Comment.Math.Univ.Carolin. 55,3 (2014) 411–420 

# Moufang semidirect products of loops with groups and inverse property extensions

Mark Greer, Lee Raney

*Abstract.* We investigate loops which can be written as the semidirect product of a loop and a group, and we provide a necessary and sufficient condition for such a loop to be Moufang. We also examine a class of loop extensions which arise as a result of a finite cyclic group acting as a group of semiautomorphisms on an inverse property loop. In particular, we consider closure properties of certain extensions similar to those as in [S. Gagola III, *Cyclic extensions of Moufang loops induced by semiautomorphisms*, J. Algebra Appl. **13** (2014), no. 4, 1350128], but from an external point of view.

*Keywords:* extensions; semidirect products; Moufang loops; inverse property loops

*Classification:* 20N05

## 1. Introduction

A *loop* $(Q, \cdot)$ consists of a set $Q$ with a binary operation $\cdot : Q \times Q \to Q$ such that (i) for all $a, b \in Q$, the equations $ax = b$ and $ya = b$ have unique solutions $x, y \in Q$, and (ii) there exists $1 \in Q$ such that $1x = x1 = x$ for all $x \in Q$. We denote these unique solutions by $x = a\backslash b$ and $y = b/a$, respectively. Standard references in loop theory are [2], [12]. All loops considered here are finite loops.

If $G$, $N$, and $H$ are groups, then $G$ is an *extension* of $H$ by $N$ if $N \trianglelefteq G$ and $G/N \simeq H$. Extensions of groups are of great interest in group theory. Of particular interest are the ideas of *internal* and *external* semidirect products of groups. That is, for $G = NH$ with $N \trianglelefteq G$ and $N \cap H = 1$, the structure of $G$ is uniquely determined by $N$, $H$, and the action of $H$ on $N$ by conjugation. In this case, $G$ is the internal *semidirect product of $H$ acting on $N$*. Alternatively, given groups $N$ and $H$ and a group homomorphism $\phi : H \to \mathrm{Aut}(N)$ from $H$ into the group of automorphisms of $N$, one can construct the external semidirect product $G = N \rtimes_\phi H$ in a standard way: namely, $G = \{(n, h) \mid n \in N, h \in H\}$ with multiplication defined by $(n_1, h_1)(n_2, h_2) = (n_1\phi(h_1)(n_2), h_1h_2)$. Note that for all $h \in H$, $\phi(h)$ corresponds to conjugation by $h$ in $N$. There is a natural equivalence between internal and external semidirect products of groups, and both are usually referred to as *semidirect products*.

Now, let $N$ be a loop and $H$ a group. Let $\phi : H \to \mathrm{Sym}(N)$ be a group homomorphism from $H$ into the symmetric group on $N$. Then, the external semidirect product $G = N \rtimes_\phi H$ is the quasigroup defined as $G = \{(x,g) \mid x \in N, g \in H\}$ with multiplication given by

$$(x,g)(y,h) = (x\phi(g)(y), gh). \tag{1}$$

Such semidirect products have been studied in [8]. Note that if $\phi : H \to \mathrm{Sym}(N)_1$, where $\mathrm{Sym}(N)_1$ is the stabilizer of 1 in $\mathrm{Sym}(N)$, then $G$ is a loop. In general, properties of the loop $N$ do not necessarily extend to all of $G$. For example, if $N$ is a Moufang loop, $G$ need not be Moufang, even in the case that $H$ acts on $N$ as a group of automorphisms.

In §2, we provide a necessary and sufficient condition on $N$ and $\phi$ in order for $G = N \rtimes_\phi H$ to be a Moufang loop (Theorem 5). We also provide an example which shows that it is possible to satisfy the hypotheses of Theorem 5.

In [7], the author studies Moufang loops that can be written as the product of a normal Moufang subloop and a cyclic subgroup. There it is shown that given a Moufang loop $G = NH$, where $N$ is a normal subloop and $H$ is a cyclic subgroup of order coprime to 3, the multiplication in $G$ is completely determined by a particular semiautomorphism of $N$. Conversely, the author notes that an arbitrary extension of a cyclic group of order coprime to 3 by a Moufang loop (with multiplication given as in the conclusion of Theorem 8) is not necessarily Moufang. In §3, our approach is to consider the *external* extension of a cyclic group $H$ (whose order is not divisible by 3) by a loop $N$ with multiplication defined as in [7]. In particular, we show that the class of IP loops is closed under such extensions (Theorem 10). We conclude with examples which illustrate that, in general, certain classes of loops (*i.e.* diassociative loops, power associative loops, flexible loops, etc.) are not closed under such extensions.

## 2. Semidirect products and Moufang loops

Throughout juxtaposition binds more tightly than an explicit $\cdot$ so that, for instance, $xy \cdot z$ means $(xy)z$. For $x \in Q$, where $Q$ is a loop, define the *right* and *left translations* by $x$ by, respectively, $R_x(y) = yx$ and $L_x(y) = xy$ for all $y \in Q$. The fact that these mappings are permutations of $Q$ follows easily from the definition of a loop. It is easy to see that $L_x^{-1}(y) = x\backslash y$ and $R_x^{-1}(y) = y/x$. We define the *multiplication group of* $Q$, $\mathrm{Mlt}(Q) = \langle L_x, R_x \mid \forall x \in Q\rangle$ and the *inner mapping group of* $Q$, $\mathrm{Mlt}(Q)_1 = \mathrm{Inn}(Q) = \{\theta \in \mathrm{Mlt}(Q) \mid \theta(1) = 1\}$, the stabilizer of $1 \in Q$.

A bijection $f : Q \to Q$ is a *semiautomorphism* of $Q$ if (i) $f(1) = 1$ and (ii) $f(x(yx)) = f(x)(f(y)f(x))$. The *semiautomorphism group of* $Q$, $\mathrm{SemiAut}(Q)$ is defined as the set of semiautomorphisms of $Q$ under composition. The following proposition will be helpful throughout.

**Proposition 1.** *Let $Q$ be a loop with two-sided inverses* (i.e. $1\backslash x = 1/x = x^{-1}$ *for all* $x \in Q$) *and* $f \in \mathrm{SemiAut}(Q)$. *Then for all* $x \in Q$, $f(x^{-1}) = (f(x))^{-1}$.

PROOF: Since $f$ is a semiautomorphism, $f(x) = f(x \cdot (x^{-1}x)) = f(x)(f(x^{-1})f(x))$. Canceling gives $f(x)^{-1} = f(x^{-1})$. □

*Moufang loops*, which are easily the most studied class of loops, are defined by any one of the following four equivalent identities:

$$(xy)(zx) = x(yz \cdot x) \qquad (xy)(zx) = (x \cdot yz)x$$

$$(xy \cdot x)z = x(y \cdot xz) \qquad (zx \cdot y)x = z(x \cdot yx).$$

In particular, Moufang loops are *diassociative* (*i.e.* every subloop generated by two elements is a group). If $Q$ is Moufang, then $\mathrm{Inn}(Q) \leq \mathrm{SemiAut}(Q)$ [1].

The following shows that if $G = N \rtimes_\phi H$ is Moufang, then $H$ acts on $N$ as a group of automorphisms.

**Lemma 2.** *Let $N$ be a loop and $H$ a group. Let $\phi : H \to \mathrm{Sym}(N)_1$ be a group homomorphism. If $G = N \rtimes_\phi H$ is Moufang, then $\phi(H) \subseteq \mathrm{Aut}(N)$.*

Proof: Since $G$ is Moufang, $(xy)(zx) = x((yz)x)$ for all $x, y, z \in G$. Therefore for any $h \in H$,

$$\begin{gathered}[(x, 1_H)(1, h)][(z, 1_H)(x, 1_H)] = (x, 1_H)[(1, h)(z, 1_H) \cdot (x, 1_H)] \qquad \Leftrightarrow \\ (x \cdot \phi(h)(zx), h) = (x \cdot \phi(h)(z)\phi(h)(x), h)\end{gathered}$$

Cancellation gives $\phi(h)(zx) = \phi(h)(z)\phi(h)(x)$. □

A loop $Q$ is an *inverse property* loop (or IP loop) if $x$ has a two-sided inverse for all $x \in Q$, and $(yx^{-1})x = y = x(x^{-1}y)$ holds for all $x, y \in Q$.

**Lemma 3.** *Let $N$ be an IP loop. Then there exists $f \in \mathrm{Aut}(N)$ such that*

$$(*) \qquad (xy)(zf(x)) = x(yz \cdot f(x))$$

*for all $x, y, z \in N$ if and only if $N$ is Moufang.*

Proof: Firstly, if $N$ is Moufang, then the trivial isomorphism satisfies $(*)$ for all $x, y, z \in N$. Conversely, let $f \in \mathrm{Aut}(N)$ such that $(*)$ holds for all $x, y, z \in N$. Note that setting $z = 1$ in $(*)$ gives

$$(2) \qquad x \cdot yf(x) = xy \cdot f(x).$$

Since $f$ is an automorphism and the above holds for all $x \in N$, replacing $x$ with $f^{-1}(x)$ yields

$$(3) \qquad f^{-1}(x) \cdot yx = f^{-1}(x)y \cdot x.$$

Now, compute

$$\begin{aligned}(x \cdot zf(x))f(y) &= [x \cdot (y \cdot y^{-1}z)f(x)]f(y) \overset{(*)}{=} [(xy)(y^{-1}z \cdot f(x))]f(y) \\ &= [(xy)(y^{-1}z \cdot f(x))](f(x)^{-1} \cdot f(xy)) \\ &\overset{(*)}{=} (xy)[(y^{-1}z \cdot f(x))f(x)^{-1} \cdot f(xy)]\end{aligned}$$

$$= (xy) \cdot (y^{-1}z)f(xy) \overset{(*)}{=} (xy \cdot y^{-1}) \cdot zf(xy)$$
$$= x \cdot zf(xy).$$

As before, replacing $x$ with $f^{-1}(x)$ yields

$$(f^{-1}(x) \cdot zx)f(y) = f^{-1}(x) \cdot (z \cdot xf(y)). \tag{4}$$

Hence, we have

$$\begin{aligned}(x \cdot f^{-1}(y)z)(yf(x)) &\overset{(*)}{=} x[(f^{-1}(y)z \cdot y)f(x)] \overset{(3)}{=} x[(f^{-1}(y) \cdot zy)f(x)] \\ &\overset{(4)}{=} x[f^{-1}(y) \cdot (z \cdot yf(x))].\end{aligned}$$

Multiplying by $f^{-1}(y)$ on the left gives

$$f^{-1}(y)[x \cdot f^{-1}(y)(z \cdot yf(x))] = f^{-1}(y)[(x \cdot f^{-1}(y)z)(yf(x))]. \tag{5}$$

Now, compute

$$\begin{aligned}f^{-1}(y)x \cdot z(yf(x)) &= f^{-1}(y)x \cdot zf(f^{-1}(y)x) \\ &= f^{-1}(y)x \cdot (x^{-1} \cdot xz)f(f^{-1}(y)x) \\ &\overset{(*)}{=} (f^{-1}(y)x \cdot x^{-1})(xz \cdot f(f^{-1}(y)x)) \\ &= f^{-1}(y)[xz \cdot yf(x)].\end{aligned}$$

Replacing $z$ with $f^{-1}(y)z$ gives

$$(f^{-1}(y)x) \cdot (f^{-1}(y)z) \cdot (yf(x)) = f^{-1}(y)[(x \cdot f^{-1}(y)z) \cdot (yf(x))]. \tag{6}$$

Combining (5) and (6), we have

$$\begin{aligned}f^{-1}(y)[x \cdot f^{-1}(y)(z \cdot yf(x))] &\overset{(5)}{=} f^{-1}(y)[(x \cdot f^{-1}(y)z) \cdot (yf(x))] \\ &\overset{(6)}{=} (f^{-1}(y)x)[f^{-1}(y)z \cdot yf(x)].\end{aligned}$$

Replacing $y$ with $f(y)$ yields

$$y \cdot x(y \cdot zf(yx)) = yx \cdot (yz \cdot f(yx)). \tag{7}$$

Finally, we have

$$\begin{aligned}(xy \cdot x)z &= (xy \cdot x) \cdot (zf(xy)^{-1})f(xy) \overset{(*)}{=} xy \cdot [(x \cdot zf(xy)^{-1})f(xy)] \\ &\overset{(7)}{=} x \cdot y(x \cdot (zf(xy)^{-1} \cdot f(xy))) \\ &= x(y \cdot xz).\end{aligned}$$

Therefore, $N$ is Moufang. $\square$

**Theorem 4.** *Let $N$ be a loop. Then $N$ is Moufang if and only if there exists some $f \in \mathrm{Aut}(N)$ which satisfies $(*)$ for all $x, y, z \in N$.*

Proof: As before, if $N$ is Moufang, then the trivial automorphism satisfies $(*)$ for all $x, y, z \in N$. Conversely, suppose there exists an $f \in \mathrm{Aut}(N)$ that satisfies $(*)$ for all $x, y, z \in N$. By Lemma 3, it is enough to show that $N$ is an IP loop. As before, we have

$$x \cdot yf(x) = xy \cdot f(x), \tag{2}$$

$$f^{-1}(x) \cdot yx = f^{-1}(x)y \cdot x. \tag{3}$$

Moreover, $f(x) = (x \cdot x\backslash 1)f(x) = x \cdot (x\backslash 1)f(x)$, implying $x\backslash f(x) = x\backslash 1 \cdot f(x)$. Applying $f^{-1}$ yields

$$f^{-1}(x)\backslash x = f^{-1}(x)\backslash 1 \cdot x.$$

We compute

$$\begin{aligned}\underbrace{(f^{-1}(x)\backslash x)}(x\backslash 1) &= [(f^{-1}(x)\backslash 1) \cdot x](x\backslash 1) = [(f^{-1}(x)\backslash 1) \cdot x] \cdot f(f^{-1}(x)\backslash 1)\\ &\overset{(2)}{=} (f^{-1}(x)\backslash 1)(x \cdot f(f^{-1}(x)\backslash 1)) = (f^{-1}(x)\backslash 1) \cdot x(x\backslash 1)\\ &= f^{-1}(x)\backslash 1.\end{aligned}$$

Finally, we compute

$$\begin{aligned}x &= f^{-1}(x)(f^{-1}(x)\backslash 1) \cdot x \overset{(3)}{=} f^{-1}(x)[\underbrace{(f^{-1}(x)\backslash 1)} \cdot x]\\ &= f^{-1}(x)[(f^{-1}(x)\backslash x)(x\backslash 1) \cdot x] = f^{-1}(x)[(f^{-1}(x)\backslash x)(x\backslash 1) \cdot f(f^{-1}(x))]\\ &\overset{(*)}{=} [f^{-1}(x)(f^{-1}(x)\backslash x)] \cdot [(x\backslash 1)(f(f^{-1}(x)))] = x \cdot (x\backslash 1)x.\end{aligned}$$

This immediately implies $1/x = x\backslash 1 = x^{-1}$.

To show that $N$ is IP, observe that

$$xy \cdot f(x) = x \cdot yf(x) = x[x^{-1}(x^{-1}\backslash y) \cdot f(x)] \overset{(*)}{=} xx^{-1} \cdot (x^{-1}\backslash y)f(x) = (x^{-1}\backslash y)f(x),$$

which implies $xy = x^{-1}\backslash y$. Hence, $x^{-1} \cdot xy = y$.

To show that $yx \cdot x^{-1} = y$, note that $y^{-1} \cdot yx = x$ implies $x/(yx) = y^{-1}$. Also note that $(*)$ is equivalent to $[x \cdot (yz \cdot f(x))]/(zf(x)) = xy$. Using these two identities combined with $x^{-1} \cdot xy = y$, we have

$$y^{-1} = f((xy)^{-1})/[yf((xy)^{-1})] = [(xy)^{-1} \cdot (xy \cdot f((xy)^{-1}))]/[yf((xy)^{-1})] = (xy)^{-1}x.$$

Hence, $xy \cdot y^{-1} = (xy)[(xy)^{-1}x] = x$. □

Theorem 4 yields an equivalent definition for a loop to be Moufang in terms of the behavior of one of its automorphisms. In the same spirit, the next theorem gives a necessary and sufficient condition for a semidirect product of a loop and group to be Moufang.

**Theorem 5.** *Let $N$ be a loop and $H$ a group. Let $\phi : H \to \mathrm{Sym}(N)_1$ be a group homomorphism. Then $G = N \rtimes_\phi H$ is a Moufang loop if and only if $\phi(H) \subseteq \mathrm{Aut}(N)$ and $\phi(h)$ satisfies $(*)$ for all $h \in H$ and for all $x, y, z \in N$.*

Proof: Suppose $\phi(H) \subseteq \mathrm{Aut}(N)$ and $\phi(h)$ satisfies $(*)$ for all $h \in H$ and for all $x, y, z \in N$, and let $(x, g), (y, h), (z, k) \in G$. Then, by hypothesis,

$$[x\phi(g)(y)] \cdot [\phi(gh)(z)\phi(ghk)(x)] \overset{(*)}{=} x[\phi(g)(y)\phi(gh)(z) \cdot \phi(ghk)(x)].$$

Hence,

$$([x\phi(g)(y)] \cdot [\phi(gh)(z)\phi(ghk)(x)], ghkg) = (x[\phi(g)(y)\phi(gh)(z) \cdot \phi(ghk)(x)], ghkg),$$

and thus

$$[(x, g)(y, h)][(z, k)(x, g)] = (x, g) \cdot [(y, h)(z, k) \cdot (x, g)],$$

which shows that $G$ is Moufang.

Conversely, if $G$ is Moufang, then by Lemma 2 we have $\phi(H) \subseteq \mathrm{Aut}(N)$. Therefore

$$[(x, 1_H)(y, 1_H)][(z, h)(x, 1_H)] = (x, 1_H)[(y, 1_H)(z, h) \cdot (x, 1_H)],$$

which is equivalent to

$$(xy \cdot z\phi(h)(x), h) = (x(yz \cdot \phi(h)(x)), h).$$

Hence, $\phi(h)$ satisfies $(*)$ for all $h \in H$, for all $x, y, z \in N$. □

Note that if $N =$ `MoufangLoop(12,1)`, then it can be verified that there are no nontrivial automorphisms $f \in \mathrm{Aut}(N)$ which satisfy $(*)$ for all $x, y, z \in N$. Hence, for $H$ a group and $\phi : H \to \mathrm{Aut}(N)$ a nontrivial group homomorphism, then $G = N \rtimes_\phi H$ is not Moufang, which shows that the converse of Lemma 2 is false. The next example illustrates that Theorem 5 can hold nontrivially.

*Example* 6. Let $N =$ `MoufangLoop(16,1)` and $H$ a group of order 2. Let $\phi : H \to \mathrm{Aut}(N)$ be a group homomorphism with $\phi(H) = \langle$`(2,6)(3,7)(10,14)(11,15)`$\rangle$. It can be verified that $\phi(h)$ satisfies $(*)$ for all $h \in H$ and for all $x, y, z \in N$. Then $G = N \rtimes_\phi H$ is a Moufang loop of order 32 (`MoufangLoop(32,18)`).

We end this section by showing that the class of IP loops is closed under semidirect products with groups acting as automorphisms.

**Proposition 7.** *Let $N$ be a loop and $H$ a group. Let $\phi : H \to \mathrm{Aut}(N)$ be a group homomorphism. Then $G = N \rtimes_\phi H$ is an IP loop if and only if $N$ is an IP loop.*

Proof: Clearly, if $G$ is IP, then $N$ is IP. Conversely, if $N$ is IP, $x^{-1}$ and $g^{-1}$ exist for all $x \in N$ and $g \in G$. It is easy to verify that $(x,g)^{-1} = (\phi(g^{-1})(x^{-1}), g^{-1})$. To show $G$ is IP, we compute

$$\begin{aligned}
(x,g)^{-1}[(x,g)(y,h)] &= (\phi(g^{-1})(x^{-1}), g^{-1})(x \cdot \phi(g)(y), gh) \\
&= (\phi(g^{-1})(x^{-1}) \cdot \phi(g^{-1})[x \cdot \phi(g)(y)], g^{-1}gh) \\
&= ((\phi(g^{-1})(x))^{-1}[\phi(g^{-1})(x) \cdot y], h) \\
&= (y,h). \\
[(y,h)(x,g)](x,g)^{-1} &= (y \cdot \phi(h)(x), hg)(\phi(g^{-1})(x^{-1}), g^{-1}) \\
&= ([y \cdot \phi(h)(x)]\phi(hg)(\phi(g^{-1})(x^{-1})), hgg^{-1}) \\
&= ([y \cdot \phi(h)(x)](\phi(h)(x))^{-1}, h) \\
&= (y,h).
\end{aligned}$$

□

## 3. Extensions and inverse property loops

We now turn our attention to extensions of loops as considered in [7].

**Theorem 8** ([7]). *Suppose $G = NH$ is a Moufang loop, where $N$ is a normal subloop of $G$ and $H = \langle u \rangle$ is a finite cyclic subgroup of $G$ whose order is coprime to 3. Then for any $xu^m, yu^n \in G$ where $x, y \in N$, we have*

$$(xu^m)(yu^n) = f^{\frac{2m+n}{3}}(f^{\frac{-2m-n}{3}}(x) f^{\frac{m-n}{3}}(y))u^{m+n}, \tag{8}$$

*where*

$$f : N \to N \qquad x \to uxu^{-1}$$

*is a semiautomorphism of $N$. Moreover, $G$ is a group if and only if $N$ is a group and $f$ is an automorphism of $N$.*

Note that if $H = \langle u \rangle \simeq \mathbb{Z}_2$, then the multiplication in $G$ given by (8) is

$$xu \cdot y = xf(y) \cdot u, \qquad x \cdot yu = f(f(x)f(y)) \cdot u, \qquad xu \cdot yu = f(f(x)y),$$

for all $x, y \in N$. Hence, multiplication in such a loop $G$ is easily seen to be equivalent to the multiplication given in [4, Lemma 1, p. 21]. These types of extensions have been well-studied, see [3], [4], [5].

*Example* 9. A loop $Q$ is said to be a *semiautomorphic, inverse property loop* (or just semiautomorphic IP loop) if

(1) $Q$ is flexible; that is, $(xy)x = x(yx)$ for all $x, y \in Q$;
(2) $Q$ is an IP loop; and
(3) every inner mapping is a semiautomorphism.

It is known that Moufang loops are semiautomorphic IP loops and that semiautomorphic IP loops are diassociative loops [9]. Let $N$ be a semiautomorphic IP loop and $H = \langle u \rangle$ have order 2, and let $\phi : H \to \mathrm{SemiAut}(N)$ be the group homomorphism such that $\phi(u)(x) = x^{-1}$ for all $x \in N$. Then $G = (N, H, \phi)$ with multiplication extended by (8) is a semiautomorphic IP loop (see [5]), but not necessarily Moufang, since it is shown in [4] that $G$ is Moufang if and only if $N$ is a group.

Here, we consider the *external* viewpoint. That is, the external extension of a cyclic group $H = \langle u \rangle$ by a loop $N$ with $\phi : H \to \mathrm{SemiAut}(N)$ a group homomorphism. We extend the multiplication from $N$ to $G = (N, H, \phi)$ as

$$(x, u^m)(y, u^n) = (f^{\frac{2m+n}{3}}(f^{\frac{-2m-n}{3}}(x) f^{\frac{m-n}{3}}(y)), u^{m+n}), \tag{9}$$

where $f(x) = \phi(u)(x)$. As the previous example shows, if $N$ is Moufang, $G$ need not be Moufang. However, as the next theorem illustrates, the class of IP loops is closed under such extensions.

**Theorem 10.** *Let $N$ be a loop, $H = \langle u \rangle$ a cyclic group, and let $\phi : H \to \mathrm{SemiAut}(N)$ be a group homomorphism. If $|\phi(u)|$ is coprime to 3, define an extension $G = (N, H, \phi)$ with multiplication given by*

$$(x, u^m)(y, u^n) = (f^{\frac{2m+n}{3}}(f^{\frac{-2m-n}{3}}(x) f^{\frac{m-n}{3}}(y)), u^{m+n}),$$

*where $f(x) = \phi(u)(x)$. Then $G$ is an IP loop if and only if $N$ is an IP loop.*

PROOF: Clearly, $N$ is an IP loop if $G$ is an IP loop. Now, suppose $N$ is an IP loop. Note that since the order of $f$ in $\mathrm{SemiAut}(N)$ is coprime to 3, $f$ must have a unique cubed root, denoted by $f^{\frac{1}{3}}$. It is clear that $G$ is a loop with identity $(1, u^0)$. For $(x, u^m) \in G$, consider the element $(f^{-m}(x^{-1}), u^{-m})$. Note that this is well-defined, since $N$ is an IP loop and $H$ is a group. Then,

$$\begin{aligned}
(x, u^m)(f^{-m}(x^{-1}), u^{-m}) &= (f^{\frac{2m-m}{3}}(f^{\frac{-2m+m}{3}}(x) f^{\frac{m+m}{3}}(f^{-m}(x^{-1}))), u^0) \\
&= (f^{\frac{m}{3}}(f^{\frac{-m}{3}}(x) f^{\frac{-m}{3}}(x^{-1})), u^0) \\
&= (f^{\frac{m}{3}}(f^{\frac{-m}{3}}(x)(f^{\frac{-m}{3}}(x))^{-1}), u^0) \\
&= (f^{\frac{m}{3}}(1), u^0) \\
&= (1, u^0) \\
&= (f^{\frac{-m}{3}}(1), u^0) \\
&= (f^{\frac{-m}{3}}((f^{\frac{-2m}{3}}(x))^{-1} f^{\frac{-2m}{3}}(x)), u^0) \\
&= (f^{\frac{-m}{3}}(f^{\frac{-2m}{3}}(x^{-1}) f^{\frac{-2m}{3}}(x)), u^0) \\
&= (f^{\frac{-2m+m}{3}}(f^{\frac{2m-m}{3}}(f^{-m}(x^{-1}) f^{\frac{-m-m}{3}}(x))), u^0) \\
&= (f^{-m}(x^{-1}), u^{-m})(x, u^m).
\end{aligned}$$

Hence, $G$ has two sided inverses for all $(x, u^m) \in G$, namely

$$(x, u^m)^{-1} = (f^{-m}(x^{-1}), u^{-m}).$$

To show that $G$ is an IP loop, we compute

$$
\begin{aligned}
&(x,u^m)[(f^{-m}(x^{-1}),u^{-m})(y,u^n)]\\
&\quad= (x,u^m)(f^{\frac{-2m+n}{3}}(f^{\frac{2m-n}{3}}(f^{-m}(x^{-1}))f^{\frac{-m-n}{3}}(y)),u^{-m+n})\\
&\quad= (f^{\frac{m+n}{3}}(f^{\frac{-m-n}{3}}(x)[f^{\frac{2m-n}{3}}(f^{\frac{-2m+n}{3}}(f^{\frac{2m-n}{3}}(f^{-m}(x^{-1}))f^{\frac{-m-n}{3}}(y)))]),u^n)\\
&\quad= (f^{\frac{m+n}{3}}(f^{\frac{-m-n}{3}}(x)[(f^{\frac{2m-n}{3}}(f^{-m}(x^{-1}))f^{\frac{-m-n}{3}}(y))]),u^n)\\
&\quad= (f^{\frac{m+n}{3}}(f^{\frac{-m-n}{3}}(x)[f^{\frac{-m-n}{3}}(x^{-1})f^{\frac{-m-n}{3}}(y)]),u^n)\\
&\quad= (f^{\frac{m+n}{3}}(f^{\frac{-m-n}{3}}(x)[(f^{\frac{-m-n}{3}}(x))^{-1}f^{\frac{-m-n}{3}}(y)]),u^n)\\
&\quad= (f^{\frac{m+n}{3}}(f^{\frac{-m-n}{3}}(y)),u^n)\\
&\quad= (y,u^n).
\end{aligned}
$$

$$
\begin{aligned}
&[(y,u^n)(f^{-m}(x^{-1}),u^{-m})](x,u^m)\\
&\quad= (f^{\frac{2n-m}{3}}[f^{\frac{-2n+m}{3}}(y)f^{\frac{n+m}{3}}(f^{-m}(x^{-1}))],u^{n-m})(x,u^m)\\
&\quad= (f^{\frac{2n-m}{3}}[f^{\frac{-2n+m}{3}}(f^{\frac{2n-m}{3}}[f^{\frac{-2n+m}{3}}(y)f^{\frac{n+m}{3}}(f^{-m}(x^{-1}))])f^{\frac{n-2m}{3}}(x)],u^n)\\
&\quad= (f^{\frac{2n-m}{3}}(([f^{\frac{-2n+m}{3}}(y)f^{\frac{n+m}{3}}(f^{-m}(x^{-1}))])f^{\frac{n-2m}{3}}(x)),u^n)\\
&\quad= (f^{\frac{2n-m}{3}}([f^{\frac{-2n+m}{3}}(y)f^{\frac{n-2m}{3}}(x^{-1})]f^{\frac{n-2m}{3}}(x)),u^n)\\
&\quad= (f^{\frac{2n-m}{3}}([f^{\frac{-2n+m}{3}}(y)(f^{\frac{n-2m}{3}}(x))^{-1}]f^{\frac{n-2m}{3}}(x)),u^n)\\
&\quad= (f^{\frac{2n-m}{3}}(f^{\frac{-2n+m}{3}}(y),u^n))\\
&\quad= (y,u^n).
\end{aligned}
$$

□

The next two examples illustrate that if certain stronger conditions are imposed on $N$, then an extension as in Theorem 10 need not inherit a similar structure.

*Example* 11. Let $N$ be the quaternion group of order 8 (GAP `SmallGroup(8,4)`) and $H$ a group of order 2. Let $\phi : H \to \mathrm{SemiAut}(N)$ a group homomorphism with $\phi(H) = \langle$`(2,5)(6,8)`$\rangle$. Then the extension $G = (N, H, \phi)$ as in Theorem 10 is a power-associative (the subloop generated by $x$ is associative for all $x \in G$), IP loop. It can be verified that $G$ is neither flexible nor diassociative, although it is clear that $N$ has both properties.

*Steiner loops*, which arise from Steiner triple systems in combinatorics, are loops satisfying the identities $xy = yx$, $x(yx) = y$. In particular, Steiner loops are semiautomorphic IP loops.

*Example* 12. Let $N$ be a Steiner loop of order 8 (`SteinerLoop(8,1)`) and $H = \langle u\rangle$ a group of order 4. Let $\phi : H \to \mathrm{SemiAut}(N)$ be a group homomorphism with $\phi(H) = \langle$`(2,4,8,3)`$\rangle$. Then the extension $G = (N, H, \phi)$ as in Theorem 10 is an

IP loop of order 32 that is neither power-associative nor flexible, although it is clear that $N$ has both properties.

**Corollary 13.** *Let $N$ be a Moufang loop, $H = \langle u \rangle$ a cyclic group, and let $\phi : H \to \mathrm{SemiAut}(N)$ be a group homomorphism. If $|\phi(u)|$ is coprime to 3, define an extension $G = (N, H, \phi)$ with multiplication given by*

$$(x, u^m)(y, u^n) = (f^{\frac{2m+n}{3}}(f^{\frac{-2m-n}{3}}(x)f^{\frac{m-n}{3}}(y)), u^{m+n}),$$

*where $f(x) = \phi(u)(x)$. Then $G$ is an IP loop.*

Proof: Since Moufang loops are IP loops, the desired result follows from Theorem 10. $\square$

**Acknowledgment.** We are grateful to the anonymous referee for many helpful comments and suggestions. Some investigations in this paper were assisted by the automated deduction tool, Prover9, and the finite model builder, Mace4, both developed by McCune [10]. Similarly, all presented examples were found and verified using the GAP system [6] together with the LOOPS package [11].

## References

[1] Bruck R.H., *Pseudo-automorphisms and Moufang loops*, Proc. Amer. Math. Soc. **2** (1952), 66–72.
[2] Bruck R.H., *A Survey of Binary Systems*, Springer, Berlin, 1971.
[3] Chein O., *Moufang loops of small order. I*, Trans. Amer. Math. Soc. **188** (1974), 31–51.
[4] Chein O., *Moufang loops of small order*, Mem. Amer. Math. Soc. **13** (1978), no. 197.
[5] Greer M., *Semiautomorphic inverse property loops*, submitted.
[6] The GAP Group, *Groups, Algorithms, and Programming*, `http://www.gap-system.org` (2008).
[7] Gagola S. III, *Cyclic extensions of Moufang loops induced by semiautomorphisms*, J. Algebra Appl. **13** (2014), no. 4, 1350128.
[8] Kinyon M.K., Jones O., *Loops and semidirect products*, Comm. Algebra **28** (2000), 4137–4164.
[9] Kinyon M.K., Kunen K., Phillips J.D., *A generalization of Moufang and Steiner loops*, Algebra Universalis **48** (2002), 81–101.
[10] McCune W.W., *Prover9, Mace4*, `http://www.cs.unm.edu/~mccune/prover9/` (2009).
[11] Nagy G.P., Vojtěchovský P., *Loops: Computing with quasigroups and loops*, `http://www.math.du.edu/loops` (2008).
[12] Pflugfelder H.O., *Quasigroups and Loops: Introduction*, Sigma Series in Pure Mathematics, 7, Heldermann, Berlin, 1990.

Department of Mathematics, One Harrison Plaza, University of North Alabama, Florence AL 35632, USA

*E-mail:* mgreer@una.edu
lraney@una.edu

(*Received* November 15, 2013)