A symplectic representation of E_7

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Abstract. We explicitly construct a particular real form of the Lie algebra $\mathfrak{e}_7$ in terms of symplectic matrices over the octonions, thus justifying the identifications $\mathfrak{e}_7 \cong \mathfrak{sp}(6,\mathbb{O})$ and, at the group level, $E_7 \cong \mathrm{Sp}(6,\mathbb{O})$. Along the way, we provide a geometric description of the minimal representation of $\mathfrak{e}_7$ in terms of rank 3 objects called *cubies*.

Keywords: exceptional Lie algebras; octonions; E_7

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1. Introduction

The Freudenthal-Tits magic square [1], [2] of Lie algebras provides a parametrization in terms of division algebras of a family of Lie algebras that includes all of the exceptional Lie algebras except $\mathfrak{g}_2$. The "half-split" version of the magic square, in which one of the division algebras is split, is given in Table 1. The interpretation of the Lie algebra real forms appearing in the first two rows of the magic square as $\mathfrak{su}(3,\mathbb{K})$ and $\mathfrak{sl}(3,\mathbb{K})$ has been discussed in [3], [4]; see also [5], [6]. Freudenthal [7] (see also Brown [8]) provided an algebraic description of the symplectic geometry of $\mathfrak{e}_7$, and Barton & Sudbery [6] advanced this description to the Lie algebra level by interpreting the third row of the magic square as $\mathfrak{sp}(6,\mathbb{K})$. We continue this process here, providing a natural symplectic interpretation of the minimal representation of $\mathfrak{e}_7 = \mathfrak{e}_{7(-25)}$. Wilson [9] has also recently given a quaternionic construction of the group E_7.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{su}(3,\mathbb{R})$	$\mathfrak{su}(3,\mathbb{C})$	$\mathfrak{c}_3 \cong \mathfrak{su}(3,\mathbb{H})$	$\mathfrak{f}_4 \cong \mathfrak{su}(3,\mathbb{O})$
$\mathbb{C}'$	$\mathfrak{sl}(3,\mathbb{R})$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{a}_{5(-7)} \cong \mathfrak{sl}(3,\mathbb{H})$	$\mathfrak{e}_{6(-26)} \cong \mathfrak{sl}(3,\mathbb{O})$
$\mathbb{H}'$	$\mathfrak{c}_{3(3)} \cong \mathfrak{sp}(6,\mathbb{R})$	$\mathfrak{su}(3,3,\mathbb{C})$	$\mathfrak{d}_{6(-6)}$	$\mathfrak{e}_{7(-25)}$
$\mathbb{O}'$	$\mathfrak{f}_{4(4)}$	$\mathfrak{e}_{6(2)}$	$\mathfrak{e}_{7(-5)}$	$\mathfrak{e}_{8(-24)}$

TABLE 1. The "half-split" 3×3 magic square of Lie algebras.

2. Freudenthal's description of $\mathfrak{e}_7$

Let $\mathcal{X},\mathcal{Y} \in \mathrm{H}_3(\mathbb{O})$ be elements of the Albert algebra, that is, 3×3 Hermitian matrices whose components are octonions. There are two natural products on the

Albert algebra, namely the *Jordan product*

$$\mathcal{X} \circ \mathcal{Y} = \frac{1}{2}\Big(\mathcal{X}\mathcal{Y} + \mathcal{Y}\mathcal{X}\Big) \tag{1}$$

and the *Freudenthal product*

$$\mathcal{X} * \mathcal{Y} = \mathcal{X} \circ \mathcal{Y} - \frac{1}{2}\Big((\mathrm{tr}\mathcal{X})\mathcal{Y} + (\mathrm{tr}\mathcal{Y})\mathcal{X}\Big) + \frac{1}{2}\Big((\mathrm{tr}\mathcal{X})(\mathrm{tr}\mathcal{Y}) - \mathrm{tr}(\mathcal{X} \circ \mathcal{Y})\Big)\mathcal{I} \tag{2}$$

which can be thought of as a generalization of the cross product on $\mathbb{R}^3$ (with the trace of the Jordan product playing the role of the dot product).

The Lie algebra $\mathfrak{e}_6 = \mathfrak{e}_{6(-26)}$ acts on the Albert algebra $\mathrm{H}_3(\mathbb{O})$. The generators of $\mathfrak{e}_6$ fall into one of three categories; there are 26 *boosts*, 14 *derivations* (of $\mathbb{O}$, that is, elements of $\mathfrak{g}_2$), and 38 remaining *rotations* (the remaining generators of $\mathfrak{f}_4$).[1] For both boosts and rotations, $\phi \in \mathfrak{e}_6$ can be treated as a 3×3, tracefree, octonionic matrix; boosts are Hermitian, and rotations are anti-Hermitian. Such matrices $\phi \in \mathfrak{e}_6$ act on the Albert algebra via

$$\mathcal{X} \longmapsto \phi\mathcal{X} + \mathcal{X}\phi^\dagger \tag{3}$$

where $\dagger$ denotes conjugate transpose (in $\mathbb{O}$). As discussed at the group level in [10], the derivations can be obtained by successive rotations (or boosts) through *nesting*, corresponding to commutators in the Lie algebra, so it suffices to consider the boosts and rotations, that is, to consider matrix transformations.[2]

The dual representation of $\mathfrak{e}_6$ is formed by the duals ϕ' of each $\phi \in \mathfrak{e}_6$, defined via

$$\mathrm{tr}\big(\phi(\mathcal{X}) \circ \mathcal{Y}\big) = -\mathrm{tr}\big(\mathcal{X} \circ \phi'(\mathcal{Y})\big) \tag{4}$$

for $\mathcal{X}, \mathcal{Y} \in \mathrm{H}_3(\mathbb{O})$. It is easily checked that $\phi' = \phi$ on rotations, but that $\phi' = -\phi$ on boosts. Thus,

$$\phi' = -\phi^\dagger \tag{5}$$

for both boosts and rotations.

We can regard $\mathfrak{e}_7$ as the conformal algebra associated with $\mathfrak{e}_6$, since $\mathfrak{e}_7$ consists of the 78 elements of $\mathfrak{e}_6$, together with 27 translations, 27 conformal translations, and a dilation. In fact, Freudenthal [7] represents elements of $\mathfrak{e}_7$ as

$$\Theta = (\phi, \rho, \mathcal{A}, \mathcal{B}) \tag{6}$$

where $\phi \in \mathfrak{e}_6$, $\rho \in \mathbb{R}$ is the dilation, and $\mathcal{A}, \mathcal{B} \in \mathrm{H}_3(\mathbb{O})$ are elements of the Albert algebra, representing (null) translations.

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[1] This terminology is based on the analogous action of $\mathfrak{sl}(2,\mathbb{O}) \cong \mathfrak{so}(9,1)$ on $\mathrm{H}_2(\mathbb{O}) \cong \mathbb{R}^{(9,1)}$.

[2] Since all rotations can be obtained from pairs of boosts, it would be enough to consider boosts alone.

What does Θ act on? Freudenthal [7] explicitly constructs the minimal representation of $\mathfrak{e}_7$, which consists of elements of the form

$$\boldsymbol{\mathcal{P}} = (\mathcal{X}, \mathcal{Y}, p, q) \tag{7}$$

where $\mathcal{X}, \mathcal{Y} \in \mathrm{H}_3(\mathbb{O})$, and $p, q \in \mathbb{R}$. But how are we to visualize these elements? Freudenthal does tell us that Θ acts on $\boldsymbol{\mathcal{P}}$ via

$$\mathcal{X} \longmapsto \phi(\mathcal{X}) + \frac{1}{3}\rho\,\mathcal{X} + 2\mathcal{B} * \mathcal{Y} + \mathcal{A}\,q \tag{8}$$

$$\mathcal{Y} \longmapsto 2\mathcal{A} * \mathcal{X} + \phi'(\mathcal{Y}) - \frac{1}{3}\rho\,\mathcal{Y} + \mathcal{B}\,p \tag{9}$$

$$p \longmapsto \mathrm{tr}(\mathcal{A} \circ \mathcal{Y}) - \rho\,p \tag{10}$$

$$q \longmapsto \mathrm{tr}(\mathcal{B} \circ \mathcal{X}) + \rho\,q \tag{11}$$

But again, how are we to visualize this action?

We conclude this section by giving two further constructions due to Freudenthal [7]. There is a "super-Freudenthal" product $*$ taking elements $\boldsymbol{\mathcal{P}}$ of the minimal representation of $\mathfrak{e}_7$ to elements of $\mathfrak{e}_7$, given by[3]

$$\boldsymbol{\mathcal{P}} * \boldsymbol{\mathcal{P}} = (\phi, \rho, \mathcal{A}, \mathcal{B}) \tag{12}$$

where

$$\phi = \langle \mathcal{X}, \mathcal{Y} \rangle \tag{13}$$

$$\rho = -\frac{1}{4}\mathrm{tr}\big(\mathcal{X} \circ \mathcal{Y} - pq\,\mathcal{I}\big) \tag{14}$$

$$\mathcal{A} = -\frac{1}{2}\big(\mathcal{Y} * \mathcal{Y} - p\,\mathcal{X}\big) \tag{15}$$

$$\mathcal{B} = \frac{1}{2}\big(\mathcal{X} * \mathcal{X} - q\,\mathcal{Y}\big) \tag{16}$$

where

$$\langle X, Y \rangle Z = Y \circ (X \circ Z) - X \circ (Y \circ Z) - (X \circ Y) \circ Z + \frac{1}{3}\mathrm{tr}(X \circ Y) Z \tag{17}$$

Finally, $\mathfrak{e}_7$ preserves the quartic invariant

$$J = \mathrm{tr}\big((\mathcal{X} * \mathcal{X}) \circ (\mathcal{Y} * \mathcal{Y})\big) - p \det \mathcal{X} - q \det \mathcal{Y} - \frac{1}{4}\big(\mathrm{tr}(\mathcal{X} \circ \mathbf{Y}) - pq\big)^2 \tag{18}$$

which can be constructed using $\boldsymbol{\mathcal{P}} * \boldsymbol{\mathcal{P}}$.

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}'$	$\mathfrak{so}(2) \cong \mathfrak{su}(2,\mathbb{R})$	$\mathfrak{so}(3) \cong \mathfrak{su}(2,\mathbb{C})$	$\mathfrak{so}(5) \cong \mathfrak{su}(2,\mathbb{H})$	$\mathfrak{so}(9) \cong \mathfrak{su}(2,\mathbb{O})$
$\mathbb{C}'$	$\mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{so}(5,1) \cong \mathfrak{sl}(2,\mathbb{H})$	$\mathfrak{so}(9,1) \cong \mathfrak{sl}(2,\mathbb{O})$
$\mathbb{H}'$	$\mathfrak{so}(3,2) \cong \mathfrak{sp}(4,\mathbb{R})$	$\mathfrak{so}(4,2) \cong \mathfrak{su}(2,2,\mathbb{C})$	$\mathfrak{so}(6,2)$	$\mathfrak{so}(10,2)$
$\mathbb{O}'$	$\mathfrak{so}(5,4)$	$\mathfrak{so}(6,4)$	$\mathfrak{so}(8,4)$	$\mathfrak{so}(12,4)$

TABLE 2. The "half-split" 2×2 magic square of Lie algebras.

3. The symplectic structure of $\mathfrak{so}(k+2,2)$

An analogous problem has been analyzed for the 2×2 magic square, which is shown in Table 2; the interpretation of the first two rows was discussed in [10]; see also [5]. Dray, Huerta, and Kincaid showed first [11] (see also [12]) how to relate $\mathrm{SO}(4,2)$ to $\mathrm{SU}(2,\mathbb{H}' \otimes \mathbb{C})$, and later [13] extended their treatment to the full 2×2 magic square of Lie groups in Table 2. In the third row, their Clifford algebra description of $\mathrm{SU}(2,\mathbb{H}' \otimes \mathbb{K})$ is equivalent to a symplectic description as $\mathrm{Sp}(4,\mathbb{K})$, with $\mathbb{K} = \mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$.

Explicitly, they represent $\mathfrak{so}(k+2,2)$, where $k = |\mathbb{K}| = 1,2,4,8$, in terms of actions on 4×4 matrices of the form

$$P_0 = \begin{pmatrix} p\,\mathbf{I} & \mathbf{X} \\ -\widetilde{\mathbf{X}} & q\,\mathbf{I} \end{pmatrix} \tag{19}$$

where $\mathbf{X}$ is a 2×2 Hermitian matrix over $\mathbb{K}$, carrying a representation of $\mathfrak{so}(k+1,1)$, $p,q \in \mathbb{R}$, $\mathbf{I}$ denotes the 2×2 identity matrix, and tilde denotes trace-reversal, that is, $\widetilde{\mathbf{X}} = \mathbf{X} - \mathrm{tr}(\mathbf{X})\,\mathbf{I}$. The matrix P_0 can be thought of as the upper right 4×4 block of an 8×8 Clifford algebra representation over $\mathbb{K}$, and the action of $\mathfrak{so}(k+2,2)$ on P_0 is obtained as usual from (the restriction of) the quadratic elements of the Clifford algebra. The generators $A \in \mathfrak{so}(k+2,2)$ can be chosen so that the action takes the form

$$P_0 \longmapsto AP_0 \pm P_0 A \tag{20}$$

where the case-dependent signs are related to the restriction from 8×8 matrices to 4×4 matrices. Following Sudbery [5], we define the elements A of the symplectic Lie algebra $\mathfrak{sp}(4,\mathbb{K})$ by the condition[4]

$$A\Omega + \Omega A^\dagger = 0 \tag{21}$$

[3] We use $*$ to denote this "super-Freudenthal" product because of its analogy to the Freudenthal product $*$, with which there should be no confusion. Neither of these products is the same as the Hodge dual map, also denoted $*$, used briefly in Sections 3 and 4.

[4] The Lie algebra $\mathfrak{sp}(4,\mathbb{K})$ also contains the isometry algebra of $\mathrm{Im}(\mathbb{K})$.

where

$$\Omega = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix} \tag{22}$$

Solutions of (21) take the form

$$A = \begin{pmatrix} \boldsymbol{\phi} - \frac{1}{2}\,\rho\,\mathbf{I} & \mathbf{A} \\ \mathbf{B} & -\boldsymbol{\phi}^\dagger + \frac{1}{2}\,\rho\,\mathbf{I} \end{pmatrix} \tag{23}$$

where both $\mathbf{A}$ and $\mathbf{B}$ are Hermitian, $\mathrm{tr}(\boldsymbol{\phi}) = 0$, and $\rho \in \mathbb{R}$. But generators of $\mathfrak{so}(k+2,2)$ take exactly the same form: $\boldsymbol{\phi}$ represents an element of $\mathfrak{so}(k+1,1)$, $\mathbf{A}$ and $\mathbf{B}$ are (null) translations, and ρ is the dilation. Direct computation shows that the generators A of $\mathfrak{so}(k+2,2)$ do indeed satisfy (21), so that $\mathfrak{so}(k+2,2)$ and $\mathfrak{sp}(4,\mathbb{K})$ can be identified as vector spaces, and hence also as Lie algebras; the above construction therefore establishes the isomorphism

$$\mathfrak{so}(k+2,2) \cong \mathfrak{sp}(4,\mathbb{K}) \tag{24}$$

as claimed.

We can bring the representation (19) into a more explicitly symplectic form by treating $\mathbf{X}$ as a vector-valued 1-form, and computing its Hodge dual $*\mathbf{X}$, defined by

$$*\mathbf{X} = \mathbf{X}\boldsymbol{\epsilon} \tag{25}$$

where

$$\boldsymbol{\epsilon} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{26}$$

is the Levi-Civita tensor in two dimensions. Using the identity

$$\boldsymbol{\epsilon}\mathbf{X}\boldsymbol{\epsilon} = \widetilde{\mathbf{X}}^T \tag{27}$$

we see that $P = P_0\,\mathbf{I} \otimes \boldsymbol{\epsilon}$ takes the form

$$P = \begin{pmatrix} p\,\boldsymbol{\epsilon} & *\mathbf{X} \\ -(*\mathbf{X})^T & q\,\boldsymbol{\epsilon} \end{pmatrix} \tag{28}$$

which is antisymmetric, and whose block structure is shown in Figure 1. The diagonal blocks, labeled 00 and 11, are antisymmetric, and correspond to p and q, respectively, whereas the off-diagonal blocks, labeled 01 and 10, contain equivalent information, corresponding to $*\mathbf{X}$. Note that $*\mathbf{X}$ does not use up all of the degrees of freedom available in an off-diagonal block; the set of *all* antisymmetric 4×4 matrices is *not* an irreducible representation of $\mathfrak{sp}(4,\mathbb{K})$.

The action of $\mathfrak{sp}(4,\mathbb{K})$ on P is given by

$$P \longmapsto AP + PA^T \tag{29}$$

Figure 1. The block structure of a 4×4 antisymmetric matrix in terms of 2×2 blocks. A binary labeling of the blocks is shown on the left; on the right, blocks with similar shading contain equivalent information.

for $A \in \mathfrak{sp}(4, \mathbb{K})$, that is, for A satisfying (21).[5] When working over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, the action (29) is just the antisymmetric square

$$v \wedge w \longmapsto Av \wedge w + v \wedge Aw \tag{30}$$

of the natural representation $v \longmapsto Av$, with $v \in \mathbb{K}^4$.

4. Cubies

Before generalizing the above construction to the 3×3 magic square, we first consider the analog of $*\mathbf{X}$. Let $\mathcal{X} \in \mathrm{H}_3(\mathbb{O})$ be an element of the Albert algebra, which we can regard as a vector-valued 1-form with components $\mathcal{X}_a{}^b$, with $a, b \in \{1, 2, 3\}$. The Hodge dual $*\mathcal{X}$ of $\mathcal{X}$ is a vector-valued 2-form with components

$$(*\mathcal{X})_{abc} = \mathcal{X}_a{}^m \epsilon_{mbc} \tag{31}$$

where ϵ_{abc} denotes the Levi-Civita tensor in three dimensions, that is, the completely antisymmetric tensor satisfying

$$\epsilon_{123} = 1 \tag{32}$$

and where repeated indices are summed over. We refer to $*\mathcal{X}$ as a *cubie*. We also introduce the dual of ϵ_{abc}, the completely antisymmetric tensor ϵ^{abc} satisfying

$$\epsilon_{mns}\epsilon^{mns} = 6 \tag{33}$$

and note the further identities

$$\epsilon_{amn}\, \epsilon^{bmn} = 2\, \delta_a{}^b \tag{34}$$

$$\epsilon_{abm}\, \epsilon^{cdm} = \delta_a{}^c\, \delta_b{}^d - \delta_a{}^d\, \delta_b{}^c \tag{35}$$

$$\begin{aligned} \epsilon_{abc}\, \epsilon^{def} &= \delta_a{}^d\, \delta_b{}^e\, \delta_c{}^f + \delta_b{}^d\, \delta_c{}^e\, \delta_a{}^f + \delta_c{}^d\, \delta_a{}^e\, \delta_b{}^f \\ &\quad - \delta_a{}^d\, \delta_c{}^e\, \delta_b{}^f - \delta_b{}^d\, \delta_a{}^e\, \delta_c{}^f - \delta_c{}^d\, \delta_a{}^e\, \delta_b{}^f \end{aligned} \tag{36}$$

[5] Thus, (29) can be used if desired to determine the signs in (20).

In particular, we have

$$(*\mathcal{X})_{amn}\epsilon^{bmn} = 2\mathcal{X}_a{}^b \tag{37}$$

Operations on the Albert algebra can be rewritten in terms of cubies. For instance,

$$\operatorname{tr}\mathcal{X} = \frac{1}{2}\,\mathcal{X}_{abc}\,\epsilon^{abc} \tag{38}$$

$$\big(*(\mathcal{X}\,\mathcal{Y})\big)_{abc} = \frac{1}{2}\,\mathcal{X}_{amn}\,\mathcal{Y}_{pbc}\,\epsilon^{mnp} \tag{39}$$

$$\big(*(\mathcal{X}\circ\mathcal{Y})\big)_{abc} = \frac{1}{4}\big(\mathcal{X}_{amn}\,\mathcal{Y}_{pbc} + \mathcal{Y}_{amn}\,\mathcal{X}_{pbc}\big)\,\epsilon^{mnp} \tag{40}$$

$$\begin{aligned}\operatorname{tr}(\mathcal{X}\circ\mathcal{Y}) &= \frac{1}{8}\big(\mathcal{X}_{amn}\,\mathcal{Y}_{pbc} + \mathcal{Y}_{amn}\,\mathcal{X}_{pbc}\big)\,\epsilon^{mnp}\,\epsilon^{bca}\\ &= \frac{1}{8}\big(\mathcal{X}_{amn}\,\mathcal{Y}_{pbc} + \mathcal{Y}_{pbc}\,\mathcal{X}_{amn}\big)\,\epsilon^{mnp}\,\epsilon^{bca}\end{aligned} \tag{41}$$

$$(\operatorname{tr}\mathcal{X})(\operatorname{tr}\mathcal{Y}) = \frac{1}{2}\,\mathcal{X}_{abc}\,\mathcal{Y}_{def}\,\epsilon^{abc}\,\epsilon^{def} \tag{42}$$

from which the components of $*(\mathcal{X}*\mathcal{Y})$ can also be worked out. In the special case where the components of $\mathcal{X}$ and $\mathcal{Y}$ commute, contracting both sides of (34) with $\mathcal{X}\otimes\mathcal{Y}$ yields

$$\frac{1}{2}\mathcal{X}_c{}^m\mathcal{Y}_d{}^n\,\epsilon_{amn}\,\epsilon^{bcd} = (\mathcal{X}*\mathcal{Y})_a{}^b \tag{43}$$

or equivalently

$$\big(*(\mathcal{X}*\mathcal{Y})\big)_{abc} = \frac{1}{2}(\mathcal{X}_b{}^m\mathcal{Y}_c{}^n - \mathcal{X}_c{}^m\mathcal{Y}_b{}^n)\,\epsilon_{amn} \tag{44}$$

providing two remarkably simple expressions for the Freudenthal product, albeit only in a very special case. We will return to this issue below.

Lemma 1. *The action of $\phi\in\mathfrak{e}_6$ on cubies is given by*

$$(*\mathcal{X})_{abc} \longmapsto \phi_a{}^m(*\mathcal{X})_{mbc} + (*\mathcal{X})_{amc}\phi'_b{}^m + (*\mathcal{X})_{abm}\phi'_c{}^m \tag{45}$$

Proof: Consider the expression

$$Q_{nbc} = \phi'_n{}^m\epsilon_{mbc} + \phi'_b{}^m\epsilon_{nmc} + \phi'_c{}^m\epsilon_{nbm} \tag{46}$$

which is completely antisymmetric, and hence vanishes unless n, b, c are distinct. But then

$$Q_{nbc} = \operatorname{tr}(\phi')\,\epsilon_{nbc} \tag{47}$$

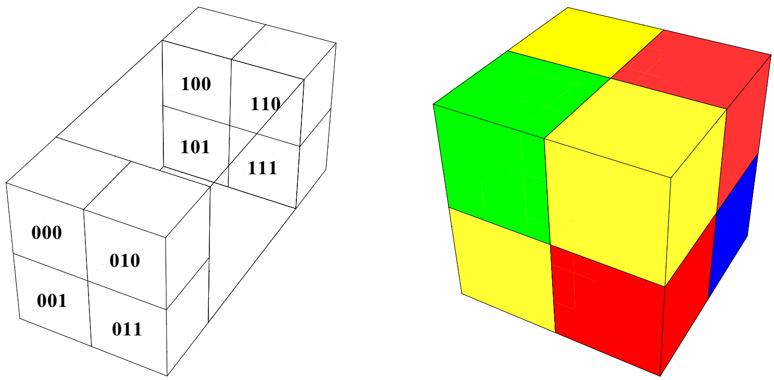

FIGURE 2. The block structure of a $6\times6\times6$ antisymmetric tensor in terms of $3\times3\times3$ "cubies". A binary labeling of the cubies is shown on the pulled-apart cube on the left; on the right, cubies with similar shading contain equivalent information.

which vanishes, since $\mathrm{tr}(\phi') = -\mathrm{tr}(\phi) = 0$. Thus, (3) becomes

$$\begin{aligned} \mathcal{X}_a{}^m \epsilon_{mbc} &\longmapsto \left(\phi_a{}^n \mathcal{X}_n{}^m + \mathcal{X}_a{}^n \phi^{\dagger}_n{}^m\right) \epsilon_{mbc} \\ &= \phi_a{}^n \mathcal{X}_n{}^m\, \epsilon_{mbc} + \mathcal{X}_a{}^n \phi'_b{}^m \epsilon_{nmc} + \mathcal{X}_a{}^n \phi'_c{}^m \epsilon_{nbm} \end{aligned} \tag{48}$$

as claimed, where we have used (5), (31), and (47). □

A similar result holds for the action of ϕ'.

5. The symplectic structure of $\mathfrak{e}_7$

The representation (6) can be written in block form, which we also call Θ, namely[6]

$$\Theta = \begin{pmatrix} \phi - \frac{1}{3}\rho\mathcal{I} & \mathcal{A} \\ \mathcal{B} & \phi' + \frac{1}{3}\rho\mathcal{I} \end{pmatrix} \tag{49}$$

where $\mathcal{I}$ denotes the 3×3 identity matrix. By analogy with Section 3, we would like Θ to act on $*\mathcal{X}$, which has 3 indices, and is antisymmetric in 2 of them, and hence has the correct symmetries to be an off-diagonal block of a rank 3 antisymmetric tensor $\boldsymbol{\mathcal{P}}$. The components of $\boldsymbol{\mathcal{P}}$ make up a $6\times6\times6$ cube, which we divide into $3\times3\times3$ cubies, as shown in Figure 2; compare Figure 1. We identify the diagonal cubies, labeled 000 and 111, with $p*\mathcal{I}$ and $q*\mathcal{I}$, respectively, the cubie labeled 011 with $*\mathcal{X}$, the cubie labeled 100 with $*\mathcal{Y}$, and then let antisymmetry do the

[6]The derivations $\mathfrak{g}_2 \subset \mathfrak{e}_6$ require nested matrix transformations of the form (49).

rest. Explicitly, we have

$$\boldsymbol{\mathcal{P}}_{abc} = \begin{cases} p\,\epsilon_{abc} & a \le 3, b \le 3, c \le 3 \\ (*\mathcal{Y})_{\hat{a}bc} & a \ge 4, b \le 3, c \le 3 \\ (*\mathcal{X})_{a\hat{b}\hat{c}} & a \le 3, b \ge 4, c \ge 4 \\ q\,\epsilon_{\hat{a}\hat{b}\hat{c}} & a \ge 4, b \ge 4, c \ge 4 \end{cases} \tag{50}$$

where we have introduced the convention that $\hat{a} = a-3$, and where the remaining components are determined by antisymmetry.[7]

In the complex case, we could begin with the natural action of Θ on 6-component complex vectors, and then take the antisymmetric cube, that is, we could consider the action

$$u \wedge v \wedge w \longmapsto \Theta u \wedge v \wedge w + u \wedge \Theta v \wedge w + u \wedge v \wedge \Theta w \tag{51}$$

with $u, v, w \in \mathbb{C}^6$, or equivalently

$$\boldsymbol{\mathcal{P}}_{abc} \longmapsto \Theta_a{}^m \boldsymbol{\mathcal{P}}_{mbc} + \Theta_b{}^m \boldsymbol{\mathcal{P}}_{amc} + \Theta_c{}^m \boldsymbol{\mathcal{P}}_{abm} \tag{52}$$

Over the octonions, however, the action (52) must be modified and reinterpreted in order to yield the Freudenthal action (8), as we now show.

Lemma 2. *The action of the dilation* $\Theta = (0, \rho, 0, 0) \in \mathfrak{e}_7$ *on* $\boldsymbol{\mathcal{P}}$ *is given by* (52).

Proof: From (49), we have

$$\Theta_a{}^b = \pm \frac{1}{3}\,\rho\,\delta_a{}^b \tag{53}$$

with the sign being negative for $a = b \le 3$ and positive for $a = b \ge 3$. Thus, (52) becomes

$$\boldsymbol{\mathcal{P}}_{abc} \longmapsto \pm\frac{1}{3}\,\rho\,\boldsymbol{\mathcal{P}}_{abc} \pm \frac{1}{3}\,\rho\,\boldsymbol{\mathcal{P}}_{abc} \pm \frac{1}{3}\,\rho\,\boldsymbol{\mathcal{P}}_{abc} \tag{54}$$

where the signs depend on which of a, b, c are "small" (≤ 3) or "large" (≥ 4). Examining (50), it is now easy to see that $p \mapsto -\rho\,p$, $q \mapsto +\rho\,p$, $\mathcal{X} \mapsto +\frac{\rho}{3}\mathcal{X}$, and $\mathcal{Y} \mapsto -\frac{\rho}{3}\mathcal{Y}$, exactly as required by (8)–(11). □

Lemma 3. *If the elements of* $\mathcal{A}, \mathcal{B} \in \mathrm{H}_3(\mathbb{O})$ *commute with those of* $\boldsymbol{\mathcal{P}}$*, then the action of the translations* $\Theta = (0, 0, \mathcal{A}, 0)$ *and* $\Theta = (0, 0, 0, \mathcal{B})$ *on* $\boldsymbol{\mathcal{P}}$ *is given by* (52).

Proof: Set $\Theta = (0, 0, \mathcal{A}, 0)$ and consider the action of Θ on p, $\mathcal{X}$, $\mathcal{Y}$, and q. We need to verify (8)–(11) with $\phi = 0$, $\rho = 0$, and $\mathcal{B} = 0$. From (49), we have

$$\Theta_a{}^b = \begin{cases} \mathcal{A}_a{}^{\hat{b}} & a \le 3, b \ge 4 \\ 0 & \text{otherwise} \end{cases} \tag{55}$$

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[7]Note that $\boldsymbol{\mathcal{P}}$ is a *cube*, and has components $\boldsymbol{\mathcal{P}}_{abc}$ with $a, b, c \in \{1, 2, 3, 4, 5, 6\}$, whereas ϵ_{abc}, $*\mathcal{X}_{abc}$, and $*\mathcal{Y}_{abc}$ are the components of *cubies*, which are subblocks of $\boldsymbol{\mathcal{P}}$, with $a, b, c \in \{1, 2, 3\}$.

Since $\mathcal{A}$ has one "small" index and one "large" index, it reduces the number of "large" indices present, e.g. mapping cubie 100 to 000, and thus maps $q \mapsto \mathcal{X} \mapsto \mathcal{Y} \mapsto p$. In particular, this confirms the lack of a term involving $\mathcal{A}$ in (11). Considering terms involving q, we look at cubie 011, where the only nonzero term of (52) is

$$(*\mathcal{X})_{abc} \longmapsto \mathcal{A}_a{}^m q\,\epsilon_{mbc} = q\,(*\mathcal{A})_{abc} \tag{56}$$

which verifies (8) in this case.

We next look at cubie 000, where (52) becomes

$$p\,\epsilon_{abc} \longmapsto \mathcal{A}_a{}^m(*\mathcal{Y})_{mbc} + \mathcal{A}_b{}^m(*\mathcal{Y})_{mca} + \mathcal{A}_c{}^m(*\mathcal{Y})_{mab} \tag{57}$$

which is clearly antisymmetric, so we can use (33) and (37) to obtain

$$p \longmapsto \frac{1}{2}\,\mathcal{A}_a{}^m(*\mathcal{Y})_{mbc}\,\epsilon^{abc} = \mathcal{A}_a{}^m\mathcal{Y}_m{}^a = \operatorname{tr}(\mathcal{A}\mathcal{Y}) = \operatorname{tr}(\mathcal{A}\circ\mathcal{Y}) \tag{58}$$

which is (10), where we have used commutativity only in the last equality.

Finally, turning to cubie 100, (52) becomes

$$\begin{aligned}(*\mathcal{Y})_{abc} \longmapsto{}& \mathcal{A}_b{}^m(*\mathcal{X})_{cam} + \mathcal{A}_c{}^m(*\mathcal{X})_{bma}\\ ={}& \mathcal{A}_b{}^m\mathcal{X}_c{}^n\,\epsilon_{nam} + \mathcal{A}_c{}^m\mathcal{X}_b{}^n\,\epsilon_{nma}\end{aligned} \tag{59}$$

or equivalently, using (37) and (43),

$$2\,\mathcal{Y}_a{}^b \longmapsto 2\,\mathcal{A}_e{}^m\mathcal{X}_f{}^n\,\epsilon_{amn}\,\epsilon^{bef} = 4\,(\mathcal{X}*\mathcal{Y})_a{}^b \tag{60}$$

which is (9).

This entire argument can be repeated with only minor changes if Θ=$(0,0,0,\mathcal{B})$. □

Over $\mathbb{R}$ or $\mathbb{C}$, we are done; Lemmas 1, 2, and 3 together suffice to show that the action (52) is the same as the Freudenthal action (8)–(11). Unfortunately, the action (52) fails to satisfy the Jacobi identity over $\mathbb{H}$ or $\mathbb{O}$. However, we can still use Lemmas 1, 2, and 3 to reproduce the Freudenthal action in those cases, as follows.

Lemma 4. *The action of* $\Theta = (\phi, 0, 0, 0) \in \mathfrak{e}_7$ *on* $\boldsymbol{\mathcal{P}}$ *is determined by*

$$\boldsymbol{\mathcal{P}}_{abc} \longmapsto \Theta_a{}^m\boldsymbol{\mathcal{P}}_{mbc} + \boldsymbol{\mathcal{P}}_{amc}\Theta_b{}^m + \boldsymbol{\mathcal{P}}_{abm}\Theta_c{}^m \tag{61}$$

when acting on elements of the form (50)*, which extends to all of* $\mathfrak{e}_7$ *by antisymmetry.*

Proof: From (49), we have

$$\Theta_a{}^b = \begin{cases} \phi_a{}^b & a \le 3, b \le 3 \\ \phi'_{\hat{a}}{}^{\hat{b}} & a \ge 4, b \ge 4 \\ 0 & \text{otherwise} \end{cases} \tag{62}$$

Inserting (62) into (61) now yields precisely (45) when acting on $\mathcal{X}$; the argument for the action on $\mathcal{Y}$ is similar. Furthermore, using an argument similar to that used to prove Lemma 1 to begin with, (52) acts on p via

$$p\,\epsilon_{abc} \longmapsto \phi_a{}^m p\,\epsilon_{mbc} + \phi_b{}^m p\,\epsilon_{amc} + \phi_c{}^m p\,\epsilon_{abm} \tag{63}$$

which is completely antisymmetric in a, b, c, and therefore proportional to $\operatorname{tr}(\phi) = 0$. The argument for the action on q is similar, with ϕ replaced by ϕ'. Although (61) itself is only antisymmetric in its last two indices, that suffices to define an action on cubies 000, 011, 100, and 111; the action on the remaining 4 cubies is uniquely determined by requiring that antisymmetry be preserved. □

We now have all the pieces, and can state our main result.

Theorem 1. *The Lie algebra $\mathfrak{e}_7$ acts symplectically on cubes, that is, $\mathfrak{e}_6 \subset \mathfrak{e}_7$ acts on cubes via* (61)*, as do real translations and the dilation, and all other $\mathfrak{e}_7$ transformations can then be constructed from these transformations using linear combinations and commutators.*

Proof: Lemmas 2 and 3 are unchanged by the use of (61) rather than (52), since the components of Θ commute with those of $\boldsymbol{\mathcal{P}}$ in both cases, and Lemma 4 verifies that $\mathfrak{e}_6$ acts via (61), as claimed. It only remains to show that the remaining generators of $\mathfrak{e}_7$ can be obtained from these elements via commutators.

Using (8)–(11), it is straightforward to compute the commutator of two $\mathfrak{e}_7$ transformations of the form (6). Letting $\phi = \mathcal{Q} \in \mathfrak{e}_6$ be a boost, so that $\mathcal{Q}^\dagger = \mathcal{Q}$ and $\operatorname{tr}(\mathcal{Q}) = 0$, and using the identity

$$-(\mathcal{A} \circ \mathcal{B}) * \mathcal{X} = \big(\mathcal{B} - \operatorname{tr}(\mathcal{B})\mathcal{I}\big) \circ (\mathcal{A} * \mathcal{X}) + \mathcal{A} * (\mathcal{B} \circ \mathcal{X}) \tag{64}$$

for any $\mathcal{A}, \mathcal{B}, \mathcal{X} \in \mathrm{H}_3(\mathbb{O})$, we obtain

$$\big[(0,0,\mathcal{A},0), (\mathcal{Q},0,0,0)\big] = (0,0,\mathcal{A} \circ \mathcal{Q},0) \tag{65}$$

We can therefore obtain the null translation $(0,0,\mathcal{Q},0)$ for any *tracefree* Albert algebra element $\mathcal{Q}$ as the commutator of $(0,0,\mathcal{I},0)$ and $(\mathcal{Q},0,0,0)$; a similar argument can be used to construct the null translation $(0,0,0,\mathcal{Q})$. □

Thus, *all* generators of $\mathfrak{e}_7$ can be implemented either as a symplectic transformation on cubes via (61), or as the commutator of two such transformations.

6. Discussion

We have showed that the algebraic description of the minimal representation of e_7 introduced by Freudenthal naturally corresponds geometrically to a symplectic structure. Along the way, we have emphasized both the similarities and differences between $\mathfrak{e}_7$ and $\mathfrak{so}(10,2)$ (see also Rohrle [14]). Both of these algebras are *conformal*; their elements divide naturally into generalized rotations ($\mathfrak{e}_6$ or $\mathfrak{so}(9,1)$, respectively), translations, and a dilation. Both act naturally on a representation built out of vectors (3×3 or 2×2 Hermitian octonionic matrices, respectively), together with two additional real degrees of freedom (p and q).[8] In the 2×2 case, the representation (19) contains just one vector; in the 3×3 case (7), there are two. This at first puzzling difference is fully explained by expressing both representations as antisymmetric tensors, as in (28) and (50), respectively, and as shown geometrically in Figures 1 and 2.

In the complex case, we have shown that the symplectic action (52) exactly reproduces the Freudenthal action (8)–(11). The analogy goes even further. In $2n$ dimensions, there is a natural map taking two n-forms to a tensor of rank 2. When acting on $\boldsymbol{\mathcal{P}}$, this map takes the form

$$\boldsymbol{\mathcal{P}} \longmapsto \boldsymbol{\mathcal{P}}_{acd}\boldsymbol{\mathcal{P}}_{efg}\,\epsilon^{cdefgb} \tag{66}$$

where ϵ now denotes the volume element in six dimensions, that is, the completely antisymmetric tensor with $\epsilon^{123456}=1$. It is not hard to verify that, in the complex case, (66) is (a multiple of) $\boldsymbol{\mathcal{P}} * \boldsymbol{\mathcal{P}}$, as given by (12)–(16). Similarly, the quartic invariant (18) can be expressed in the complex case as

$$J \sim \boldsymbol{\mathcal{P}}_{gab}\boldsymbol{\mathcal{P}}_{cde}\boldsymbol{\mathcal{P}}_{fhi}\boldsymbol{\mathcal{P}}_{jkl}\,\epsilon^{abcdef}\,\epsilon^{ghijkl} \tag{67}$$

up to an overall factor.

Neither the form of the action (52), nor the expressions (66) and (67), hold over $\mathbb{H}$ or $\mathbb{O}$. This failure should not be a surprise, as trilinear tensor products are not well defined over $\mathbb{H}$, let alone $\mathbb{O}$. Nonetheless, Theorem 1 does tell us how to extend (52) to the octonions. Although it is also possible to write down versions of (66) and (67) that hold over the octonions, by using case-dependent algorithms to determine the order of multiplication, it is not clear that such expressions have any advantage over the original expressions (12)–(16) and (18) given by Freudenthal.

Despite these drawbacks, it is clear from our construction that $\mathfrak{e}_7$ should be regarded as a natural generalization of the traditional notion of a symplectic Lie algebra, and fully deserves the name $\mathfrak{sp}(6,\mathbb{O})$.

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[8]The relations (31) and (37) can be used to identify *any* 3×3 matrix over $\mathbb{K}$ with a "cubie-like" piece of a "cube-like" 3-form. However, the set of all such 3-forms does not carry an irreducible representation of e_7; in fact, it does not carry a representation of e_7 at all, as the algebra fails to close. Similar statements hold in the 2×2 case.

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