Mapping theorems on countable tightness and a question of F. Siwiec

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Abstract. In this paper ss-quotient maps and ssq-spaces are introduced. It is shown that (1) countable tightness is characterized by ss-quotient maps and quotient maps; (2) a space has countable tightness if and only if it is a countably bi-quotient image of a locally countable space, which gives an answer for a question posed by F. Siwiec in 1975; (3) ssq-spaces are characterized as the ss-quotient images of metric spaces; (4) assuming $2^\omega < 2^{\omega_1}$, a compact T_2-space is an ssq-space if and only if every countably compact subset is strongly sequentially closed, which improves some results about sequential spaces obtained by M. Ismail and P. Nyikos in 1980.

Keywords: countable tightness; strongly sequentially closed sets; sequentially closed sets; quotient maps; countably bi-quotient maps; locally countable spaces

Classification: 54B15, 54D55, 54E40

1. Introduction

Topologists obtained many interesting characterizations of spaces by mappings, in particular some images of metric spaces [13]. For example, a space is a sequential space if and only if it is a quotient image of a metric space [7]. E. Michael [19] gave a systematic study for certain quotient images of metric spaces. F. Siwiec [27] gave a survey about first-countable spaces, and posed some questions about the images of metric spaces. One of the most basic and natural generalizations of first countability is countable tightness. Every sequential space has countable tightness, and countable tightness can be characterized as a quotient image of a locally countable space as follows.

Theorem 1.1 ([27])**.** *The following are equivalent for a space X.*

(1) *X has countable tightness.*
(2) *X is a pseudo-open image of the topological sum of some countable spaces.*
(3) *X is a quotient image of the topological sum of some countable spaces.*

Recently, some questions of Siwiec in [27] caused attention once again [14], [16], [17], [18], [21], [25]. Every open map is countably bi-quotient, and every

The project is supported by the NSFC (No. 11171162, 11201414).
DOI 10.14712/1213-7243.014.403

countably bi-quotient map is pseudo-open. The following question was posed by F. Siwiec in [27, p. 33].

Question 1.2. *How can one characterize the image of the topological sum of some countable spaces under countably bi-quotient maps?*

A key to the question above is the characterization of countable tightness by certain maps. W.C. Hong [11] introduced the concepts of strongly sequentially closed sets, and the spaces of countable tightness can be characterized as follows.

Theorem 1.3 ([11])**.** *The following are equivalent for a space X.*

(1) *X has countable tightness.*
(2) *Every strongly sequentially closed set in X is closed.*
(3) *Every strongly sequentially open set in X is open.*

Maps related to quotient maps and sequentially quotient maps are introduced in this paper; we call them *ss-quotient maps* (see Definition 2.2). Countable tightness is characterized by these maps. On the other hand, it is obvious that in any space closed subsets are strongly sequentially closed, and strongly sequentially closed subsets are sequentially closed. The following question is natural.

Question 1.4. *How can one characterize by maps the spaces in which each sequentially closed subset is strongly sequentially closed?*

It is proved in this paper that the spaces in which each sequentially closed subset is strongly sequentially closed can be characterized as the ss-quotient images of metric spaces, so these spaces are called *ssq-spaces* in this paper (see Definition 3.1).

In this paper all spaces are topological spaces, and they are not required to satisfy any axioms of separation. All maps are continuous and onto. Readers may refer to [6] for unstated notations and terminologies.

2. Countable tightness

In this section countable tightness is characterized by certain maps, and Question 1.2 is answered.

Definition 2.1. Let X be a space and $A \subset X$.

(1) A is a *sequentially closed subset* [7] in X if no sequence of points of A converges to a point not in A; A is *sequentially open* [7] in X if $X - A$ is sequentially closed.

(2) A is a *strongly sequentially closed subset* [11] in X if no sequence of points of A accumulates to a point not in A; A is *strongly sequentially open* [11] in X if $X - A$ is strongly sequentially closed.

Definition 2.2. Let $f : X \to Y$ be a map.

(1) f is an *almost-open map* if for every $y \in Y$ there is $x \in f^{-1}(y)$ such that $f(U)$ is a neighborhood of y in Y when U is a neighborhood of x in X.

(2) f is a *quotient map* [6] if, whenever $F \subset Y$ and $f^{-1}(F)$ is closed in X, then F is closed in Y.

(3) f is a *sequentially quotient map* [4] if, whenever $F \subset Y$ and $f^{-1}(F)$ is sequentially closed in X, then F is sequentially closed in Y.

(4) f is an *ss-quotient map* if, whenever $F \subset Y$ and $f^{-1}(F)$ is strongly sequentially closed in X, then F is strongly sequentially closed in Y.

Open maps are not necessarily ss-quotient maps, and ss-quotient maps are not necessarily sequentially quotient maps (see Examples 5.4 and 5.1).

Definition 2.3. Let X be a space.

(1) X is *determined by countable subsets* [20] or has *countable tightness* [19] if for each subset A of X and each $x \in \overline{A}$, there exists a countable subset C of A such that $x \in \overline{C}$.

(2) X is a *sequential space* [7] if each sequentially closed subset of X is closed.

Lemma 2.4 ([19])**.** (1) *Every sequential space has countable tightness.*

(2) *Spaces of countable tightness are preserved by quotient maps.*

Lemma 2.5. *Every space is an ss-quotient image of the topological sum of some countable spaces.*

Proof: Let X be a space and let $\{C_\alpha : \alpha \in \Lambda\}$ be the family of all countable subsets in X. For each $\alpha \in \Lambda$, put $M_\alpha = C_\alpha \times \{\alpha\}$ and define $f_\alpha : M_\alpha \to C_\alpha$ by $f_\alpha(x, \alpha) = x$ for each $x \in C_\alpha$. Then M_α is countable and f_α is homeomorphic. Set the topological sum $M = \bigoplus_{\alpha \in \Lambda} M_\alpha$ and define $f : M \to X$ by $f|_{M_\alpha} = f_\alpha$ for each $\alpha \in \Lambda$. It is obvious that f is continuous and onto. We will show that f is an ss-quotient map.

Let $H \subset X$ and $f^{-1}(H)$ be strongly sequentially closed in M. If a sequence $\{x_n\}$ in H has an accumulation point x in X, put $C = \{x_n : n \in \mathbb{N}\}$. Then C is countable, and there exists $\alpha \in \Lambda$ such that $C_\alpha = \{x\} \cup C$. It follows that (x, α) is an accumulation point of the sequence $\{(x_n, \alpha)\}$ in M and each $(x_n, \alpha) \in f^{-1}(H)$, thus $(x, \alpha) \in f^{-1}(H)$, since $f^{-1}(H)$ is strongly sequentially closed. So, $x \in H$. Hence, H is strongly sequentially closed in X. Therefore, f is an ss-quotient map. □

Some relations between ss-quotient maps and quotient maps give the characterization of countable tightness as follows. A space X is called *locally countable* if for each $x \in X$, there is a neighborhood U of x in X such that U is a countable set. Every locally countable space has countable tightness.

Theorem 2.6. *The following are equivalent for a space X.*

(1) *X has countable tightness.*
(2) *Every ss-quotient map onto X is quotient.*
(3) *Every quotient map of X is ss-quotient.*
(4) *Every almost-open map of X is ss-quotient.*

Proof: (1) $\Rightarrow$ (2). Let $f : M \to X$ be an ss-quotient map, where X has countable tightness. If $F \subset X$ and $f^{-1}(F)$ is closed in M, then $f^{-1}(F)$ is strongly sequentially closed in M. Since f is ss-quotient, F is strongly sequentially closed in X. Since X has countable tightness, F is closed in X by Theorem 1.3. Hence, f is quotient.

(2) $\Rightarrow$ (3). Suppose that every ss-quotient map onto X is quotient. Let $g : X \to Y$ be a quotient map. By Lemma 2.5, there are a locally countable space M and an ss-quotient map $f : M \to X$. Then f is quotient, and M has countable tightness. Thus X has countable tightness by Lemma 2.4. If $F \subset Y$ and $g^{-1}(F)$ is strongly sequentially closed in X, then $g^{-1}(F)$ is closed in X by Theorem 1.3. Thus F is closed in Y, and it is strongly sequentially closed in Y. Hence, g is ss-quotient.

(3) $\Rightarrow$ (4) is obvious. Next, we show that (4) $\Rightarrow$ (1). If X is not of countable tightness, there is a non-closed subset H in X which is strongly sequentially closed by Theorem 1.3. Define a function $f : X \to \{0,1\}$ as follows: Put $f(H) = \{0\}$, and $f(X - H) = \{1\}$. Let $Y = \{0,1\}$ have the quotient topology induced by f. Then f is an almost-open map. In fact, since H is not closed, there is a point $x_1 \in \overline{H} - H$. Then $f(U_1) = \{0,1\}$ for each open neighborhood U_1 of x_1 in X. If H is open in X, then $\{0\}$ is open in Y, thus $f(U)$ is open in Y for each open neighborhood U of $x_0 \in H$. If H is not open in X, there is a point $x_0 \in H - \mathrm{int}(H)$, then $f(U_0) = \{0,1\}$ for each open neighborhood U_0 of x_0 in X. Thus f is almost-open.

Since 1 is an accumulation point of the set $\{0\}$, $\{0\}$ is not strongly sequentially closed in Y. Since $f^{-1}(\{0\}) = H$ is strongly sequentially closed, f is not ss-quotient. □

Next, more maps are related to countable tightness and locally countable spaces. Recall some of them.

Definition 2.7. Let $f : X \to Y$ be a map.

(1) f is *pseudo-open* [1] if whenever $y \in Y$ and $f^{-1}(y) \subset U$ with U being open in X, then $f(U)$ is a neighborhood of y in Y.

(2) f is *countably bi-quotient* [26] if for each $y \in Y$ and for each countable cover $\mathcal{U}$ of $f^{-1}(y)$ by open subsets of X there exists some finite family $\mathcal{U}'$ of $\mathcal{U}$ such that $y \in \mathrm{int}(f(\bigcup \mathcal{U}'))$.

(3) f is *strictly countably bi-quotient* [16] or a *w-map* [29] if for each $y \in Y$ and for each countable cover $\mathcal{U}$ of $f^{-1}(y)$ by open subsets of X there exists $U \in \mathcal{U}$ such that $y \in \mathrm{int}(f(U))$.

It is obvious that almost-open maps $\Rightarrow$ strictly countably bi-quotient maps $\Rightarrow$ countably bi-quotient maps $\Rightarrow$ pseudo-open maps $\Rightarrow$ quotient maps.

Theorem 2.8. *The following are equivalent for a space X.*

(1) *X has countable tightness.*
(2) *X is a strictly countably bi-quotient image of the topological sum of some countable spaces.*
(3) *X is a strictly countably bi-quotient image of a locally countable space.*
(4) *X is a countably bi-quotient image of a locally countable space.*

(5) *X is a pseudo-open image of a locally countable space.*
(6) *X is a quotient image of a locally countable space.*

Proof: By Lemma 2.4 and Definition 2.7, it is enough to prove that (1) $\Rightarrow$ (2). Suppose a space X has countable tightness. Let $\{C_\alpha : \alpha \in \Lambda\}$ be the family of all countable subsets in X. By Lemma 2.5, there are the topological sum $M = \bigoplus_{\alpha\in\Lambda} M_\alpha$ and the ss-quotient map $f : M \to X$. Then f is strictly countably bi-quotient.

In fact, let $x \in X$ and $\{U_n : n \in \mathbb{N}\}$ be a cover of $f^{-1}(x)$ by open subsets of M. If $x \notin \mathrm{int}(f(U_n))$ for each $n \in \mathbb{N}$, then $x \in \overline{X - f(U_n)}$, and there exists a countable subset C_n of $X - f(U_n)$ such that $x \in \overline{C_n}$. Put $C = \{x\} \cup \bigcup_{n\in\mathbb{N}} C_n$. Then C is countable, and $C = C_\alpha$ for some $\alpha \in \Lambda$, and $x \in \overline{C_n} \subset \overline{C_\alpha - f(U_n)}$ for each $n \in \mathbb{N}$. Since $(x,\alpha) \in f^{-1}(x) \subset \bigcup_{n\in\mathbb{N}} U_n$, $(x,\alpha) \in U_k$ for some $k \in \mathbb{N}$. Then in X we have

$$x \in \overline{C_\alpha - f(U_k)} \cap C_\alpha \subset \overline{C_\alpha - f_\alpha(U_k \cap M_\alpha)} \cap C_\alpha = \mathrm{cl}_{C_\alpha}(C_\alpha - f_\alpha(U_k \cap M_\alpha)).$$

Therefore, $(x,\alpha) \in \overline{M_\alpha - U_k \cap M_\alpha}$ in M, thus $U_k \cap (M_\alpha - U_k) \neq \emptyset$, a contradiction. □

Remark 2.9. (1) The equivalences (1) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) in Theorem 2.8 were obtained in [27].

(2) Theorem 2.8 gives a positive answer for Question 1.2.

(3) A map $f : X \to Y$ is *bi-quotient* [19] if for each $y \in Y$ and for each cover $\mathcal{U}$ of $f^{-1}(y)$ by open subsets of X there exists some finite family $\mathcal{U}'$ of $\mathcal{U}$ such that $y \in \mathrm{int}(f(\bigcup \mathcal{U}'))$. Every bi-quotient map is countably bi-quotient. It is easy to see that the following are equivalent for a space X.

(a) X is a locally countable space.
(b) X is an open image of the topological sum of some countable spaces.
(c) X is an open image of a locally countable space.
(d) X is an almost-open image of a locally countable space.
(e) X is a bi-quotient image of a locally countable space.

Example 2.10. There exists an ss-quotient map $f : M \to X$ such that f is not pseudo-open, where M is a countable metric space and X has countable tightness.

Put $X = \{0\} \cup \mathbb{N} \cup (\mathbb{N} \times \mathbb{N})$. Define a topology for X as follows: each point in $\mathbb{N} \times \mathbb{N}$ is isolated; $U \subset X$ is a neighborhood of $n \in \mathbb{N}$ in X if and only if $V(n,m) = \{n\} \cup \{(n,k) \in \mathbb{N} \times \mathbb{N} : k \geq m\} \subset U$ for some $m \in \mathbb{N}$; $U \subset X$ is a neighborhood of 0 in X if and only if $\{0\} \cup \bigcup_{n\geq i} V(n, m_n) \subset U$ for some $i, m_n \in \mathbb{N}$. The set X endowed with this topology is called *Arens' space* S_2 [9, Example 9.10].

There exist a countable metric space M and a quotient map $f : M \to X$ such that f is not pseudo-open [9, Example 9.10]. Since X is countable, X has countable tightness. Finally, f is ss-quotient by Theorem 2.6.

3. *ssq*-spaces

In this section we will answer Question 1.4, and characterize the *ss*-quotient images of metric spaces. The following concept is introduced.

Definition 3.1. A space X is an *ssq-space* if and only if each sequentially closed subset of X is strongly sequentially closed.

By Theorem 1.3 and Definition 2.3 the following are obvious.

Proposition 3.2. *A space is sequential if and only if it is an ssq-space with countable tightness.*

Proposition 3.3. *Let $f : X \to Y$ be a map.*

(1) *If f is an ss-quotient map and X is an ssq-space, then f is sequentially quotient and Y is an ssq-space.*

(2) *If f is a sequentially quotient map and Y is an ssq-space, then f is ss-quotient.*

Proof: (1) If $G \subset Y$ and $f^{-1}(G)$ is sequentially closed in X, then $f^{-1}(G)$ is strongly sequentially closed in X because X is an *ssq*-space. Since f is an *ss*-quotient map, G is strongly sequentially closed in Y, then G is sequentially closed in Y. Hence, f is a sequentially quotient map.

If H is sequentially closed in Y, then $f^{-1}(H)$ is sequentially closed in X because f is continuous. Since X is an *ssq*-space, $f^{-1}(H)$ is strongly sequentially closed in X, then H is strongly sequentially closed in Y because f is an *ss*-quotient map, thus Y is an *ssq*-space.

(2) If $H \subset Y$ and $f^{-1}(H)$ is strongly sequential closed in X, then $f^{-1}(H)$ is sequentially closed in X. Since f is sequentially quotient, H is sequentially closed in Y. Since Y is an *ssq*-space, H is strongly sequentially closed in Y, then f is *ss*-quotient. □

Corollary 3.4. (1) *ssq-spaces are preserved by ss-quotient maps.*

(2) *Every ss-quotient map of an ssq-space is sequentially quotient.*

Lemma 3.5 ([15]). *Every space is a sequentially quotient image of a locally compact metric space.*

Theorem 3.6. *The following are equivalent for a space X.*

(1) *X is an ssq-space.*
(2) *Every sequentially quotient map onto X is an ss-quotient map.*
(3) *X is an image of a locally compact metric space under an ss-quotient map.*
(4) *X is an image of a metric space under an ss-quotient map.*

Proof: (1) ⇒ (2). Let $f : M \to X$ be a sequentially quotient map, where X is an *ssq*-space. By Proposition 3.3, f is an *ss*-quotient map.

(2) ⇒ (3). By Lemma 3.5, X is a sequentially quotient image of a locally compact metric space. The map is an *ss*-quotient map by (2).

(3) ⇒ (4) is obvious, and (4) ⇒ (1) follows from Corollary 3.4. □

Question 3.7. *Are ssq-spaces preserved by quotient maps?*

A space X is *Fréchet* [7] if, whenever $x \in \overline{A} \subset X$ there is a sequence $\{x_n\}$ in A such that $\{x_n\}$ converges to x in X.

Every Fréchet space is sequential, and every Fréchet space is preserved by a pseudo-open map [19]. Next, a functional characterization of Fréchet spaces is given using ss-quotient maps.

Theorem 3.8. *A T_2-space X is Fréchet if and only if every ss-quotient map onto X is pseudo-open.*

Proof: Suppose that X is a T_2 and Fréchet space. Let $f : M \to X$ be an ss-quotient map. By Theorem 2.6, f is quotient. Since X is T_2 and Fréchet, f is pseudo-open [1].

Conversely, suppose that every ss-quotient map onto X is pseudo-open. By Theorem 2.6, X has countable tightness. By Lemma 3.5, there exist a locally compact metric space M and a sequentially quotient map $f : M \to X$. By Theorem 3.6, f is ss-quotient. Thus f is pseudo-open and X is Fréchet. □

Example 3.9. There are a first-countable T_1-space X and an ss-quotient map $f : M \to X$ such that f is not pseudo-open.

Let X be the set $\mathbb{N}$ endowed with the finite-complement topology. Then X is a first-countable T_1-space. Take $X_0 = X - \{0\}$ and $X_1 = \{2n : n \in \mathbb{N}\}$ as the subspaces of X. Let $M = X_0 \oplus X_1$ and $f : M \to X$ be the obvious map. It is easy to check that f is a non-pseudo-open, quotient map. But f is ss-quotient by Theorem 2.6.

Recall that a class of maps is said to be *hereditary* [1], [4] if whenever $f : X \to Y$ is in the class, then for each subspace H of Y, the restriction of f to $f^{-1}(H)$ is in the class. Pseudo-open maps are precisely *hereditarily quotient* [1]. Sequentially quotient maps are *hereditarily sequentially quotient* [4].

Example 3.10. We show that ss-quotient maps are not hereditary.

Consider a quotient $f : M \to X$ from Example 2.10. Since M is metric, X is sequential, thus X is an ssq-space. Let $X_0 = X - \mathbb{N}$, $M_0 = f^{-1}(X_0)$ and $g = f|_{M_0} : M_0 \to X_0$. The subspace X_0 of X is called the *Arens-Fort space* in [28, Example 26]. Since $\mathbb{N}\times\mathbb{N}$ is sequentially closed, and non-strongly sequentially closed in X_0, X_0 is not an ssq-space. By Corollary 3.4, g is not ss-quotient. Thus ss-quotient maps are not hereditary, and ssq-spaces are not hereditary.

Countable tightness is hereditary. It is well-known that every hereditarily sequential T_2-space is Fréchet. The following question is raised.

Question 3.11. *How can one characterize hereditary ssq-spaces?*

Example 3.12. There is a compact, Hausdorff and hereditary ssq-space which has not countable tightness.

Let $X = [0, \omega_1]$ be endowed with the usual ordinal topology. Then X is a compact, Hausdorff space which has not countable tightness. Next, we will show that X is a hereditary *ssq*-space. Let $A \subset Y \subset X$ and A be sequentially closed in Y. If A is not strongly sequentially closed in Y, there exists a sequence $\{x_n\}$ in A such that $\{x_n\}$ has an accumulation point $\alpha \in Y - A$. If $\alpha = \omega_1$, then $x_n \neq \omega_1$ for each $n \in \mathbb{N}$, thus $x_n < \omega_1$. There is $\beta < \omega_1$ such that $x_n < \beta$ for each $n \in \mathbb{N}$. Then a neighborhood $(\beta, \omega_1] \cap Y$ of α in Y contains no x_n for each $n \in \mathbb{N}$, a contradiction. Therefore, $\alpha < \omega_1$. Since $[0, \omega_1) \cap Y$ is open and first-countable in Y, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to \alpha$ in Y, then $\alpha \in A$ because A is sequentially closed, a contradiction. Hence, A is strongly sequentially closed in Y. Therefore, Y is an *ssq*-space, and X is a hereditary *ssq*-space.

4. Strongly sequentially closed sets in compact spaces

It is well-known that every countably compact and sequential space is sequentially compact. The most classic problem about countable tightness is the following Moore-Mrowka Problem [20], [22]: Is every compact T_2-space of countable tightness sequential?

It was shown that the answer to the Moore-Mrowka Problem is **No** from the set-theoretic principle $\diamondsuit$ [24]. On the other hand, the answer of the Moore-Mrowka Problem is **Yes** under the Shelah's Proper Forcing Axiom (PFA) as follows.

Lemma 4.1 ([2, Theorem 2.3]). *Assuming* (PFA). *If X is a compact T_2-space of countable tightness, then every non-isolated point in X has a non-trivial sequence converging to the point.*

It is also a question whether there is a compact space of countable tightness that is not sequentially compact [22].

It is obvious that (1) if X is countably compact, then each strongly sequentially closed subset of X is countably compact; (2) each countably compact subset of a T_2-space is sequentially closed. We can discuss strongly sequentially closed sets in countably compact or compact spaces.

Theorem 4.2. *Let X be a countably compact and T_2-space. Then X is an ssq-space if and only if X is sequentially compact space in which every countably compact subset is strongly sequentially closed.*

Proof: Let X be a countably compact and *ssq*-space. If X is not sequentially compact, there exists a sequence $\{x_n\}$ in X which has not any convergent subsequence. Put $A = \{x_n : n \in \mathbb{N}\}$. Then A is sequentially closed, thus A is strongly sequentially closed. Since X is countably compact, $\{x_n\}$ has an accumulation point x in X, we can assume that $x_n \neq x$ for each $n \in \mathbb{N}$. Since A is strongly sequentially closed, $x \in A$, a contradiction. Hence, X is sequentially compact. Let B be countably compact in X. If B is not strongly sequentially closed in X, then B is not sequentially closed, i.e., there is a sequence $\{y_n\}$ in B such that $y_n \to y \notin B$. Since X is T_2, $\{y_n\}$ has no accumulation point in B, thus B is not countably compact, a contradiction.

Conversely, let X be sequentially compact space in which every countably compact subset is strongly sequentially closed. Suppose that a subset S of X is not strongly sequentially closed. Then S is not countably compact, there is a sequence $\{z_n\}$ in S such that $\{z_n\}$ has no accumulation point in S. Since X is sequentially compact, there is a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ converging to a point $z \in X$, then $z \notin S$, and S is not sequentially closed. Hence, X is an *ssq*-space. $\square$

Remark 4.3. By the proof of Theorem 4.2, the T_2-condition is used only to show that every countably compact subset is strongly sequentially closed in an *ssq*-space. The condition is essential. There is a sequentially compact and *ssq*-space X such that X has a countably compact subset which is not strongly sequentially closed. For example, let X be the set $\mathbb{N}$ endowed with the finite-complement topology. Then X is a compact, first-countable and T_1-space. Thus X is a sequentially compact and *ssq*-space, and $\{2n : n \in \mathbb{N}\}$ is a countably compact subset in X which is not strongly sequentially closed in X.

Example 4.4. (1) Assuming CH, an example of a sequentially compact, T_2-space X is given such that X has countable tightness and is not sequential in [8, Example 1.2]. By Proposition 3.2, X is not an *ssq*-space.

(2) There is a separable, sequentially compact, sequential T_2-space which is not compact by [23, Theorem 3.2]

A space X is called *C-closed* [12] if every countably compact subset of X is closed in X. C-closed property is discussed in [12]. By Theorems 1.3 and 4.2, the following result is obtained.

Corollary 4.5 ([12]). *A countably compact T_2-space is sequential if and only if it is a sequentially compact and C-closed space.*

PROOF: Let X be a countably compact T_2-space. If X is sequential, then X is an *ssq*-space of countable tightness. Thus X is a sequentially compact space in which every countably compact subspace is closed by Theorems 4.2 and 1.3.

Conversely, if X is a sequentially compact space in which every countably compact subspace is closed, then X is an *ssq*-space by Theorem 4.2. Let A be strongly sequentially closed in X. Since X is countably compact, then A is countably compact in X, thus A is closed. Hence, X has countable tightness by Theorem 1.3. Therefore, X is sequential. $\square$

Lemma 4.6. *Let X be a countably compact T_1-space in which every countably compact subset of X is strongly sequentially closed. If X is separable, then $|X| \leq 2^\omega$.*

PROOF: If A is an infinite subset of X, denote an accumulation point in X for some sequence of points of A by $\alpha(A)$. Let $T(B) = \{\alpha(A) : A \subset B, \text{and } |A| \leq \omega\}$ for each $B \subset X$. It is easy to see that $|T(B)| \leq |B|^\omega$.

Now, let D be a countable dense subset of X, and put $B_0 = D$. If $\alpha < \omega_1$ and for every $\beta < \alpha$ we have defined B_β with $|B_\beta| \leq 2^\omega$, let $B_\alpha = \bigcup_{\beta<\alpha} B_\beta \cup T(\bigcup_{\beta<\alpha} B_\beta)$. Then $|B_\alpha| \leq 2^\omega$. Let $C = \bigcup_{\alpha<\omega_1} B_\alpha$. Then C is countably compact and $|C| \leq 2^\omega$, thus C is strongly sequentially closed. Since X is a T_1-space and D is countable, $\overline{D} \subset C$. Hence $X = C$, and $|X| \leq 2^\omega$. $\square$

Remark 4.7. Since the Stone-Čech compactification $\beta\mathbb{N}$ is a separable compact T_2-space with $|\beta\mathbb{N}| = 2^{\mathbf{c}}$ [6], there is a countably compact subset which is not strongly sequentially closed in $\beta\mathbb{N}$ by Lemma 4.6.

Lemma 4.8 ([5] (see [6, 3.12.11])). *Let X be a compact T_2-space. If $\chi(x,X) \geq \mathbf{m} \geq \omega$ for each $x \in X$, then $|X| \geq 2^{\mathbf{m}}$.*

Theorem 4.9. *Let X be a compact T_2-space in which every countably compact subset of X is strongly sequentially closed. Then X is an ssq-space if one of the following conditions is satisfied.*

(1) *Assuming $2^\omega < 2^{\omega_1}$.*
(2) $|X| < 2^{\omega_1}$.

Proof: Let A be a non-strongly sequentially closed subset in X. Then A is not countably compact in X, and there is a sequence $\{x_n\}$ in A which has no accumulation point in A. Put $B = \{x_n : n \in \mathbb{N}\}$, and $Y = \overline{B}$. Then Y is a separable compact T_2-subspace in which every countably compact subset of X is strongly sequentially closed. By Lemma 4.6, $|Y| \leq 2^\omega$. Note that $Y - B$ is a compact G_δ-subset of Y, $\chi(y,Y) = \chi(y, Y - B)$ for each $y \in Y - B$ [6].

Suppose one of the conditions above holds, then $|Y| < 2^{\omega_1}$. There is $y \in Y - B$ such that $\chi(y,Y) \leq \omega$ by Lemma 4.8. Therefore, there is a subsequence of $\{x_n\}$ converging to y. Since $\{x_n\}$ has no accumulation point in A, $y \in X - A$. Thus, A is not sequentially closed. Hence, X is an *ssq*-space. □

Corollary 4.10 ([12]). *Assuming $2^\omega < 2^{\omega_1}$. A compact T_2-space is sequential if and only if it is C-closed.*

Proof: Let X be a compact T_2-space which is C-closed. X is an *ssq*-space by Theorem 4.9. Let A be strongly sequentially closed in X. Since X is compact, then A is countably compact in X, thus A is closed. Hence, X has countable tightness by Theorem 1.3. Therefore, X is sequential. □

Let X be a space. A set $H \subset X$ is called *k-closed* [3] in X if $H \cap K$ is relatively closed in K for each compact subset K of X. A space is a *k-space* [6] if each of its k-closed subset is closed. A space is a *kq-space* [15] if each of its sequentially closed subset is k-closed. It is obvious that a space is sequential if and only if it is a k-space and kq-space.

Proposition 4.11. *Assuming* (PFA). *Every T_2-space of countable tightness is a kq-space.*

Proof: Let X be a T_2-space of countable tightness, and $A \subset X$. If A is not k-closed in X, there is a compact subset K of X such that $K \cap A$ is not closed in K. Since countable tightness is hereditary, K has countable tightness. By Lemma 4.1, there is a sequence $\{x_n\}$ in $K \cap A$ converging to a point $x \in K - A$, thus A is not sequentially closed in X. Hence, X is a kq-space. □

Example 4.12. There is a countably compact, kq-space which is not sequentially compact.

There is an infinite, completely regular, countably compact space X in which all compact subsets are finite by [9, Example 9.1]. It is obvious that X is not sequentially compact. Since each compact subset of X is finite, each subset of X is k-closed, then X is a kq-space. Moreover, X is not an ssq-space by Theorem 4.2.

Example 4.13. (1) k-spaces $\not\Rightarrow$ sq-spaces, kq-spaces, C-closed spaces, spaces of countable tightness.

$\beta\mathbb{N}$ is a compact space, thus it is a k-space. Since $\mathbb{N}$ is sequentially closed, non-strongly sequentially closed and non-k-closed in $\beta\mathbb{N}$, $\beta\mathbb{N}$ is not an ssq-space, not a kq-space. $\beta\mathbb{N}$ is not a C-closed space by Remark 4.7. $\beta\mathbb{N}$ is not of countable tightness by [10, Example 7.22].

(2) C-closed, ssq-spaces, kq-spaces $\not\Rightarrow$ k-spaces, spaces of countable tightness.

Let X be an uncountable set, $p \in X$ be a particular point. Define a topology on X by declaring sets whose complement is either countable or includes p to be open. The space is called the *Fortissimo space* in [28, Example 25]. Since every countable subset of X is closed, then X has not countable tightness, every subset of X is strongly sequentially closed, and each countably compact subset is finite. Thus X is a C-closed, ssq, and kq-space which is not a k-space.

(3) ssq-spaces, k-spaces $\not\Rightarrow$ C-closed, kq-spaces, spaces of countable tightness.

Let X be the ordinal topological space $[0, \omega_1]$. Then X is a non-sequential, k-space. Thus X is not a kq-space. By Example 3.12, X is an ssq-space which has not countable tightness. Since $[0, \omega_1)$ is a countably compact subset which is not closed in X, X is not C-closed.

(4) kq-spaces, C-closed spaces, spaces of countable tightness $\not\Rightarrow$ k-spaces, ssq-spaces.

Continue Example 2.10. Let $X_0 = X - \mathbb{N}$. It is obvious that X_0 has countable tightness. Since each countably compact subset of X_0 is finite, X_0 is a C-closed, kq-space which is not a k-space. Since $\mathbb{N} \times \mathbb{N}$ is sequentially closed, and non-strongly sequentially closed in X_0, X_0 is not an ssq-space.

The authors do not know whether (1) there is a space of countable tightness which is not C-closed; (2) there is a C-closed space which is not a kq-space; (3) there is a kq-space which is not C-closed.

5. Examples and questions about maps

Let $f : X \to Y$ be a map. The map f is a *k-quotient map* [3] if, whenever $F \subset Y$ and $f^{-1}(F)$ is k-closed in X, then F is k-closed in Y.

Example 5.1. ss-quotient maps $\not\Rightarrow$ sequentially quotient.

Let $X = \beta\mathbb{N}$ be the Stone-Čech compactification of $\mathbb{N}$, and $Y = \mathbb{N} \cup \{\infty\}$ the one point compactification of $\mathbb{N}$. Define a function $f : X \to Y$ as follows: $f(n) = n$ for each $n \in \mathbb{N}$, and $f(x) = \infty$ for each $x \in \beta\mathbb{N} - \mathbb{N}$. Then f is a map. Since the sequence $\{n\}$ converges to ∞ in Y and X has no non-trivial convergent sequence, f is not sequentially quotient. On the other hand, if $F \subset Y$ is not strongly sequentially closed, then F is infinite and $\infty \notin F$, thus $f^{-1}(F) \subset \mathbb{N}$ and each non-trivial sequence in $f^{-1}(F)$ has an accumulation point in $X - \mathbb{N}$, i.e., $f^{-1}(F)$ is not strongly sequentially closed in X. Therefore, f is ss-quotient.

Example 5.2. Sequentially quotient maps $\not\Rightarrow$ ss-quotient.

Let X be the set $\beta\mathbb{N}$ endowed with the discrete topology. Define a map $f : X \to \beta\mathbb{N}$ by $f(x) = x$ for each $x \in X$. Here $\beta\mathbb{N}$ is endowed with the topology of Stone-Čech compactifications. Since $\beta\mathbb{N}$ has no non-trivial convergent sequence, f is sequentially quotient. Since $f^{-1}(\mathbb{N})$ is strongly sequentially closed in X and $\mathbb{N}$ is not strongly sequentially closed in $\beta\mathbb{N}$, f is not ss-quotient.

Example 5.3. ss-quotient maps $\not\Rightarrow$ quotient, k-quotient.

Let $Y = [0, \omega_1]$ be endowed with the ordinal topology. Put $X = [0, \omega_1]$ endowed with the following topology: ω_1 is isolated; the rest points in X have the usual neighborhoods of the ordinal topology. Define a function $f : X \to Y$ by the identity map. Then f is a map. Since $f^{-1}(\{\omega_1\}) = \{\omega_1\}$ is open in X and $\{\omega_1\}$ is not open in Y, f is not quotient. Let $F = [0, \omega_1)$. Since Y is compact, F is not k-closed in Y. Since $f^{-1}(F)$ is closed in X, it is k-closed in X. Thus f is not k-quotient.

On the other hand, if $B \subset Y$ is not strongly sequentially closed in Y, there is a sequence $\{y_n\}$ in B which has an accumulation point $b \in Y - B$. Since $[0, \omega_1)$ is strongly sequentially closed in Y, $b < \omega_1$. We can assume that $y_n < \omega_1$ for each $n \in \mathbb{N}$. Then the sequence $\{y_n\}$ has an accumulation point $y \notin f^{-1}(B)$ in X, thus $f^{-1}(B)$ is not strongly sequentially closed in X. Therefore, f is ss-quotient.

Example 5.4. Open maps $\not\Rightarrow$ ss-quotient.

Let $X = [0, \omega_1]$ be endowed with the following topology: ω_1 has the usual neighborhoods of the ordinal topology; each point in $[0, \omega_1)$ is isolated. Define a function $f : X \to \{0, 1\}$ by $f(\omega_1) = 1$ and $f(x) = 0$ for each $x \in [0, \omega_1)$. The set $\{0, 1\}$ is endowed with the quotient topology induced by f, then f is open. Since 1 is an accumulation point of the set $\{0\}$, $\{0\}$ is not strongly sequentially closed in $\{0, 1\}$. Since $f^{-1}(\{0\}) = [0, \omega_1)$ is strongly sequentially closed in X, f is not ss-quotient.

Example 5.5. k-quotient maps $\not\Rightarrow$ ss-quotient.

Continue Example 2.10. Let X be the Arens' space S_2, and $X_0 = X - \mathbb{N}$. Then each compact subset of X_0 is finite. Take D as the set X_0 with the discrete topology. Define a map $h : D \to X_0$ by the identity map. Since each subset of X_0 is k-closed, h is k-quotient. Since $\mathbb{N} \times \mathbb{N}$ is not strongly sequentially closed in X_0 and $h^{-1}(\mathbb{N} \times \mathbb{N})$ is closed in D, h is not ss-quotient.

Question 5.6. *How can one characterize a space such that every ss-quotient map of the space is sequentially quotient?*

Question 5.7. *How can one characterize a space such that every ss-quotient map onto the space is sequentially quotient?*

Question 5.8. *How can one characterize a space such that every quotient map onto the space is ss-quotient?*

Question 5.9. *How can one characterize a space such that every quotient map onto the space is sequentially quotient?*

Question 5.10. *How can one characterize a space with C-closed property by certain maps?*

Acknowledgments. We wish to thank the reviewers for the detailed list of corrections, suggestions to the paper, and all her/his efforts in order to improve the paper.

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(*Received* September 14, 2013*, revised* December 17, 2013)