM(f_2) \succeq \cdots \succeq LM(f_n) \succeq \cdots$, and so $LM(f_n) = LM(f_{n+1}) = \cdots$ for some $n \geq 1$. Hence the algorithm terminates, and its output f_n is a normal form of f with respect to G. We have $f_n + I = f + I$ in $F\langle X\rangle/I$; that is, f_n is congruent to f modulo the ideal I generated by G.*

A normal form of f with respect to G is not uniquely determined by the algorithm of Figure 1: the output depends on the choices of v and w in step (2)(b). Hence the output does not necessarily equal $NF_I(f)$, which is uniquely determined by Corollary 4.4.

Example 4.9. Let $X = \{a, b, c\}$ and let $I \subset F\langle X\rangle$ be the ideal generated by

$$G = \{g_1, \ldots, g_6\} = \{a^2 - a,\, ba + ab,\, b^2 - b,\, ca + ac - c,\, cb + bc - c,\, c^2 - b - a\}.$$

(We have seen this set before in Example 3.3.) We compute the normal form of $f_1 = c^2 b$ with respect to G in two different ways, and obtain two different answers. Neither of these is $NF_I(c^2 b) = b$; see Example 6.7.

Choosing g_6 we obtain $f_2 = f_1 - g_6 b = c^2 b - (c^2 b - b^2 - ab) = b^2 + ab$. Next choosing $g_3 = b^2 - b$ we obtain $f_3 = f_2 - g_3 = b^2 + ab - (b^2 - b) = ab + b$. No further reductions are possible, so the algorithm returns $ab + b$.

Making different choices at each step, we obtain the following results:

$$
\begin{aligned}
f_2 &= f_1 - c g_5 = c^2 b - (c^2 b + cbc - c^2) = -cbc + c^2, \\
f_3 &= f_2 + g_5 c = -cbc + c^2 + (cbc + bc^2 - c^2) = bc^2, \\
f_4 &= f_3 - b g_6 = bc^2 - (bc^2 - b^2 - ba) = b^2 + ba, \\
f_5 &= f_4 - g_3 = b^2 + ba - (b^2 - b) = ba + b, \\
f_6 &= f_5 - g_2 = ba + b - (ba + ab) = -ab + b.
\end{aligned}
$$

No further reductions are possible, so the algorithm returns $-ab + b$.

5. Gröbner bases for ideals in free associative algebras

If the set G of generators of the ideal I is a Gröbner basis, then the output of Figure 1 is uniquely determined, and equals the normal form of f modulo I.

Definition 5.1. Let X be a finite set and let $I \subseteq F\langle X\rangle$ be an ideal. The set G of generators for I is a **Gröbner basis** for I if the following condition holds: For every nonzero $f \in I$ there exists $g \in G$ such that $LM(g)$ is a subword of $LM(f)$.

Remark 5.2. A Gröbner basis is not a basis for I as a vector space over F, but rather a set of generators for I as a (two-sided) ideal in $F\langle X\rangle$.

The next theorem explains the importance of Gröbner bases. Recall that the set $N(I)$ of all normal words modulo I is the complement of $LM(I)$ in X^*. If G is a Gröbner basis for I, then we can easily compute $N(I)$ using part (a), and we can easily compute $NF_I(f)$ for all $f \in F\langle X\rangle$ using part (b).

Theorem 5.3. *If G is a Gröbner basis for the ideal $I \subseteq F\langle X\rangle$ then:*

(a) $N(I) = \{w \in X^* \mid$ *for all* $g \in G$, $LM(g)$ *is not a subword of* $w\}$;

(b) *for all* $f \in F\langle X\rangle$ *we have* $NF_I(f) = NF(f, G)$: *the normal form of f modulo I equals the normal form of f with respect to G.*

Proof: Part (a) follows immediately from Definitions 4.2 and 5.1. For part (b), consider $f \in F\langle X\rangle$ and let $h = NF(f, G)$ be its normal form with respect to G computed by Figure 1. For any $w \in \mathrm{support}(h)$, since $h \in I$ and G is a Gröbner basis for I, Definition 4.7 implies that $LM(g)$ is not a subword of w for any $g \in G$. Part (a) of the theorem now shows that $w \in N(I)$; since this holds for all $w \in \mathrm{support}(h)$, we have $h \in C(I)$. By the last statement of Lemma 4.8 we have $f - h \in I$. Since $f = (f - h) + h \in I \oplus C(I)$, the uniqueness in Corollary 4.4 implies $h = NF_I(f)$. $\square$

Theorem 5.3 is very powerful, but we have to find an algorithm whose input is a set G of generators for the ideal $I \subseteq F\langle X\rangle$, and whose output is a Gröbner basis of I. This requires defining overlaps and compositions for generators (Definition 5.8), and proving the Composition (Diamond) Lemma (Lemma 6.3).

Definition 5.4. A finite subset $G \subset F\langle X\rangle$ is **self-reduced** if (1) every $g \in G$ is in normal form with respect to $G \setminus \{g\}$, and (2) every $g \in G$ is in standard form.

Remark 5.5. Condition (1) in Definition 5.4 is stronger than [48, Definition 6.1.5], which requires only that $LM(h)$ is not a subword of $LM(g)$ for all $g \in G$, $h \in G \setminus \{g\}$.

Using Figure 1, we can construct an algorithm whose input is a finite subset $G \subset F\langle X\rangle$ generating an ideal $I \subseteq F\langle X\rangle$ and whose output is a self-reduced set generating the same ideal. However, the set $\{\, NF(g, G \setminus \{g\}) \mid g \in G\}$ may not generate the same ideal, and may not be self-reduced; so we have to be careful.

Example 5.6. Let $X = \{a, b, c\}$ with $a \prec b \prec c$, and let $G = \{c-a, c-b\}$. Then G is not self-reduced; computing the normal form of each element with respect to the other gives $c-a-(c-b) = b-a$ and $c-b-(c-a) = -b+a$ (with standard form $b-a$). Clearly the set $\{b-a\}$ does not generate the same ideal as G.

Let $X = \{a, b, c, d\}$ with $a \prec b \prec c \prec d$, and let $G = \{d-a, d-b, d-c\}$. Then G is not self-reduced; one way of computing the normal form of each element with respect to the others is as follows, replacing each result by its standard form:

$$d-a-(d-b) = b-a, \quad d-b-(d-c) = c-b, \quad d-c-(d-a) = a-c \to c-a.$$

Clearly the set $\{b-a,\ c-b,\ c-a\}$ is not self-reduced.

Exercise 5.7. Construct an algorithm whose input is a finite subset $G \subset F\langle X\rangle$ generating an ideal $I \subseteq F\langle X\rangle$ and whose output is a self-reduced set generating I. (Sort G using deglex order of leading monomials.)

Definition 5.8. Consider two nonzero elements $g_1, g_2 \in F\langle X\rangle$ in standard form; we allow $g_1 = g_2$. Set $w_1 = LM(g_1)$ and $w_2 = LM(g_2)$. Assume that:

(1) w_1 is not a proper subword of w_2, and w_2 is not a proper subword of w_1. This condition is satisfied if g_1, g_2 belong to a self-reduced set.
(2) For some words $u_1, u_2, v \in X^*$ with $v \neq 1$ we have $w_1 = u_1 v$ and $w_2 = v u_2$. Condition (1) implies that $u_1 \neq 1$ and $u_2 \neq 1$.

We call v an **overlap** between w_1 and w_2; we have $w_1 u_2 = u_1 v u_2 = u_1 w_2$, where u_1 (resp. u_2) is a proper left (resp. right) subword of w_1 (resp. w_2). We call $g_1 u_2 - u_1 g_2$ a **composition** of g_1 and g_2; the common term cancels, since both g_1 and g_2 are monic. (Compositions are sometimes called S-polynomials.)

Example 5.9. Consider $w_1 = a^2bcba$ and $w_2 = bacba^2$ in X^* where $X = \{a, b, c\}$:

- w_1 has a self-overlap: $w_1 = u_1 v = v u_2$ for $u_1 = a^2bcb$, $v = a$, $u_2 = abcba$.
- w_1 and w_2 overlap: $w_1 = u_1 v$, $w_2 = v u_2$ for $u_1 = a^2bc$, $v = ba$, $u_2 = cba^2$.
- w_2 and w_1 have overlaps of length 1 and length 2:
  - $w_2 = u_2 v$, $w_1 = v u_1$ for $u_2 = bacba$, $v = a$, $u_1 = abcba$.
  - $w_2 = u_2 v$, $w_1 = v u_1$ for $u_2 = bacb$, $v = a^2$, $u_1 = bcba$.

Example 5.10. Consider the generators $g_5 = cb + bc - c$ and $g_6 = c^2 - b - a$ from Example 3.3. There is a composition with $w_6 = c^2$, $w_5 = cb$, $u_6 = c$, $u_5 = b$, $v = c$:

$$\begin{aligned} g_6 u_5 - u_6 g_5 &= (c^2 - b - a)b - c(cb + bc - c) = c^2 b - b^2 - ab - c^2 b - cbc + c^2 \\ &= -b^2 - ab - cbc + c^2 \xrightarrow{\text{sf}} cbc - c^2 + b^2 + ab, \end{aligned}$$

where the arrow denotes replacing the polynomial by its standard form (sf).

Remark 5.11. To motivate considering compositions, suppose that g_1, g_2 belong to a set G of generators of the ideal I, and that $s = g_1 u_2 - u_1 g_2$ is a composition. If $h = NF(s, G)$ is nonzero, then $h \in I$ and $LM(h)$ is not divisible by any $w \in LM(G)$. If we replace G by $G \cup \{h\}$, then we are one step closer to a Gröbner basis for I.

6. The composition (diamond) lemma

This lemma is fundamental to the theory of Gröbner bases, and leads to an algorithm for constructing a Gröbner basis for an ideal from a given set of generators. The reason for the name is as follows. We have $f \in F\langle X \rangle$, and we want to compute its normal form (Figure 1) with respect to a subset $G \subset F\langle X \rangle$. At every step, there may be different choices: many leading monomials of elements of G may occur as subwords of many monomials in f. We want to be sure that whatever choices we make, the result will be the same. This condition is called the "resolution of ambiguities", and is illustrated by this "diamond":

$$\begin{array}{ccccc} & & f = f_0 & & \\ & \swarrow & & \searrow & \\ f'_i & & & & f''_j \\ & \searrow & & \swarrow & \\ & & f'_s = f''_t & & \end{array}$$

Definition 6.1. Let $G = \{g_1, \dots, g_n\}$ be a set of generators for the ideal $I \subseteq F\langle X \rangle$. For any $w \in X^*$ we define $I(G, w)$ to be the subspace of I spanned by the elements ugv where $g \in G$ and $u, v \in X^*$ and $LM(ugv) \prec w$:

$$I(G, w) = \left\{ \sum_{i=1}^{n} \alpha_i u_i g_i v_i \;\middle|\; \alpha_i \in F;\ u_i, v_i \in X^*;\ LM(u_i g_i v_i) \prec w \right\}.$$

Thus $I(G, w)$ is the subspace of I, relative to the generating set G, consisting of the elements all of whose monomials precede w in the total order on X^*.

Remark 6.2. Condition (2) in the next lemma implies that every composition of elements of G is a linear combination of elements $u_i g_i v_i$ where $g_i \in G$ and $u_i, v_i \in X^*$ with $u_i LM(g_i) v_i \prec LM(g)u = vLM(h)$.

Lemma 6.3. Composition (Diamond) Lemma. *If G is a monic self-reduced set generating the ideal $I \subseteq F\langle X \rangle$, then these conditions are equivalent:*

(1) *G is a Gröbner basis for I;*
(2) *for every pair $g, h \in G$, if $LM(g)u = vLM(h)$ for some $u, v \in X^*$, then $gu - vh \in I(G, t)$ where $t = LM(g)u = vLM(h)$.*

PROOF: We follow [48, Theorem 6.1.6] but fill in some details.

(1) $\Longrightarrow$ (2): Assume G is a Gröbner basis. For $g, h \in G$ let $f = gu - vh$ where $LM(g)u = vLM(h)$ for some $u, v \in X^*$. Clearly $f \in I$ and so $NF_I(f) = 0$. For $t = LM(g)u = vLM(h)$ we have $LM(f) \prec t$ since the leading terms of $LM(g)u$ and $vLM(h)$ cancel. When we apply Figure 1 to compute $NF(f, G)$, we repeatedly subtract terms $\alpha u_1 k u_2$ where $\alpha \in F$, $k \in G$, $u_1, u_2 \in X^*$, and $LM(u_1 k u_2) \prec t$. All these terms belong to I and hence to $I(G, t)$. Since G is a Gröbner basis, $NF(f, G) = NF_I(f) = 0$. Hence f is a sum of terms in $I(G, t)$, and so $f \in I(G, t)$.

(2) $\Longrightarrow$ (1): We assume condition (2) and prove that G is a Gröbner basis for I. Let $f \in I$ be arbitrary; for some $\alpha_i \in F$, $u_i, v_i \in X^*$, and $g_i \in G$ we have

$$f = \sum_{i=1}^{n} \alpha_i u_i g_i v_i. \tag{1}$$

We need to show that $LM(g)$ is a subword of $LM(f)$ for some $g \in G$. We write $s_i = LM(u_i g_i v_i)$. Renumbering the generators in G if necessary, we assume that

$$s_1 = \cdots = s_\ell \succ s_{\ell+1} \succeq \cdots \succeq s_n. \tag{2}$$

Thus ℓ is the number of equal highest monomials in deglex order; the remaining monomials strictly precede these and are sorted in weak reverse deglex order.

If $\ell = 1$ then $s_1 \succ s_2$ and so $LM(f) = u_1 s_1 v_1 = u_1 LM(g_1) v_1$ as required.

We now assume $\ell \geq 2$. In this case we can rewrite equation (1) as follows:

$$f = \alpha_1 (u_1 g_1 v_1 - u_2 g_2 v_2) + (\alpha_1 + \alpha_2) u_2 g_2 v_2 + \sum_{i=3}^{n} \alpha_i u_i g_i v_i.$$

Since $\ell \geq 2$, we have $u_1 LM(g_1) v_1 = u_2 LM(g_2) v_2$. If $u_1 = u_2$ then $LM(g_1) v_1 = LM(g_2) v_2$, so either $LM(g_1)$ is a left subword of $LM(g_2)$ or $LM(g_2)$ is a left subword of $LM(g_1)$, contradicting the assumption that G is self-reduced. Hence $u_1 \neq u_2$, and so either u_1 is a proper left subword of u_2 or u_2 is a proper left subword of u_1.

Assume that u_1 is a proper left subword of u_2; the other case is similar. We have $u_2 = u_1 u_2'$ where $u_2' \neq 1$. Then $u_1 LM(g_1) v_1 = u_1 u_2' LM(g_2) v_2$ and so $LM(g_1) v_1 = u_2' LM(g_2) v_2$. If v_1 is a right subword of v_2 then $LM(g_2)$ is a subword of $LM(g_1)$, again contradicting the assumption that G is self-reduced. Hence v_2 is a right subword of v_1, giving $v_1 = v_1' v_2$ where $v_1' \neq 1$. Then $LM(g_1) v_1' v_2 = u_2' LM(g_2) v_2$ and so $LM(g_1) v_1' = u_2' LM(g_2)$. Condition (2) implies $g_1 v_1' - u_2' g_2 \in I(G, s)$ where $s = LM(g_1) v_1' = u_2' LM(g_2)$. Therefore

$$u_1 (g_1 v_1' - u_2' g_2) v_2 = u_1 g_1 v_1' v_2 - u_1 u_2' g_2 v_2 = u_1 g_1 v_1 - u_2 g_2 v_2.$$

But $u_1 LM(g_1) v_1 = u_2 LM(g_2) v_2$ since $\ell \geq 2$; cancellation gives $u_1 g_1 v_1 - u_2 g_2 v_2 \in I(G,t)$ where $t = u_1 LM(g_1) v_1$. Therefore we can rewrite equation (1) to obtain an expression for f of the same form where

i) either $LM(u_1 g_1 v_1)$ is lower in deglex order (if $\ell = 2$ and $\alpha_1 + \alpha_2 = 0$),
ii) or the number ℓ defined by the order relations (2) has decreased.

Since the total order on X^* satisfies the DCC, after finitely many steps we obtain an expression for f where $\ell = 1$, and then $LM(f) = u_1 s_1 v_1 = u_1 LM(g_1) v_1$. □

Lemma 6.4. *Let $g, h \in G$ be in standard form and let $s \in X^*$. Set $u = sLM(h)$, $v = LM(g)s$, $t = LM(g)u = vLM(h) = LM(g)sLM(h)$. Then $gu - vh \in I(G,t)$.*

PROOF: Separate the leading monomials: $g = LM(g) + g_0$, $h = LM(h) + h_0$, where either $g_0 = 0$ or $LM(g_0) \prec LM(g)$, and either $h_0 = 0$ or $LM(h_0) \prec LM(h)$. Then

$$\begin{aligned} gu - vh &= \big(LM(g) + g_0\big) sLM(h) - LM(g)s\big(LM(h) + h_0\big) \\ &= g_0 sLM(h) - LM(g)sh_0 = g_0 s(h - h_0) - (g - g_0)sh_0 = g_0 sh - gsh_0, \end{aligned}$$

hence $gu - vh = (g_0 s)h - g(sh_0) \in I(G,t)$ where $t = LM(g)sLM(h)$. □

Theorem 6.5. Main Theorem. *If G is a monic self-reduced set of generators for the ideal $I \subseteq F\langle X\rangle$ then these conditions are equivalent:*

(1) *G is a Gröbner basis for I;*
(2) *for every composition f of the generators in G we have $NF(f,G) = 0$.*

PROOF: (1) $\Longrightarrow$ (2): Let G be a Gröbner basis for I, and let $f = g_1 u_2 - u_1 g_2$ be a composition of $g_1, g_2 \in G$ where $u_1, u_2 \in X^*$. Then $f \in I$, and by definition of Gröbner basis, $LM(f) = v_1 LM(g) v_2$ for some $g \in G$ and $v_1, v_2 \in X^*$. If we set $f_1 = f - \alpha v_1 g v_2$ where $\alpha = lc(f)$ and the subtracted element belongs to I, then either $f_1 = 0$ or $LM(f_1) \prec LM(f)$. Repeating this calculation, and using the DCC on X^*, we obtain $NF(f,G) = 0$ after a finite number of steps.

(2) $\Longrightarrow$ (1): Let $f = g_1 u_2 - u_1 g_2$ be a composition of $g_1, g_2 \in G$ with $u_1, u_2 \in X^*$, set $t = LM(g_1) u_2 = u_1 LM(g_2)$, and assume $NF(f,G) = 0$. Definition 5.8 implies $u_2 \neq LM(g_2)$ and $u_1 \neq LM(g_1)$. If u_2 is longer than $LM(g_2)$ then also u_1 is longer than $LM(g_1)$, and by Lemma 6.4 we have $f \in I(G,t)$. If u_2 is shorter than $LM(g_2)$ then u_1 is shorter than $LM(g_1)$. Since $NF(f,G) = 0$, Figure 1 outputs zero after a finite number of steps. During each iteration, we set $f_{n+1} \longleftarrow f_n - \alpha u_1 g u_2$, where $LM(u_1 g u_2) = LM(f_n) \preceq LM(f) \prec t$. Thus f is a linear combination of terms $u_1 g u_2$ which strictly precede t in deglex order, and hence $f \in I(G,t)$. In both cases we have $f \in I(G,t)$, and now Lemma 6.3 completes the proof. □

Remark 6.6. Theorem 6.5 motivates to the algorithm in Figure 2 whose input is a set G generating the ideal $I \subseteq F\langle X\rangle$ and whose output (if the algorithm terminates) is a Gröbner basis for I. For a different approach, emphasizing Shirshov's point of view as developed in Novosibirsk, see [13], [15], [24], [23, Chapter 1], [100].

Example 6.7. Let $X = \{a,b,c\}$ and let $I \subset F\langle X\rangle$ be the ideal defining the universal associative envelope of the Jordan algebra $S_2(F)$ of symmetric 2×2 matrices; by Example 3.3 we know that I is generated by the set

$$\{g_1,\ldots,g_6\} = \{\ a^2-a,\ ba+ab,\ b^2-b,\ ca+ac-c,\ cb+bc-c,\ c^2-b-a\ \}.$$

The first iteration of the algorithm produces 10 compositions:

$$\begin{aligned}
&g_1a-ag_1 \xrightarrow{\text{sf}} 0, \quad g_2a-bg_1 \xrightarrow{\text{sf}} s_1 = aba+ba, \quad g_3a-bg_2 \xrightarrow{\text{sf}} s_2 = bab+ba,\\
&g_3b-bg_3 \xrightarrow{\text{sf}} 0, \quad g_4a-cg_1 \xrightarrow{\text{sf}} s_3 = aca, \quad g_5a-cg_2 \xrightarrow{\text{sf}} s_4 = cab-bca+ca,\\
&g_5b-cg_3 \xrightarrow{\text{sf}} s_5 = bcb, \quad g_6a-cg_4 \xrightarrow{\text{sf}} s_6 = cac-c^2+ba+a^2,\\
&g_6b-cg_5 \xrightarrow{\text{sf}} s_7 = cbc-c^2+b^2+ab, \quad g_6c-cg_6 \xrightarrow{\text{sf}} s_8 = cb+ca-bc-ac.
\end{aligned}$$

Computing normal forms of these compositions using Figure 1, we obtain:

$$\begin{aligned}
&s_1-ag_2+g_1b-g_2 = -2ab \xrightarrow{\text{sf}} ab, \quad s_2-g_2b+ag_3-g_2 = -2ab \xrightarrow{\text{sf}} ab,\\
&s_3-ag_4+g_1c = 0 \xrightarrow{\text{sf}} 0,\\
&s_4-g_4b+bg_4-g_2c+ag_5-g_5-g_4 = -2bc-2ac+2c \xrightarrow{\text{sf}} bc+ac-c,\\
&s_5-bg_5+g_3c = 0 \xrightarrow{\text{sf}} 0, \quad s_6-g_4c+ag_6-g_2 = -2ab \xrightarrow{\text{sf}} ab,\\
&s_7-g_5c+bg_6+g_2 = 2ab \xrightarrow{\text{sf}} ab,\\
&s_8-g_5-g_4 = -2bc-2ac+2c \xrightarrow{\text{sf}} bc+ac-c.
\end{aligned}$$

We include the new generators $t_1 = ab$, $t_2 = bc+ac-c$ with the original set and sort the result by deglex order of leading monomials:

$$\begin{aligned}
&g_1 = a^2-a, \quad t_1 = ab, \quad g_2 = ba+ab, \quad g_3 = b^2-b, \quad t_2 = bc+ac-c,\\
&g_4 = ca+ac-c, \quad g_5 = cb+bc-c, \quad g_6 = c^2-b-a.
\end{aligned}$$

We compute the normal form of each element with respect to those preceding it; the new elements are $g_2' = g_2 - t_1 = ba$ and $g_5' = g_5 - t_2 = cb - ac$:

$$\begin{aligned}
&g_1 = a^2-a, \quad t_1 = ab, \quad g_2' = ba, \quad g_3 = b^2-b, \quad t_2 = bc+ac-c,\\
&g_4 = ca+ac-c, \quad g_5' = cb-ac, \quad g_6 = c^2-b-a.
\end{aligned}$$

This is a Gröbner basis: all compositions of these elements have normal form 0. We can now compute the normal form of any element of $F\langle X\rangle$; see Theorem 5.3(b). In particular, for the element $f = c^2b$ from Example 4.9 we have:

$$f_1 - g_6b - g_3 - t_1 = c^2b - (c^2-b-a)b - (b^2-b) - ab = b.$$

Example 6.8. A generating set [48, p. 226] for which Figure 2 never terminates: $X = \{a,b\}$ and $G_0 = \{g_1 = aba - ba\}$. The first iteration produces one composition:

$$g_1ba - abg_1 = (aba-ba)ba - ab(aba-ba) = -baba + ab^2a \xrightarrow{\text{sf}} baba - ab^2a.$$

GrobnerBasis(G)

Input: A finite subset $G \subset F\langle X\rangle$ generating an ideal $I \subseteq F\langle X\rangle$.

Output: If step (2) terminates, the output is a Gröbner basis of I.

(1) Set `newcompositions` $\leftarrow$ `true`.
(2) While `newcompositions` do:
   (a) Convert the elements of G to standard form.
   (b) Sort G by deglex order of leading monomials: $G = \{g_1, \dots, g_n\}$.
   (c) Convert G to a self-reduced set:
      - Set `selfreduced` $\leftarrow$ `false`.
      - While not `selfreduced` do:
         (i) Set `selfreduced` $\leftarrow$ `true`.
         (ii) Set $H \leftarrow \{\,\}$.
         (iii) For $i = 1, \dots, n$ do:
         set $H \leftarrow H \cup \{NF(g_i, \{g_1, \dots, g_{i-1}\})\}$.
         (iv) Convert the elements of H to standard form.
         (v) Sort H by deglex order of leading monomials.
         (vi) If $G \neq H$ then set `selfreduced` $\leftarrow$ `false`.
         (vii) Set $G \leftarrow H$.
   (d) Set `compositions` $\leftarrow \{\,\}$.
   (e) Set `newcompositions` $\leftarrow$ `false`.
   (f) For $g \in G$ do for $h \in G$ do:
      - If $LM(g)$ and $LM(h)$ have an overlap w then:
         (i) Define u, v by $LM(g) = vw$ and $LM(h) = wu$.
         (ii) Set $s \leftarrow gu - vh$.
         (iii) Replace s by its standard form.
         (iv) Set $t \leftarrow NF(s, G)$.
         (v) Replace t by its standard form.
         (vi) If $t \neq 0$ and $t \notin$ `compositions` then
            * Set `newcompositions` $\leftarrow$ `true`.
            * Set `compositions` $\leftarrow$ `compositions` $\cup \{t\}$.
   (g) Set $G \leftarrow G \cup$ `compositions`.
(3) Return G.

FIGURE 2. Algorithm for a Gröbner basis of the ideal $I \subseteq F\langle X\rangle$

Computing the normal form gives:

$$(baba - ab^2a) - b(aba - ba) = -ab^2a + b^2a \xrightarrow{\text{sf}} ab^2a - b^2a = g_2.$$

We obtain a new self-reduced generating set: $G_1 = \{g_1 = aba - ba, g_2 = ab^2a - b^2a\}$. The second iteration produces three compositions:

$$
\begin{aligned}
g_1b^2a - abg_2 &= (aba - ba)b^2a - ab(ab^2a - b^2a) \xrightarrow{\text{sf}} bab^2a - ab^3a, \\
g_2ba - ab^2g_1 &= (ab^2a - b^2a)ba - ab^2(aba - ba) \xrightarrow{\text{sf}} b^2aba - ab^3a, \\
g_2b^2a - ab^2g_2 &= (ab^2a - b^2a)b^2a - ab^2(ab^2a - b^2a) \xrightarrow{\text{sf}} b^2ab^2a - ab^4a.
\end{aligned}
$$

Computing the normal forms of these compositions with respect to G_1 gives

$$
\begin{aligned}
&(bab^2a - ab^3a) - b(ab^2a - b^2a) = -ab^3a + b^3a \xrightarrow{\text{sf}} ab^3a - b^3a = g_3, \\
&(b^2aba - ab^3a) - b^2(aba - ba) = -ab^3a + b^3a \xrightarrow{\text{sf}} ab^3a - b^3a, \\
&(b^2ab^2a - ab^4a) - b^2(ab^2a - b^2a) = -ab^4a + b^4a \xrightarrow{\text{sf}} ab^4a - b^4a = g_4.
\end{aligned}
$$

We obtain a new self-reduced generating set:

$$
G_2 = \{ \, g_1 = aba - ba, \; g_2 = ab^2a - b^2a, \; g_3 = ab^3a - b^3a, \; g_4 = ab^4a - b^4a \, \}.
$$

We can write down explicitly the elements of the set G_n obtained at the end of the n-th iteration, and verify that the algorithm never terminates.

Example 6.9. Another example in which self-compositions play an essential role: $X = \{a, b\}$ and $G = \{g_1 = aba - a^2b - a, \; g_2 = bab - ab^2 - b\}$. The first iteration of the Gröbner basis algorithm produces three compositions:

$$
\begin{aligned}
s_1 &= g_1ba - abg_1 = (aba - a^2b - a)ba - ab(aba - a^2b - a) \\
&= ababa - a^2b^2a - aba - ababa + aba^2b + aba = aba^2b - a^2b^2a, \\
s_2 &= g_2a - bg_1 = (bab - ab^2 - b)a - b(aba - a^2b - a) \\
&= baba - ab^2a - ba - baba + ba^2b + ba = ba^2b - ab^2a, \\
s_3 &= g_2ab - bag_2 = (bab - ab^2 - b)ab - ba(bab - ab^2 - b) \\
&= babab - ab^2ab - bab - babab + ba^2b^2 + bab = ba^2b^2 - ab^2ab.
\end{aligned}
$$

Computing the normal forms with respect to $\{g_1, g_2\}$ gives:

$$
\begin{aligned}
s_1 - g_1ab - a^2g_2 &= aba^2b - a^2b^2a - (aba - a^2b - a)ab - a^2(bab - ab^2 - b) \\
&= aba^2b - a^2b^2a - aba^2b + a^2bab + a^2b - a^2bab + a^3b^2 + a^2b \\
&= -a^2b^2a + a^3b^2 + 2a^2b \xrightarrow{\text{sf}} a^2b^2a - a^3b^2 - 2a^2b = h_1, \\
s_2 &= h_2, \\
s_3 + abg_2 + g_1b^2 &= ba^2b^2 - ab^2ab + ab(bab - ab^2 - b) + (aba - a^2b - a)b^2 \\
&= ba^2b^2 - ab^2ab + ab^2ab - abab^2 - ab^2 + abab^2 - a^2b^3 - ab^2 \\
&= ba^2b^2 - a^2b^3 - 2ab^2 = h_3.
\end{aligned}
$$

We combine these compositions with the original generators and sort them:

$$
\begin{aligned}
&g_1 = aba - a^2b - a, \qquad g_2 = bab - ab^2 - b, \qquad h_2 = ba^2b - ab^2a, \\
&h_1 = a^2b^2a - a^3b^2 - 2a^2b, \qquad h_3 = ba^2b^2 - a^2b^3 - 2ab^2.
\end{aligned}
$$

Self-reducing this set eliminates h_3 since $h_3 - h_2 b - abg_2 - g_1 b^2 = 0$. The second iteration produces five compositions with these normal forms:

$$h_4 = ba^3b - ab^2a^2 + ba^2, \quad h_5 = ba^3b^2 - a^2b^3a, \quad h_6 = a^3b^3a - a^4b^3 - 3a^3b^2,$$
$$h_7 = ba^4b^2 - ab^2a^3b + 2ba^3b, \quad h_8 = a^4b^4a - a^5b^4 + 2a^3b^3a - 6a^4b^3 - 6a^3b^2.$$

Combining these compositions with g_1, g_2, h_2, h_1 and self-reducing the resulting set eliminates h_5 and replaces h_7 and h_8 with these elements:

$$h_7' = ba^4b^2 - a^2b^3a^2 + 2ab^2a^2 - 2ba^2, \qquad h_8' = a^4b^4a - a^5b^4 - 4a^4b^3.$$

The third iteration of the algorithm produces 18 compositions.

Remark 6.10. A rich source of examples of the Gröbner basis algorithm comes from the construction of universal associative envelopes for nonassociative triple systems obtained from the trilinear operations classified in [30]; see §10 below.

7. The Poincaré-Birkhoff-Witt Theorem

We present the combinatorial proof of the PBW Theorem discovered by Bokut [13] and Bergman [10]. We follow the exposition of [48, Theorem 6.2.1]. The assumption that the Lie algebra is finite dimensional is not essential.

Theorem 7.1. PBW Theorem. *If L is a finite dimensional Lie algebra over a field F with ordered basis $X = \{x_1, \dots, x_n\}$, then a basis of its universal associative envelope $U(L)$ consists of the monomials $x_1^{e_1} \cdots x_n^{e_n}$ with $e_1, \dots, e_n \geq 0$. Therefore:*

(i) *$U(L)$ is infinite dimensional,*
(ii) *the natural map $L \to U(L)$ is injective,*
(iii) *L is isomorphic to a subalgebra of $U(L)^-$.*

Proof: The structure constants $c_{ij}^k \in F$ satisfy $c_{ji}^k = -c_{ij}^k$ and $c_{ii}^k = 0$:

$$[x_i, x_j] = \sum_{k=1}^{n} c_{ij}^k x_k.$$

The universal associative envelope $U(L)$ is the quotient of the free associative algebra $F\langle X\rangle$ by the ideal I generated by the elements

$$g_{ij} = x_i x_j - x_j x_i - [x_i, x_j] = x_i x_j - x_j x_i - \sum_{k=1}^{n} c_{ij}^k x_k.$$

If $i = j$ then $g_{ii} = 0$. If $i \neq j$ then by anticommutativity of the Lie bracket, we may assume $i > j$, and hence $x_i x_j$ is the leading monomial of g_{ij}. We will show that the set $G = \{g_{ij} \mid 1 \leq j < i \leq n\}$ is a Gröbner basis for I. Consider two leading monomials, $LM(g_{ij}) = x_i x_j$ $(i > j)$ and $LM(g_{\ell k}) = x_\ell x_k$ $(\ell > k)$. The only possible compositions of these generators occur when either $j = \ell$ or $k = i$.

By symmetry we may assume $j = \ell$, so we consider g_{ij} and g_{jk} where $i > j > k$. We have $LM(g_{ij})\, x_k = x_i x_j x_k = x_i\, LM(g_{jk})$, which produces the composition

$$\begin{aligned} g_{ij}x_k - x_i g_{jk} &= \big(x_i x_j - x_j x_i - [x_i, x_j]\big)x_k - x_i\big(x_j x_k - x_k x_j - [x_j, x_k]\big) \\ &= x_i x_j x_k - x_j x_i x_k - [x_i, x_j]x_k - x_i x_j x_k + x_i x_k x_j + x_i[x_j, x_k] \\ &= -x_j x_i x_k - [x_i, x_j]x_k + x_i x_k x_j + x_i[x_j, x_k] \\ &= x_i x_k x_j - x_j x_i x_k - [x_i, x_j]x_k + x_i[x_j, x_k]. \end{aligned}$$

(It is convenient to avoid explicit structure constants in this calculation; recall that $[x_i, x_j]$ is a homogeneous polynomial of degree 1.) To compute the normal form of this composition with respect to G, we subtract $g_{ik}x_j$ and add $x_j g_{ik}$:

$$\begin{aligned} & x_i x_k x_j - x_j x_i x_k - [x_i, x_j]x_k + x_i[x_j, x_k] \\ &\quad - \big(x_i x_k - x_k x_i - [x_i, x_k]\big)x_j + x_j\big(x_i x_k - x_k x_i - [x_i, x_k]\big) \\ &= x_i x_k x_j - x_j x_i x_k - [x_i, x_j]x_k + x_i[x_j, x_k] \\ &\quad - x_i x_k x_j + x_k x_i x_j + [x_i, x_k]x_j + x_j x_i x_k - x_j x_k x_i - x_j[x_i, x_k] \\ &= -[x_i, x_j]x_k + x_i[x_j, x_k] + x_k x_i x_j + [x_i, x_k]x_j - x_j x_k x_i - x_j[x_i, x_k] \\ &= -x_j x_k x_i + x_k x_i x_j - [x_i, x_j]x_k + x_i[x_j, x_k] + [x_i, x_k]x_j - x_j[x_i, x_k]. \end{aligned}$$

We next add $g_{jk}x_i$ and subtract $x_k g_{ij}$:

$$\begin{aligned} & -x_j x_k x_i + x_k x_i x_j - [x_i, x_j]x_k + x_i[x_j, x_k] + [x_i, x_k]x_j - x_j[x_i, x_k] \\ &\quad + \big(x_j x_k - x_k x_j - [x_j, x_k]\big)x_i - x_k\big(x_i x_j - x_j x_i - [x_i, x_j]\big) \\ &= -x_j x_k x_i + x_k x_i x_j - [x_i, x_j]x_k + x_i[x_j, x_k] + [x_i, x_k]x_j - x_j[x_i, x_k] \\ &\quad + x_j x_k x_i - x_k x_j x_i - [x_j, x_k]x_i - x_k x_i x_j + x_k x_j x_i + x_k[x_i, x_j] \\ &= -[x_i, x_j]x_k + x_i[x_j, x_k] + [x_i, x_k]x_j - x_j[x_i, x_k] - [x_j, x_k]x_i + x_k[x_i, x_j] \\ &= x_i[x_j, x_k] - [x_j, x_k]x_i + x_j[x_k, x_i] - [x_k, x_i]x_j + x_k[x_i, x_j] - [x_i, x_j]x_k. \end{aligned}$$

The last expression equals $[x_i, [x_j, x_k]] + [x_j, [x_k, x_i]] + [x_k, [x_i, x_j]]$, which is zero by the Jacobi identity. Thus every composition has normal form zero, proving that we have a Gröbner basis. The leading monomials of this Gröbner basis are $x_i x_j$ where $i > j$. A basis for $U(L)$ consists of all monomials w which do not have any of these leading monomials as a subword. That is, if w contains a subword $x_i x_j$ then $i \le j$. It follows that the monomials in the statement of the theorem form a basis for $U(L)$. In particular, the monomials $x_1, \dots, x_n$ of degree 1 are linearly independent in $U(L)$, and hence the natural map from L to $U(L)$ is injective. $\square$

Corollary 7.2. *Every polynomial identity satisfied by the Lie bracket in every associative algebra is a consequence of anticommutativity and the Jacobi identity.*

Proof: If $p(a_1, \dots, a_n) \equiv 0$ is a polynomial identity which is not a consequence of anticommutativity and the Jacobi identity, then $p(a_1, \dots, a_n)$ is a nonzero element of the free Lie algebra L generated by $\{a_1, \dots, a_n\}$. If A is any associative algebra, and $\epsilon\colon L \to A^-$ is any morphism of Lie algebras, then by definition of polynomial identity, $\epsilon(p) = 0$. If we take $A = U(L)$ and let ϵ be the injective map $L \to U(L)^-$ from the PBW theorem, then $p \neq 0$ implies $\epsilon(p) \neq 0$, a contradiction. $\square$

Remark 7.3. Lie algebras arose originally as tangent algebras of Lie groups. Weakening the requirement of associativity in the definition of Lie group gives rise to various classes of nonassociative analytic loops, such as Moufang loops, Bol loops, and monoassociative loops. The corresponding tangent algebras are known respectively as Malcev algebras, Bol algebras, and BTQ algebras. For universal nonassociative envelopes of Malcev and Bol algebras, see [106], [108]. This problem is still open for BTQ algebras [29]. All these tangent algebras are special cases of Sabinin algebras; for the universal nonassociative envelopes of Sabinin algebras, see [107], [112].

The PBW theorem shows that for every Lie algebra L, the ideal generators obtained from the structure constants form a Gröbner basis. These generators can be interpreted as rewriting rules in $U(L)$ as follows:

$$x_i x_j - x_j x_i - \sum_{k=1}^{n} c_{ij}^k x_k \in I \qquad \Longleftrightarrow \qquad x_i x_j = x_j x_i + \sum_{k=1}^{n} c_{ij}^k x_k \in U(L).$$

Repeated application of these rules allows us to work out multiplication formulas for monomials in $U(L)$.

Example 7.4. Let L be the 2-dimensional solvable Lie algebra with basis $\{a, b\}$ where $[a, b] = b$. The basis of $U(L)$ from the PBW theorem consists of the monomials $a^i b^j$ for $i, j \geq 0$. The ideal I is generated by $ab - ba - b$, and so in $U(L)$ we have $ba = ab - b$. Using this and induction on the exponents we can work out a formula for the product $(a^i b^j)(a^k b^\ell)$ as a linear combination of basis monomials.

Example 7.5. Let L be the 3-dimensional nilpotent Lie algebra with basis $\{a, b, c\}$ where $[a, b] = c$, $[a, c] = [b, c] = 0$. The PBW basis of $U(L)$ consists of $a^i b^j c^k$ for $i, j, k \geq 0$. In $U(L)$ we have $ba = ab - c$, $ac = ca$, $bc = cb$. We can use these to prove a formula for $(a^i b^j c^k)(a^\ell b^m c^n)$ as a linear combination of basis monomials.

Example 7.6. Let L be the 3-dimensional simple Lie algebra $\mathfrak{sl}_2(F)$ with basis $\{e, f, h\}$ where $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. The PBW basis of $U(L)$ consists of $f^i h^j e^k$ for $i, j, k \geq 0$, and $eh = he - 2e$, $hf = fh - 2f$, $ef = fe + h$. Using these we can express $(f^i h^j e^k)(f^\ell h^m e^n)$ as a linear combination of basis monomials.

8. Jordan structures on 2×2 matrices

In this section we study two examples of nonassociative structures whose universal associative envelopes are finite dimensional. The underlying vector space in both cases is $M_2(F)$, the 2×2 matrices over F. We use this notation:

$$a = E_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \;\; b = E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \;\; c = E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \;\; d = E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

8.1 The Jordan algebra of 2×2 matrices. We make $M_2(F)$ into a special Jordan algebra J using the product $x \circ y = xy + yx$ (we omit the scalar $\frac{1}{2}$). The universal associative envelope $U(J)$ is $F\langle a, b, c, d\rangle / I$ where the ideal I is generated by the following 10 elements, obtained from the structure constants of J:

$$\begin{gathered} g_1 = a^2 - a, \;\; g_2 = ba + ab - b, \;\; g_3 = b^2, \;\; g_4 = ca + ac - c, \;\; g_5 = cb + bc - d - a, \\ g_6 = c^2, \;\; g_7 = da + ad, \;\; g_8 = db + bd - b, \;\; g_9 = dc + cd - c, \;\; g_{10} = d^2 - d. \end{gathered}$$

We obtain three distinct compositions from (g_5, g_2), (g_5, g_3), (g_6, g_5); with normal forms $s_1 = ad$, $s_2 = bd - ab$, $s_3 = cd - ac$. We obtain a new set of 13 generators:

$$\begin{gathered} a^2 - a, \;\; ad, \;\; ba + ab - b, \;\; b^2, \;\; bd - ab, \;\; ca + ac - c, \;\; cb + bc - d - a, \\ c^2, \;\; cd - ac, \;\; da, \;\; db + ab - b, \;\; dc + ac - c, \;\; d^2 - d. \end{gathered}$$

Every composition of these generators has normal form 0: we have a Gröbner basis. There are only 9 monomials in $F\langle a, b, c, d\rangle$ which do not have the leading monomial of one of the Gröbner basis elements as a subword:

$$u_1 = 1, \;\; u_2 = a, \;\; u_3 = b, \;\; u_4 = c, \;\; u_5 = d, \;\; u_6 = ab, \;\; u_7 = ac, \;\; u_8 = bc, \;\; u_9 = abc.$$

The cosets of these monomials modulo I form a basis for $U(J)$. Table 1 contains the multiplication table for $U(J)$, where u_i is denoted i and dot indicates 0. This table was obtained by computing the normal form of each product of basis elements. We can now show using [26] that $U(J) \approx F \oplus M_2(F) \oplus M_2(F)$. For a general discussion of the representation theory of Jordan algebras; see [78].

	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	2	6	7	·	6	7	9	9
3	3	3−6	·	8	6	·	8−9	·	·
4	4	4−7	2+5−8	·	7	5−8+9	·	4	4−7
5	5	·	3−6	4−7	5	·	·	8−9	·
6	6	·	·	9	6	·	·	·	·
7	7	·	2−9	·	7	·	·	7	·
8	8	9	3	·	8−9	6	·	8	9
9	9	9	6	·	·	6	·	9	9

TABLE 1. Structure constants for $U(J)$ where $J = M_2(F)^+$

Remark 8.1. If J is a finite dimensional Jordan algebra then $\dim U(J) < \infty$. If J is an n-dimensional Jordan algebra with zero product, then $U(J)$ is the exterior algebra of an n-dimensional vector space. If J is the Jordan algebra of a symmetric bilinear form, then $U(J)$ is the corresponding Clifford algebra.

8.2 The Jordan triple system of 2×2 matrices. This subsection introduces multilinear operations, which we discuss in general in §9. We consider $T = M_2(F)$ with the trilinear operation $\langle x, y, z \rangle = xyz + zyx$. From the structure constants, we find that $U(T) \approx F\langle a, b, c, d \rangle / I$ where I is generated by these 40 elements:

$$\begin{gathered} a^3 - a,\ aba,\ aca,\ ada,\ ba^2 + a^2b - b,\ bab,\ b^2a + ab^2,\ b^3,\ bca + acb - a,\ bcb - b, \\ bda + adb,\ bdb,\ ca^2 + a^2c - c,\ cab + bac - d,\ cac,\ cba + abc - a,\ cb^2 + b^2c, \\ cbc - c,\ c^2a + ac^2,\ c^2b + bc^2,\ c^3,\ cda + adc,\ cdb + bdc - a,\ cdc,\ da^2 + a^2d, \\ dab + bad,\ dac + cad,\ dad,\ dba + abd - b,\ db^2 + b^2d,\ dbc + cbd - d,\ dbd, \\ dca + acd - c,\ dcb + bcd - d,\ dc^2 + c^2d,\ dcd,\ d^2a + ad^2,\ d^2b + bd^2 - b, \\ d^2c + cd^2 - c,\ d^3 - d. \end{gathered}$$

These elements produce 36 distinct nonzero compositions:

$$\begin{gathered} ad,\ b^2,\ bd - ab,\ c^2,\ cd - ac,\ da,\ db - ba,\ dc - ca,\ d^2 - cb - bc + a^2,\ a^2d,\ ab^2, \\ abd - a^2b,\ acb + abc - a,\ ac^2,\ acd - a^2c,\ adb,\ adc,\ ad^2,\ bad,\ b^2c,\ b^2d,\ bc^2, \\ bcd - bac,\ bdc + acb - a,\ bdc - abc,\ bd^2 - b^2c - a^2b,\ bd^2 - abd,\ bd^2 - a^2b,\ cad, \\ cbd + bcd - a^2d - d,\ cbd + bcd - d,\ cbd + bac - d,\ c^2d,\ cd^2 - acd, \\ cd^2 + bc^2 - a^2c,\ cd^2 - a^2c. \end{gathered}$$

Self-reducing the union of these two sets produces these 22 elements:

$$\begin{gathered} ad,\ b^2,\ bd - ab,\ c^2,\ cd - ac,\ da,\ db - ba,\ dc - ca,\ d^2 - cb - bc + a^2,\ a^3 - a, \\ aba,\ aca,\ acb + abc - a,\ ba^2 + a^2b - b,\ bab,\ bca - abc,\ bcb - b,\ ca^2 + a^2c - c, \\ cab + bac - d,\ cac,\ cba + abc - a,\ cbc - c. \end{gathered}$$

All compositions of these elements have normal form 0, so we have a Gröbner basis. There are only 17 monomials in $F\langle a, b, c, d \rangle$ which are not divisible by the leading monomial of one of these elements; their cosets form a basis of $U(T)$:

$$1,\ a,\ b,\ c,\ d,\ a^2,\ ab,\ ac,\ ba,\ bc,\ ca,\ cb,\ a^2b,\ a^2c,\ abc,\ bac,\ a^2bc.$$

We leave to the interested reader the task of calculating the multiplication table for $U(T)$, and verifying that $U(T) \approx F \oplus M_2(F) \oplus M_2(F) \oplus M_2(F) \oplus M_2(F)$. For the structure theory of Jordan triple systems, see [92], [99].

9. Multilinear operations

We now consider generalizations of the Lie bracket and the Jordan product to n-linear operations for any integer $n \geq 2$; see [30], [31].

9.1 Multilinear operations. An n-linear operation $\omega(a_1, \ldots, a_n)$ is a linear combination of permutations of the monomial $a_1 \cdots a_n$. We may regard ω either as a multilinear element of degree n in the free associative algebra on n generators, or as an element of the group algebra FS_n of the symmetric group S_n:

$$\omega(a_1, \ldots, a_n) = \sum_{\sigma \in S_n} x_\sigma a_{\sigma(1)} \cdots a_{\sigma(n)} = \sum_{\sigma \in S_n} x_\sigma \sigma \qquad (x_\sigma \in F).$$

The group S_n acts by permuting subscripts: $\sigma \cdot a_{\tau(1)} \cdots a_{\tau(n)} = a_{\sigma\tau(1)} \cdots a_{\sigma\tau(n)}$. Two operations are said to be equivalent if each is a linear combination of permutations of the other; in other words, they generate the same left ideal in FS_n. We assume that the characteristic of F is either 0 or a prime $p > n$; this is necessary and sufficient for FS_n to be semisimple. In this case, FS_n is the direct sum of simple two-sided ideals, each isomorphic to a matrix algebra $M_d(F)$, and the projections of S_n to these matrix algebras define the irreducible representations of S_n.

9.2 The case $n = 2$. Every bilinear operation is equivalent to either the Lie bracket $[x, y] = xy - yx$, the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$, the original associative operation xy, or the zero operation. In other words, the only left ideals in $FS_2 \approx F \oplus F$ are $\{0\} \oplus F$, $F \oplus \{0\}$, $F \oplus F$, and $\{0\} \oplus \{0\}$. The first copy of F corresponds to the unit representation of S_2, and a basis is $S = \frac{1}{2}(xy + yx)$. The second copy corresponds to the sign representation, and a basis is $A = \frac{1}{2}(xy - yx)$. These elements are orthogonal idempotents: $S^2 = S$, $A^2 = A$, $SA = AS = 0$.

9.3 The case $n = 3$. Faulkner [58] classified the polynomial identities satisfied by a large class of nearly simple triple systems. Twenty years later, trilinear operations were classified up to equivalence in [30]; that paper also determined the polynomial identities of degree 5 satisfied by these operations. In this case we have $FS_3 \approx F \oplus M_2(F) \oplus F$. The first and last copies of F correspond to the unit and sign representations; bases for these summands are the following idempotents:

$$S = \tfrac{1}{6}(abc+acb+bac+bca+cab+cba), \quad A = \tfrac{1}{6}(abc-acb-bac+bca+cab-cba).$$

The summand $M_2(F)$ corresponds to the irreducible 2-dimensional representation. To find a basis for $M_2(F)$ corresponding to the matrix units E_{ij} $(i, j = 1, 2)$ we use the representation theory of the symmetric group; for methods which apply to all n, see [45], [111], [118]. Any trilinear operation can be represented as a triple of matrices; as representatives of the equivalence classes we may take the triples in which each matrix is in row canonical form:

$$\left[a, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, c \right].$$

There are exactly 19 trilinear operations satisfying polynomial identities in degree 5 which do not follow from their identities in degree 3; see [27], [54]. Together with these operations, we include the symmetric, alternating and cyclic sums [28]. These 22 trilinear operations are given in Table 2: the first column gives the name of the operation; the second column gives the row canonical forms of the representation matrices of the corresponding element of the group algebra; the third column gives the simplest representative of the equivalence class as a linear combination of permutations. (The parameter q represents the $(1,2)$ entry of the 2×2 matrix.)

9.4 Associative n-ary algebras. For the classification of simple associative triple systems, see [73], [86], [93]; for simple associative n-ary systems, see [38]. The classification of n-ary systems can be reformulated as follows. Let $(d_1, \dots, d_{n-1})$ be a sequence of $n-1$ positive integers; two sequences are equivalent if they differ by a cyclic permutation. For $i = 1, \dots, n-1$, let V_i be a vector space of dimension d_i, and consider the direct sum $V = V_1 \oplus \cdots \oplus V_{n-1}$. Let A be the subspace of $\mathrm{End}_F(V)$ consisting of the linear operators $T \colon V \to V$ satisfying

$$T(V_1) \subseteq V_2, \quad T(V_2) \subseteq V_3, \quad \dots, \quad T(V_{n-2}) \subseteq V_{n-1}, \quad T(V_{n-1}) \subseteq V_1.$$

Then A is a simple associative n-ary system, and every such system has this form. If we choose bases of $V_1, \dots, V_{n-1}$ then we can represent elements of A as $D \times D$ block matrices where $D = d_1 + \cdots + d_{n-1}$. The block in position (i,j) has size $d_i \times d_j$, and nonzero entries may appear only in blocks $(2,1)$, …, $(n-1,n)$, $(n,1)$. For $n = 3, 4, 5$ we obtain the following matrices, where T_{ij} is a block of size $d_i \times d_j$:

$$\left[\begin{array}{cc} 0 & T_{12} \\ T_{21} & 0 \end{array} \right], \qquad \left[\begin{array}{ccc} 0 & 0 & T_{13} \\ T_{21} & 0 & 0 \\ 0 & T_{32} & 0 \end{array} \right], \qquad \left[\begin{array}{cccc} 0 & 0 & 0 & T_{14} \\ T_{21} & 0 & 0 & 0 \\ 0 & T_{32} & 0 & 0 \\ 0 & 0 & T_{43} & 0 \end{array} \right].$$

9.5 Special nonassociative n-ary systems. If A is an associative n-ary system and $\omega(a_1, \dots, a_n)$ is an n-linear operation, then we obtain a nonassociative n-ary system A^ω by interpreting each monomial in ω as the corresponding product in A. We call A^ω a "special" n-ary system[1] (by analogy with special Jordan algebras) since it comes from a multilinear operation on an associative system. To understand these systems better, we construct their universal associative envelopes using noncommutative Gröbner bases; the goal is to classify their irreducible finite dimensional representations. This generalizes the universal enveloping algebras of Lie and Jordan algebras, where a basic dichotomy arises: a finite dimensional Lie algebra has an infinite dimensional universal envelope, but a finite dimensional Jordan algebra has a finite dimensional universal envelope.

9.6 Universal associative envelopes. We give the precise definition of the universal associative envelope of a (special) nonassociative n-ary system relative to an n-linear operation. The earliest discussion of this construction appears to be [12, §2]; see also [27, §7.2]. Suppose that B is a subspace, of an associative

[1]Perhaps "representable" would be a more meaningful term.

operation	$F \oplus M_2(F) \oplus F$	FS_3
symmetric sum	$\left[1, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, 0\right]$	$abc+acb+bac+bca+cab+cba$
alternating sum	$\left[0, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, 1\right]$	$abc-acb-bac+bca+cab-cba$
cyclic sum	$\left[1, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, 1\right]$	$abc+bca+cab$
Lie $q=\infty$	$\left[0, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, 0\right]$	$abc-acb-bca+cba$
Lie $q=\frac{1}{2}$	$\left[0, \begin{bmatrix}1 & \frac{1}{2}\\0 & 0\end{bmatrix}, 0\right]$	$abc+acb-bca-cba$
Jordan $q=\infty$	$\left[1, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, 0\right]$	$abc+cba$
Jordan $q=0$	$\left[1, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, 0\right]$	$abc+bac$
Jordan $q=1$	$\left[1, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, 0\right]$	$abc+acb$
Jordan $q=\frac{1}{2}$	$\left[1, \begin{bmatrix}1 & \frac{1}{2}\\0 & 0\end{bmatrix}, 0\right]$	$abc+2acb+2cab+cba$
anti-Jordan $q=\infty$	$\left[0, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, 1\right]$	$abc-2acb+2cab-cba$
anti-Jordan $q=-1$	$\left[0, \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}, 1\right]$	$abc-acb$
anti-Jordan $q=\frac{1}{2}$	$\left[0, \begin{bmatrix}1 & \frac{1}{2}\\0 & 0\end{bmatrix}, 1\right]$	$abc-cba$
anti-Jordan $q=2$	$\left[0, \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}, 1\right]$	$abc-bac$
fourth family $q=\infty$	$\left[1, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, 1\right]$	$abc-acb-bac$
fourth family $q=0$	$\left[1, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, 1\right]$	$abc-acb+bca$
fourth family $q=1$	$\left[1, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, 1\right]$	$abc-bac+cab$
fourth family $q=-1$	$\left[1, \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}, 1\right]$	$abc+bac+cab$
fourth family $q=2$	$\left[1, \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}, 1\right]$	$abc+acb+bca$
fourth family $q=\frac{1}{2}$	$\left[1, \begin{bmatrix}1 & \frac{1}{2}\\0 & 0\end{bmatrix}, 1\right]$	$abc+acb+bac$
cyclic commutator	$\left[0, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, 0\right]$	$abc-bca$
weakly commutative	$\left[1, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, 0\right]$	$abc+acb+bac-cba$
weakly anticommutative	$\left[0, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, 1\right]$	$abc+acb-bca-cab$

TABLE 2. The twenty-two trilinear operations

n-ary system A, which is closed under the n-linear operation ω. Let $d = \dim B$ and let $X = \{b_1, \ldots, b_d\}$ be a basis of B; then we have the structure constants for B^ω:

$$\omega(b_{i_1}, \ldots, b_{i_n}) = \sum_{j=1}^{d} c^j_{i_1 \cdots i_n} b_j \qquad (1 \leq i_1, \ldots, i_n \leq d).$$

Let $F\langle X \rangle$ be the free associative algebra generated by X and let $I \subseteq F\langle X \rangle$ be the ideal generated by the following d^n elements:

$$\sum_{\sigma \in S_n} x_\sigma b_{i_{\sigma(1)}} \cdots b_{i_{\sigma(n)}} - \sum_{j=1}^{d} c^j_{i_1 \cdots i_n} b_j \qquad (1 \leq i_1, \ldots, i_n \leq d).$$

The universal associative envelope of B^ω is the quotient algebra $U(B^\omega) = F\langle X \rangle / I$. Since the system B^ω is special, the natural map $B^\omega \to U(B^\omega)$ will be injective. From this set of generators for I, we compute a Gröbner basis for I, and use this Gröbner basis to obtain a monomial basis for $U(B^\omega)$. The multiplication table for $U(B^\omega)$ is obtained by computing normal forms of products of basis monomials.

10. Special nonassociative triple systems

The three smallest simple associative triple systems consist of matrices of the following forms, where $*$ represents an arbitrary scalar:

$$A_1 = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & * & * & * \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{bmatrix}.$$

See [53], [54] for a detailed study using noncommutative Gröbner bases of the universal associative envelopes of the nonassociative triple systems A_1^ω obtained by applying the trilinear operations ω of Table 2 to the 2-dimensional system A_1. There are two classes of operations: Lie type, for which the envelopes are infinite dimensional; and Jordan type, for which the envelopes are finite dimensional. For the operations of Lie type, the envelopes are down-up algebras [9] or their quotients. For the operations of Jordan type, the Wedderburn decomposition [26] of the universal envelope permits a classification of the irreducible representations. In this section we study the 4- and 6-dimensional systems A_2^ω and A_3^ω. The computations are described in detail for A_2^ω; the results for A_1^ω, A_2^ω and A_3^ω are summarized in Table 3; for an explanation of the notation, see subsection 10.14. All calculations were done using the computer algebra system Maple.

10.1 Symmetric sum. There are 20 generators obtained from the structure constants which form a Gröbner basis for the ideal defining the universal envelope:

operation	$U(A_1^\omega)$	$U(A_2^\omega)$	$U(A_3^\omega)$
Sym sum	4 1,2,4,**4,5**, ...	20 1,4,16,44,131,344, ...	56 1,6,36,160,750,3240, ...
Alt sum	0 1,2,4,8,16,32, ...	4 1,4,16,60,225,840, ...	20 1,6,36,196,1071,5796, ...
Cyc sum	4 1,2,4,**4,5**, ...	24,40 $\mid$ 59,724 $\mid$ 62 26	unable to complete
Lie $q=\infty$	2 1,2,4,6,9,12, ...	20,24 $\mid$ 16 1,4,14,36,85,176, ...	70,140 $\mid$ 51 1,6,30,110,360,1026, ...
Lie $q=\frac{1}{2}$	2 1,2,4,6,9,12, ...	20,26 $\mid$ 12 1,4,10,20,35,56, ...	70,147 $\mid$ 39 1,6,24,74,195,456, ...
Jor $q=\infty$	6,4 $\mid$ 4 5	40,32 $\mid$ 20 19	126,107 $\mid$ 54 69
Jor $q=0$	6 9	40,20 $\mid$ 27,4 $\mid$ 15 10	126,97 $\mid$ 71,9 $\mid$ 32 17
Jor $q=1$	6 9	40,19 $\mid$ 27,6 $\mid$ 15 10	126,93 $\mid$ 71,18 $\mid$ 32 17
Jor $q=\frac{1}{2}$	6,4 $\mid$ 4 5	40,94 $\mid$ 15 10	126,542 $\mid$ 32 17
AJ $q=\infty$	2 1,2,4,6,9,12, ...	24,76 $\mid$ 15 10	90,513 $\mid$ 32 17
AJ $q=-1$	2,2 $\mid$ 4,2 $\mid$ 4 5	24,37 $\mid$ 23,6 $\mid$ 15 10	90,135 $\mid$ 62,18 $\mid$ 32 17
AJ $q=\frac{1}{2}$	2 1,2,4,6,9,12, ...	24,32 $\mid$ 12 1,4,8,12,18,24, ...	90,107 $\mid$ 36 1,6,18,36,72,120, ...
AJ $q=2$	2,2 $\mid$ 4,2 $\mid$ 4 5	24,37 $\mid$ 23,4 $\mid$ 15 10	90,137 $\mid$ 62,9 $\mid$ 32 17
4th $q=\infty$	6,4 $\mid$ 4 5	40,140 $\mid$ 15 10	146,1065 $\mid$ 32 17
4th $q=0$	6 9	44,88 $\mid$ 15 10	146,737 $\mid$ 32 17
4th $q=1$	6 9	44,76 $\mid$ 15 10	146,618 $\mid$ 32 17
4th $q=-1$	6,5 $\mid$ 4 5	44,209 $\mid$ 15 10	146,1432 $\mid$ 32 17
4th $q=2$	6,5 $\mid$ 4 5	44,227 $\mid$ 15 10	146,1601 $\mid$ 32 17
4th $q=\frac{1}{2}$	6,4 $\mid$ 4 5	44,184 $\mid$ 15 10	146,1347 $\mid$ 32 17
Cyc com	4,4 $\mid$ 4 5	40,86 $\mid$ 15 10	140,396 $\mid$ 32 17
Weak C	8,2 $\mid$ 4 5	60,15 $\mid$ 15 10	196,58 $\mid$ 32 17
Weak AC	4,4 $\mid$ 4 5	44,41 $\mid$ 15 10	160,124 $\mid$ 32 17

TABLE 3. Universal associative envelopes of nonassociative triple systems

$$
\begin{aligned}
&a^3,\ ba^2+aba+a^2b,\ b^2a+bab+ab^2,\ b^3,\ ca^2+aca+a^2c-a,\\
&cba+cab+bca+bac+acb+abc-b,\ cb^2+bcb+b^2c,\ c^2a+cac+ac^2-c,\\
&c^2b+cbc+bc^2,\ c^3,\ da^2+ada+a^2d,\ dba+dab+bda+bad+adb+abd-a,\\
&db^2+bdb+b^2d-b,\ dca+dac+cda+cad+adc+acd-d,\\
&dcb+dbc+cdb+cbd+bdc+bcd-c,\ dc^2+cdc+c^2d,\ d^2a+dad+ad^2,\\
&d^2b+dbd+bd^2-d,\ d^2c+dcd+cd^2,\ d^3.
\end{aligned}
$$

There are infinitely many monomials in $F\langle a,b,c,d\rangle$ which do not contain the leading monomial of one of these generators as a subword, and so the universal envelope is infinite dimensional. The first few dimensions of the homogeneous components of the associated graded algebra are: 1, 4, 16, 44, 131, 344, 972, 2592, ... This sequence did not appear in the OEIS [105] at the time of writing.

10.2 Alternating sum. There are 4 generators, which form a Gröbner basis:

$$
\begin{aligned}
&cba-cab-bca+bac+acb-abc-b,\ dba-dab-bda+bad+adb-abd+a,\\
&dca-dac-cda+cad+adc-acd+d,\ dcb-dbc-cdb+cbd+bdc-bcd-c.
\end{aligned}
$$

There are infinitely many monomials in $F\langle a,b,c,d\rangle$ which do not contain the leading monomial of a generator, and so again the universal envelope is infinite dimensional. The first few dimensions of the homogeneous components of the associated graded algebra are 1, 4, 16, 60, 225, 840, 3136, 11704, ... This is sequence A072335 in the OEIS [105], which suggests the generating function $1/((1-x^2)(1-4x+x^2))$.

10.3 Cyclic sum. There are 24 generators, which do not form a Gröbner basis:

$$
\begin{aligned}
&a^3,\ ba^2+aba+a^2b,\ b^2a+bab+ab^2,\ b^3,\ ca^2+aca+a^2c-a,\ cab+bca+abc,\\
&cba+bac+acb-b,\ cb^2+bcb+b^2c,\ c^2a+cac+ac^2-c,\ c^2b+cbc+bc^2,\ c^3,\\
&da^2+ada+a^2d,\ dab+bda+abd-a,\ dac+cda+acd-d,\ dba+bad+adb,\\
&db^2+bdb+b^2d-b,\ dbc+cdb+bcd,\ dca+cad+adc,\ dcb+cbd+bdc-c,\\
&dc^2+cdc+c^2d,\ d^2a+dad+ad^2,\ d^2b+dbd+bd^2-d,\ d^2c+dcd+cd^2,\ d^3.
\end{aligned}
$$

There are 40 distinct nonzero compositions. Applying self-reduction to the combined set of 64 elements produces 59 elements, for which there are 724 distinct nonzero compositions. Applying self-reduction to the combined set of 783 elements produces 62 elements, which form a Gröbner basis:

$$
\begin{aligned}
&a^3,\ a^2b,\ a^2d,\ aba,\ ab^2,\ abc,\ abd-a^2c,\ aca+a^2c-a,\ ada,\ adb,\ adc,\ ad^2,\\
&ba^2,\ bab,\ bac+acb-b,\ bad,\ b^2a,\ b^3,\ b^2c,\ b^2d+acb-b,\ bca,\ bcb,\ bc^2,\ bcd,\\
&bda+a^2c-a,\ bdb-acb,\ bdc-ac^2,\ bd^2-acd,\ ca^2,\ cab,\ cac+ac^2-c,\ cad,\\
&cba,\ cb^2,\ cbc,\ cbd+ac^2-c,\ c^2a,\ c^2b,\ c^3,\ c^2d,\ cda,\ cdb,\ cdc,\ cd^2,\ da^2,\ dab,\\
&dac+acd-d,\ dad,\ dba,\ db^2,\ dbc,\ dbd+acd-d,\ dca,\ dcb,\ dc^2,\ dcd,\ d^2a,\\
&d^2b,\ d^2c,\ d^3,\ a^2cb-ab,\ a^2cd-ad.
\end{aligned}
$$

Only finitely many monomials in $F\langle a,b,c,d\rangle$ do not contain the leading monomial of a generator as a subword. A basis of the universal envelope consists of the cosets of these 26 monomials:

$$1,\ a,\ b,\ c,\ d,\ a^2,\ ab,\ ac,\ ad,\ ba,\ b^2,\ bc,\ bd,\ ca,\ cb,\ c^2,\ cd,\ da,\ db,\ dc,\ d^2,\ a^2c,\ acb,\ ac^2,\ acd,\ a^2c^2.$$

It is left as an exercise to determine the radical of the universal envelope, and the decomposition of the semisimple quotient into a direct sum of simple ideals.

10.4 Lie $q=\infty$. In this case we have a simple Lie triple system; see [85], [74]. There are 24 generators which self-reduce to 20 elements:

$$\begin{gathered}
ba^2-2aba+a^2b,\ b^2a-2bab+ab^2,\ ca^2-2aca+a^2c+2a,\\
cab-bca+bac-acb+b,\ cba-bca-acb+abc+b,\ cb^2-2bcb+b^2c,\\
c^2a-2cac+ac^2+2c,\ c^2b-2cbc+bc^2,\ da^2-2ada+a^2d,\\
dab-bda+bad-adb+a,\ dac-cda+cad-adc-d,\\
dba-bda-adb+abd+a,\ db^2-2bdb+b^2d+2b,\ dbc-cdb+cbd-bdc-c,\\
dca-cda-adc+acd,\ dcb-cdb-bdc+bcd,\ dc^2-2cdc+c^2d,\\
d^2a-2dad+ad^2,\ d^2b-2dbd+bd^2+2d,\ d^2c-2dcd+cd^2.
\end{gathered}$$

There are 24 distinct nonzero compositions; the combined set of 44 elements self-reduces to 16 elements, which is a Gröbner basis:

$$\begin{gathered}
ba-ab,\ dc-cd,\ ca^2-2aca+a^2c+2a,\ cab-bca-acb+abc+b,\\
cb^2-2bcb+b^2c,\ c^2a-2cac+ac^2+2c,\ c^2b-2cbc+bc^2,\ da^2-2ada+a^2d,\\
dab-bda-adb+abd+a,\ dac-cda+cad-acd-d,\ db^2-2bdb+b^2d+2b,\\
dbc-cdb+cbd-bcd-c,\ d^2a-2dad+ad^2,\ d^2b-2dbd+bd^2+2d,\\
cbca-cacb-bcac+acbc+cb+bc,\ dbda-dadb-bdad+adbd-da-ad.
\end{gathered}$$

There are infinitely many monomials in $F\langle a,b,c,d\rangle$ which do not contain the leading monomial of a generator, and so the universal envelope is infinite dimensional. The first few dimensions of the homogeneous components of the associated graded algebra are 1, 4, 14, 36, 85, 176, 344, 624, 1086, 1800, 2892, 4488, ... This is sequence A038164 in the OEIS [105], which suggests the generating function $1/((1-x)^4(1-x^2)^4)$.

10.5 Lie $q=\frac{1}{2}$. In this case we have a simple anti-Lie triple system. There are 40 generators which self-reduce to 20 elements:

$$\begin{gathered}
ba^2-a^2b,\ b^2a-ab^2,\ ca^2-a^2c,\ cab-bca-bac+acb-b,\\
cba+bca-acb-abc+b,\ cb^2-b^2c,\ c^2a-ac^2,\ c^2b-bc^2,\ da^2-a^2d,\\
dab-bda-bad+adb+a,\ dac-cda-cad+adc-d,\ dba+bda-adb-abd-a,\\
db^2-b^2d,\ dbc-cdb-cbd+bdc+c,\ dca+cda-adc-acd,\\
dcb+cdb-bdc-bcd,\ dc^2-c^2d,\ d^2a-ad^2,\ d^2b-bd^2,\ d^2c-cd^2.
\end{gathered}$$

There are 26 distinct nonzero compositions; the combined set of 46 generators self-reduces to 12 elements, which is a Gröbner basis:

$$a^2,\ ba+ab,\ b^2,\ c^2,\ dc+cd,\ d^2,\ cab-bca+acb+abc-b,$$
$$dab-bda+adb+abd+a,\ dac-cda-cad-acd-d,\ dbc-cdb-cbd-bcd+c,$$
$$cbca-cacb-bcac+acbc+cb-bc,\ dbda-dadb-bdad+adbd-da+ad.$$

There are infinitely many monomials in $F\langle a,b,c,d\rangle$ which do not contain the leading monomial of a generators, and so the universal envelope is infinite dimensional. The first few dimensions of the homogeneous components of the associated graded algebra are 1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286, 364, 455, 560, 680, 816, 969, ... These are the tetrahedral numbers $\binom{n+2}{3}$, sequence A000292 in the OEIS [105]. Every anti-Lie triple system can be embedded into a Lie superalgebra as the odd subspace; for Lie superalgebras and their enveloping algebras, see [103].

10.6 Jordan $q=\infty$. In this case we have a simple Jordan triple system. The original set of 40 generators is already self-reduced:

$$a^3,\ aba,\ aca-a,\ ada,\ ba^2+a^2b,\ bab,\ b^2a+ab^2,\ b^3,\ bca+acb-b,\ bcb,$$
$$bda+adb-a,\ bdb-b,\ ca^2+a^2c,\ cab+bac,\ cac-c,\ cba+abc,\ cb^2+b^2c,$$
$$cbc,\ c^2a+ac^2,\ c^2b+bc^2,\ c^3,\ cda+adc,\ cdb+bdc,\ cdc,\ da^2+a^2d,\ dab+bad,$$
$$dac+cad-d,\ dad,\ dba+abd,\ db^2+b^2d,\ dbc+cbd-c,\ dbd-d,\ dca+acd,$$
$$dcb+bcd,\ dc^2+c^2d,\ dcd,\ d^2a+ad^2,\ d^2b+bd^2,\ d^2c+cd^2,\ d^3.$$

There are 32 distinct nonzero compositions; the combined set of 72 generators self-reduces to 20, forming a Gröbner basis:

$$a^2,\ ab,\ ba,\ b^2,\ c^2,\ cd,\ dc,\ d^2,\ aca-a,\ ada,\ bca+acb-b,\ bcb,\ bda+adb-a,$$
$$bdb-b,\ cac-c,\ cbc,\ dac+cad-d,\ dad,\ dbc+cbd-c,\ dbd-d.$$

Only finitely many monomials in $F\langle a,b,c,d\rangle$ do not contain the leading monomial of a generator as a subword. A basis of the universal envelope consists of the cosets of these 19 monomials:

$$1,\ a,\ b,\ c,\ d,\ ac,\ ad,\ bc,\ bd,\ ca,\ cb,\ da,\ db,\ acb,\ adb,\ cad,\ cbd,\ acbd,\ cadb.$$

It is left as an exercise to compute the Wedderburn decomposition of the universal associative envelope, and to verify that it is isomorphic to $F \oplus M_3(F) \oplus M_3(F)$.

10.7 Jordan $q=0$. The original set of generators has 40 elements. There are 20 distinct nonzero compositions, and the combined set self-reduces to 27 elements. This set produces 4 distinct nonzero compositions, and the combined set self-reduces to 15 elements, forming a Gröbner basis:

$$\begin{gathered} a^2,\ ab,\ ad,\ ba,\ b^2,\ bc,\ bd-ac,\ c^2,\ cd,\ dc, \\ d^2,\ aca-a,\ acb-b,\ cac-c,\ dac-d. \end{gathered} \tag{3}$$

A basis of the universal associative envelope consists of the cosets of these 10 elements: 1, a, b, c, d, ac, ca, cb, da, db. It is left as an exercise to verify that the universal envelope is isomorphic to $F \oplus M_3(F)$.

10.8 Jordan $q = 1$. The original 40 generators produce 19 distinct nonzero compositions, and the combined set self-reduces to 27 elements. This set produces 6 distinct nonzero compositions, and the combined set self-reduces to 15 elements, forming a Gröbner basis which coincides with (3).

10.9 Jordan $q = \frac{1}{2}$. The original 40 generators produce 94 distinct nonzero compositions, and the combined set self-reduces to 15 elements, forming a Gröbner basis which coincides with (3).

10.10 Anti-Jordan $q = \infty, -1, 2$. For $q = \infty$, the original 24 generators produce 76 distinct nonzero compositions, and the combined set self-reduces to 15 elements which coincide with the Gröbner basis (3). For $q = -1$, the original 24 generators produce 37 distinct nonzero compositions, and the combined set self-reduces to 23 elements; this set has 6 distinct nonzero compositions, and the combined set self-reduces to 15 elements which coincide with (3). For $q = 2$, the original 24 generators produce 37 distinct nonzero compositions, and the combined set self-reduces to 23 elements; this set has 4 distinct nonzero compositions, and the combined set self-reduces to 15 elements which coincide with (3).

10.11 Anti-Jordan $q = \frac{1}{2}$. In this case we have a simple anti-Jordan triple system; see [7], [59]. The original set of 24 generators is as follows:

$$\begin{aligned}
&ba^2 - a^2b,\ b^2a - ab^2,\ bca - acb + b,\ bda - adb - a,\ ca^2 - a^2c,\ cab - bac,\\
&cba - abc,\ cb^2 - b^2c,\ c^2a - ac^2,\ c^2b - bc^2,\ cda - adc,\ cdb - bdc,\ da^2 - a^2d,\\
&dab - bad,\ dac - cad - d,\ dba - abd,\ db^2 - b^2d,\ dbc - cbd + c,\ dca - acd,\\
&dcb - bcd,\ dc^2 - c^2d,\ d^2a - ad^2,\ d^2b - bd^2,\ d^2c - cd^2.
\end{aligned}$$

There are 32 distinct nontrivial compositions, and the combined set self-reduces to a Gröbner basis of 12 elements: a^2, ab, ba, b^2, c^2, cd, dc, d^2, $bca - acb + b$, $bda - adb - a$, $dac - cad - d$, $dbc - cbd + c$. The universal associative envelope is infinite dimensional; the dimensions of the homogeneous components of the associated graded algebra are $\frac{1}{2}(n+1)(n+3)$ for n odd, and $\frac{1}{2}(n+2)^2$ for n even.

10.12 Fourth family. For $q = \infty$, the original 52 generators self-reduce to 44 elements, which have 140 distinct nonzero compositions; the combined set self-reduces to the Gröbner basis (3). For $q = 0$, the original 52 generators self-reduce to 44 elements, which have 88 distinct nonzero compositions; the combined set self-reduces to (3). For $q = 1$, the original 52 generators self-reduce to 44 elements, which have 76 distinct nonzero compositions; the combined set self-reduces to (3). For $q = -1$, the original 64 generators self-reduce to 44 elements, which have 209 distinct nonzero compositions; the combined set self-reduces to (3). For $q = 2$, the original 64 generators self-reduce to 44 elements, which have 227 distinct nonzero compositions; the combined set self-reduces to (3). For $q = \frac{1}{2}$, the original 64 generators self-reduce to 44 elements, which have 184 distinct nonzero compositions; the combined set self-reduces to (3).

10.13 The last three operations. For the cyclic commutator, the original 60 generators self-reduce to 40 elements, which have 86 distinct nonzero compositions; the combined set self-reduces to the Gröbner basis (3). For the weakly commutative operation, the original 64 generators self-reduce to 60 elements, which have 15 distinct nonzero compositions; the combined set self-reduces to (3). For the weakly anticommutative operation, the original 60 generators self-reduce to 44 elements, which have 41 distinct nonzero compositions; the combined set self-reduces to (3).

10.14 Summary. Table 3 summarizes the results of [53], [54] for $U(A_1^\omega)$, the results of this section for $U(A_2^\omega)$, and the author's further computations for $U(A_3^\omega)$. Each entry has the form

$$\begin{cases} \text{algorithm} \\ \text{dimension} \end{cases}$$

The "algorithm" data is a sequence of pairs x, y corresponding to the iterations of the Gröbner basis algorithm; x is the number of self-reduced generators at the start of the iteration, and y is the number of distinct nonzero compositions at the end of the iteration (if $y = 0$ it is omitted). The "dimension" data consists either of a single number (if the universal envelope is finite dimensional), or a sequence of numbers giving the first few dimensions of the homogeneous components of the associated graded algebra (if the universal envelope is infinite dimensional). Dimensions in boldface indicate values that repeat indefinitely. For example, for row "Lie $q = \infty$" and column "$U(A_3^\omega)$", referring to the Lie triple product on the 6-dimensional simple associative triple system, we have the entry

$$\begin{cases} 70, 140 \,|\, 51 \\ 1, 6, 30, 110, 360, 1026, \dots \end{cases}$$

which means that:

(a) the algorithm terminated after two iterations: the original self-reduced set of 70 generators produced 140 nontrivial compositions; the combined set of 210 elements self-reduced to a Gröbner basis of 51 elements;

(b) the universal associative envelope is infinite dimensional; the generating function for the dimensions of the homogeneous components of the associated graded algebra begins $1 + 6z + 30z^2 + 110z^3 + 360z^4 + 1026z^5 + \cdots$.

In one case, the cyclic sum on $U(A_3^\omega)$, the computations were so complicated that Maple 13 on a MacBook Pro was unable to complete them in a reasonable time. (The polynomial identities for the cyclic sum are extremely complicated; see [32].)

10.15 Conclusions. The results of Table 3 suggest a slightly different classification of operations into "Lie type" and "Jordan type" from that of [53], [54]. Two operations, the cyclic sum and the anti-Jordan $q = \infty$ operation, produce infinite dimensional envelopes for A_1^ω but finite dimensional envelopes for A_2^ω. It seems likely that A_1^ω is exceptional, owing to its small dimension, and that the universal associative envelopes will be finite dimensional when either of these operations is applied to a simple associative triple system of dimension > 2. If this is correct, then these two operations should be classified as "Jordan type".

Four operations produced a non-semisimple envelope for A_1^ω: Jordan $q = 0, 1$ and fourth family $q = 0, 1$. In these cases, the 9-dimensional envelope has a 4-dimensional radical and a semisimple quotient isomorphic to $F \oplus M_2(F)$. On the other hand, it seems likely that $U(A_n^\omega)$ for $n = 2, 3$ is semisimple and is isomorphic to $F \oplus M_{n+1}(F)$: the dimension (10 for $n = 2$, and 17 for $n = 3$) is the sum of the squares of the dimensions of the 1-dimensional representation and the $(n+1)$-dimensional natural representation. The same result seems to hold for most of the operations, since they produce the same Gröbner basis (3).

Conjecture 10.1. *Let $A_{p,q}$ $(p \leq q)$ be the simple associative triple system consisting of $(p+q) \times (p+q)$ block matrices of the form $\left[\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right]$. Let ω be one of the following trilinear operations from Table 2: Jordan $(q = 0, 1, \frac{1}{2})$, anti-Jordan $(q = \infty, -1, 2)$, fourth family (all cases), cyclic commutator, weakly commutative, weakly anticommutative. Then, with finitely many exceptions, $U(A^\omega)$ is finite dimensional, semisimple, and isomorphic to $F \oplus M_{p+q}(F)$.*

The operations not included in this conjecture are the symmetric, alternating, and cyclic sums, together with the Lie, anti-Lie, Jordan, and anti-Jordan triple products. These seem likely to be the operations producing nonassociative triple systems with the most interesting representations. This is well-known for the four classical operations, owing to their close connection with Lie and Jordan algebras and superalgebras. Very little is known about the representations of triple systems arising from the symmetric, alternating, and cyclic sums.

11. Literature survey

The historical origins of the theory of Gröbner bases are complex, with similar ideas discovered in different contexts at different times by different people.

11.1 The commutative case. The most important branch of the theory, owing to its close connections with algebraic geometry, is commutative Gröbner bases. Many of the basic ideas can be traced back to Macaulay's 1916 monograph *The Algebraic Theory of Modular Systems* [94]. Gröbner's original 1939 paper on linear differential equations [70] is often cited as the origin of the theory; this has appeared in English translation [1], [71]. The modern form of the theory, emphasizing the algorithmic aspects, dates from Buchberger's 1965 Ph.D. thesis, which has been translated into English [33], [34], [36]. There are many textbooks on commutative Gröbner bases and their applications; see Adams and Loustaunau [2], Becker and Weispfennig [8], Cox et al. [47], Ene and Herzog [56], Fröberg [60].

11.2 The noncommutative case. The theory of noncommutative Gröbner bases originated with the Russian school of nonassociative algebra; see Zhukov [119] and especially Shirshov [113], [114], [115]. The first statements of the Composition (Diamond) Lemma in the noncommutative case, and its application to the PBW theorem, were published almost simultaneously by Bokut [13] and Bergman [10]. The computational complexity of constructing noncommutative Gröbner bases has been studied by Mora [101]. Keller's Ph.D. thesis [80], [81] led to the software package Opal [68]. More recent software, with extensive online documentation, has been developed by Cohen and Gijsbers [46]. Some important papers on noncommutative Gröbner bases are Borges-Trenard et al. [25], Gerritzen [61], Green et al. [69], Kang et al. [79]. For a connection between commutative and noncommutative Gröbner bases, see Eisenbud et al. [52]. For an extension to noncommutative power series, see Gerritzen and Holtkamp [64]. Monographs on the noncommutative case are Bokut and Kukin [23], Bueso et al. [37], Li [84]. For surveys of commutative and noncommutative Gröbner bases, see Mora [102], Green [66, 67], Ufnarovski [116].

11.3 The nonassociative case. Gröbner-Shirshov bases for free Lie algebras are discussed by Bokut and Chibrikov [21], Bokut and Chen [15]. A theory of Gröbner-Shirshov bases in free nonassociative algebras has been developed by Gerritzen [62], [63], Rajaee [110]. For related work on Sabinin algebras, see Shestakov and Umirbaev [112], Pérez-Izquierdo [107], Chibrikov [42]. For surveys of Gröbner-Shirshov bases in associative and nonassociative algebras, see Bokut [14], Bokut-Kolesnikov [22], Bokut-Shum [24].

11.4 Term rewriting. This seems to be an appropriate place to mention a parallel development in theoretical computer science: the Knuth-Bendix algorithm for computing normal forms of words in general algebras. The origins of this topic lie in the work of Church [43] and Kleene [82] on the foundations of logic, and the subsequent work of Church and Rosser [44] on normal forms in the λ-calculus. The Church-Rosser theorem states that the reduction rules of the λ-calculus are confluent, meaning that if two distinct sequences of reductions can be applied to the same term x, producing two other terms y_1 and y_2, then there exists a term z which can be obtained from both y_1 and y_2 by some sequences of reductions. (The diagram illustrating this situation is the origin of the name "diamond lemma".) The work of these logicians was recast in a more abstract and applicable form by Newman [104] in his paper on theories with a combinatorial definition of equivalence. At the same time as the logicians were obtaining their results, Birkhoff [11] was laying the foundations for universal algebra, and this led to Evans' work [57] on the word problem for abstract algebras. The culmination of these developments is the paper by Knuth and Bendix [83] which introduced the notion of critical pairs and the superposition process for pairs of reductions in universal algebra (the analogue of the composition process for polynomials in commutative and noncommutative algebra). Since that time this area has evolved into an independent branch of theoretical computer science, known as term rewriting. An introductory monograph on this topic has been written by Baader and Nipkow [5]. The historical survey by Buchberger [35] clarifies the relations between these topics and Gröbner bases; see also Marché [96].

11.5 Loday algebras. An active area of research is the extension of the Composition (Diamond) Lemma from associative algebras to the dialgebras and dendriform algebras introduced by Loday [87], [88], [89]. For associative dialgebras, see Bokut et al. [19]. For dendriform algebras, see Bokut et al. [17], Chen and Wang [41], and for Rota-Baxter algebras, see Bokut et al. [16], [20], Chen and Mo [40], Qiu [109], Guo et al. [72]. It is an open problem to extend these results further to the quadri-algebras of Aguiar and Loday [3], and to the Koszul dual of quadri-algebras introduced by Vallette [117, §5.6]. For Leibniz algebras, the analogues of Lie algebras in the setting of dialgebras, see Loday and Pirashvili [90], Aymon and Grivel [4], Casas et al. [39], Insua and Ladra [77]. For pre-Lie algebras, the analogues of Lie algebras in the setting of dendriform algebras, see Bokut et al. [18]. For L-dendriform algebras, the analogues of Lie algebras in the setting of quadri-algebras, see Bai et al. [6]. (For corresponding generalizations of Jordan algebras, see Hou et al. [75], [76].) Loday algebras are a special case of the general theory of algebraic operads [97], [91], [120]. The theory of Gröbner bases has recently been extended to this setting by Dotsenko, Khoroshkin, and Vallette [49], [50], [51]. For an application to quadri-algebras, see Madariaga [95].

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