Comment.Math.Univ.Carolin. 55,4 (2014) 471–484

Some regular quasivarieties of commutative binary modes

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Abstract. Irregular (quasi)varieties of groupoids are (quasi)varieties that do not contain semilattices. The regularization of a (strongly) irregular variety $\mathcal{V}$ of groupoids is the smallest variety containing $\mathcal{V}$ and the variety $\mathcal{S}$ of semilattices. Its quasiregularization is the smallest quasivariety containing $\mathcal{V}$ and $\mathcal{S}$.

In an earlier paper the authors described the lattice of quasivarieties of cancellative commutative binary modes, i.e. idempotent commutative and entropic (or medial) groupoids. They are all irregular and the lattice contains all irregular varieties of such groupoids. This paper extends the earlier result, by investigating some regular quasivarieties. It provides a full description of the lattice of subquasivarieties of the regularization of any irregular variety of commutative binary modes.

Keywords: regular quasivarieties; regular quasi-identity; modes; affine spaces; commutative binary modes

Classification: 08A62, 08C15, 20N02, 20N05

1. Introduction

Commutative binary modes were intensively investigated by Ježek and Kepka (see [3], [4], [5], [6]) under the name of commutative idempotent medial groupoids, by Romanowska and Smith (see [10], [11], [12]), and by Matczak, Romanowska and Smith (see[9]). In particular, the lattice of varieties of such groupoids was first described in [4], and a structural characterization was provided in [11]. The lattice of varieties splits into two parts. The irregular varieties are known to be equivalent to varieties of commutative quasigroup modes and to varieties of affine spaces over the rings $\mathbb{Z}_{2k+1}$. (See [10] and [12].) They form a lattice isomorphic to the lattice of odd natural numbers under division. The remaining varieties are regular, and with the exception of the variety of all commutative binary modes, they are regularizations of irregular ones.

Quasivarieties of commutative binary modes were investigated by Hogben and Bergman [2], and by Matczak and Romanowska [7], [8]. Hogben and Bergman have shown that the irregular varieties are the only *deductive* quasivarieties of commutative binary modes, quasivarieties that do not contain subquasivarieties that are not varieties. The next step in investigating the lattice of quasivarieties of commutative binary modes was taken by Matczak and Romanowska [7]. In

Research supported by the Warsaw University of Technology under grant number 504P/1120/0021/000.

that paper the (isomorphic) lattices of quasivarieties of commutative quasigroup modes (equivalent to quasivarieties of affine spaces over the principal ideal domain $\mathbb{D}$ of rational dyadic numbers) as well as the lattice of quasivarieties of cancellative commutative binary modes were described. The quasivariety of cancellative binary modes is contained in the largest quasivariety not containing the variety of semilattices, i.e. the largest irregular quasivariety. Some irregular quasivarieties were investigated in [8].

In this paper we continue our investigations of quasivarieties of commutative binary modes, but focus our attention on some regular quasivarieties, i.e. quasivarieties containing semilattices. We define the quasi-regularization of a quasivariety $\mathcal{Q}$ as the smallest quasivariety containing both $\mathcal{Q}$ and the variety of semilattices. Our first result (Theorem 3.4) provides a full description of the lattice of subquasivarieties of the quasi-regularization $\widetilde{\mathcal{C}}_m^q$ of any irregular variety $\mathcal{C}_m$ of commutative binary modes. We show that this lattice is isomorphic to the direct product of the lattice $\mathcal{L}(\mathcal{C}_m)$ of subvarieties of $\mathcal{C}_m$ and the 2-element lattice. Our method extends the method used by Bergman and Romanowska [1], who described the lattice of subquasivarieties of the regularization of a (strongly) irregular variety that is minimal as a variety and as a quasivariety. In the last section, we provide a full description of the lattice of subquasivarieties of the regularization $\widetilde{\mathcal{C}}_m$ of any irregular variety $\mathcal{C}_m$ of commutative binary modes. Figure 4 provides a large scale view of this lattice. The lattice is finite and distributive. However, unlike the case of minimal varieties, where the quasi-regularization covers the given variety and is covered by its regularization, there are quasivarieties containing the quasi-regularization of a given variety which are different from its quasi-regularization and its regularization. We also provide quasi-equational bases for all these quasivarieties.

Though our results concern only commutative binary modes, they can be easily extended to any idempotent variety containing only varieties as subquasivarieties and generated by a finite number of finite subdirectly irreducible algebras.

Basic facts on modes, commutative binary modes, commutative quasigroup modes and affine spaces are recalled briefly in our papers [7], [8]. For more information see the monographs [10] and [12]. As for information concerning regularizations and quasi-regularizations of (strongly) irregular varieties, we refer the reader to [12, Chapter 4].

2. Commutative binary modes

Most algebras considered in this paper are *modes* in the sense of [10] and [12], algebras in which each element forms a singleton subalgebra, and for which each operation is a homomorphism. For algebras (A, Ω) of a given type $\tau : \Omega \to \mathbb{N}$, these two properties are equivalent to satisfaction of the identity

$$x \dots x\psi = x \tag{2.1}$$

of *idempotence* for each operation ψ in Ω, and the identity

$$(2.2)\quad (x_{11}\dots x_{1m}\psi)\dots(x_{n1}\dots x_{nm}\psi)\phi = (x_{11}\dots x_{n1}\phi)\dots(x_{1m}\dots x_{nm}\phi)\psi$$

of *entropicity* for any two operations (m-ary) ψ and (n-ary) ϕ in Ω. In particular, *binary* (*or groupoid*) *modes* are algebras with one binary idempotent multiplication satisfying the identity

$$(x\cdot y)\cdot(z\cdot t) = (x\cdot z)\cdot(y\cdot t).$$

Important models of modes are provided by *affine R-spaces*, affine spaces over a commutative unital ring R and their subreducts (subalgebras of reducts). In many cases, affine R-spaces are described as algebras $(A, \underline{R})$, where for each $r\in R$, the binary operation $\underline{r}\in\underline{R}$ is defined by $xy\underline{r} = x(1-r)+yr$. See [10] and [12]. In particular, affine spaces appearing in this paper are of this type. *Commutative binary modes* $(A,\cdot)$ are binary modes with a commutative multiplication, i.e. satisfying the identity

$$(2.3)\qquad x\cdot y = y\cdot x.$$

The class $\mathcal{CBM}_{cl}$ of cancellative commutative binary modes is a subquasivariety of the variety $\mathcal{CBM}$ of commutative binary modes defined by the quasi-identity

$$(2.4)\qquad xy = xz \to y = z.$$

Subquasivarieties of $\mathcal{CBM}_{cl}$ are called *cancellative*. Recall that cancellative commutative binary modes are $\underline{2^{-1}}$-subreducts of affine $\mathbb{D}$-spaces (i.e. affine spaces over the ring $\mathbb{D}$ of rational dyadic numbers) [12, Chapter 6].

The class $\mathcal{CBM}_{ir}$ of *irregular* commutative binary modes is a subquasivariety of the variety $\mathcal{CBM}$ defined by the quasi-identity

$$(2.5)\qquad xy = x \to x = y.$$

As shown in [8], the quasivariety $\mathcal{CBM}_{ir}$ is the largest subquasivariety of $\mathcal{CBM}$ not containing the variety $\mathcal{S}$ of semilattices. Subquasivarieties of $\mathcal{CBM}_{ir}$ (and their members) are called *irregular*. Subquasivarieties of $\mathcal{CBM}$ that are not irregular are called *regular*. It is known that all cancellative subquasivarieties of $\mathcal{CBM}$ are irregular.

It is also well-known that irregular subvarieties of the variety $\mathcal{CBM}$ are all contained in $\mathcal{CBM}_{cl}$, and form a lattice isomorphic to the lattice of odd natural numbers under division. They will be denoted by $\mathcal{C}_{2k+1}$, or if it is known that $m = 2k+1$, then simply by $\mathcal{C}_m$. Each such variety $\mathcal{C}_{2k+1}$ is generated by exactly one (cancellative) groupoid $(\mathbb{Z}_{2k+1}, \underline{2^{-1}}) = (\mathbb{Z}_{2k+1}, \underline{k+1})$, and is equivalent to the variety $\underline{\underline{\mathbb{Z}}}_{2k+1}$ of affine $\mathbb{Z}_{2k+1}$-spaces. As a subvariety of $\mathcal{CBM}$, the variety

$\mathcal{C}_m = \mathcal{C}_{2k+1}$ is defined by one additional identity

$$(\iota_m) \qquad x \star_m y = x,$$

where $x \star_m y$ is a certain binary word[1].

The only subdirectly irreducible members of $\mathcal{C}_{2k+1}$ are the groupoids $\mathbb{Z}_{p_i^{n_i}}$, where p_i is an odd prime number, n_i is a positive integer, and $p_i^{n_i}$ divides $2k+1$. The varieties $\mathcal{C}_p$, for prime numbers p, are minimal as varieties and as quasivarieties. For more details see [4], [10] and [12]. Recall also that each variety $\mathcal{C}_{2k+1}$ is deductive, i.e. it has no subquasivarieties that are not varieties [2].

3. The quasi-regularization and its subquasivarieties

The *regularization* $\widetilde{\mathcal{V}}$ of an irregular idempotent variety $\mathcal{V}$ of groupoids is the smallest variety containing both $\mathcal{V}$ and the variety $\mathcal{S}$ of semilattices. It is known that $\widetilde{\mathcal{V}}$ consists of Płonka sums of $\mathcal{V}$-groupoids, and satisfies precisely the regular identities true in $\mathcal{V}$. (See e.g. [12, Chapter 4]).

The *quasi-regularization* $\widetilde{\mathcal{Q}}^q$ of an irregular idempotent quasivariety $\mathcal{Q}$ of groupoids is the smallest quasivariety containing both $\mathcal{Q}$ and $\mathcal{S}$. If $\mathcal{Q}$ is a variety $\mathcal{V}$, its quasi-regularization $\widetilde{\mathcal{V}}^q$ does not necessary coincide with its regularization $\widetilde{\mathcal{V}}$ (see e.g. [1]).

The irregular subvarieties $\mathcal{C}_m = \mathcal{C}_{2k+1}$ of $\mathcal{CBM}$ and their regularizations $\widetilde{\mathcal{C}}_m$ form all the non-trivial subvarieties of $\mathcal{CBM}$ ([4] and [10]). Each irregular variety $\mathcal{C}_m$ is defined by a strongly irregular identity $x \star_m y = x$, which determines a trivial left-zero operation $x \star_m y$ in each member of $\mathcal{C}_m$. In the regularization $\widetilde{\mathcal{C}}_m$, this operation becomes the operation of a left-normal band. The regularization $\widetilde{\mathcal{C}}_m$ is defined (as a subvariety of $\mathcal{CBM}$) also by one additional identity, namely the identity

$$(\iota'_m) \qquad x \cdot (x \star_m y) = x \star_m y.$$

A description of the quasi-regularization $\widetilde{\mathcal{C}}^q_m$ may be deduced easily from results of [1, §3]. (See also [12, §4.4].) Consider the quasi-identity

$$\begin{aligned}(\alpha_m) \qquad & (x \star_m y = x \mathbin{\&} y \star_m x = y \mathbin{\&} \\ & x \star_m z = z \star_m x = z \mathbin{\&} y \star_m z = z \star_m y = z) \\ & \rightarrow (x = y).\end{aligned}$$

Let K be one of the operators V, Q, P, S, $\mathsf{P_s}$. Denote by $\mathsf{K}(A_1, \ldots, A_n)$ the class generated under the operator K by the algebras $A_1, \ldots, A_n$. Recall that if $A_1, \ldots, A_n$ are finite, then $\mathsf{Q}(A_1, \ldots, A_n) = \mathsf{SP}(A_1, \ldots, A_n)$. The symbol $\mathbf{2}_s$ will denote the 2-element semilattice $\{0 <_{\cdot} 1\}$. In what follows, m in the symbols like

[1] As for $x \star_m y$ one can take a binary word $w_m = w_m(x, y)$ defined as follows: $w_0(x_1) = x_1$, then $w_{n+1}(x_1, \ldots, x_{2^{n+1}}) = w_n(x_1, \ldots, x_{2^n}) \cdot w_n(x_{2^n+1}, \ldots, x_{2^{n+1}})$, where n is the least integer greater than $\log_2 m$, and all x_i with $1 \leq i \leq 2^n - m$ are equal to x and all remaining x_i are equal to y.

$\mathcal{C}_m$, (α_m) or $\mathbb{Z}_m$ will always be an odd positive integer. Moreover $m = p_1^{j_1} \dots p_r^{j_r}$ will denote the decomposition of m into the product of powers of pairwise different odd prime numbers p_i.

Proposition 3.1. *For any variety $\mathcal{C}_m = \mathcal{C}_{2k+1}$, the following conditions are equivalent:*

(a) *the quasi-regularization $\widetilde{\mathcal{C}}_m^q$ consists of Płonka sums of groupoids in $\mathcal{C}_m$ with injective Płonka homomorphisms;*
(b) $\widetilde{\mathcal{C}}_m^q = \mathsf{SP}(\mathcal{C}_m \cup \mathcal{S})$;
(c) *$\widetilde{\mathcal{C}}_m^q$ is the subquasivariety of $\widetilde{\mathcal{C}}_m$ defined by the quasi-identity (α_m);*
(d) *$\widetilde{\mathcal{C}}_m^q = \mathsf{P_s}(\{\mathbb{Z}_{p^j} \mid p^j|m\} \cup \{\mathbf{2}_s\})$, i.e. $\widetilde{\mathcal{C}}_m^q$ is generated by all its subdirectly irreducible algebras.*

For a groupoid A and the trivial groupoid $\mathbf{1} = (\{\infty\}, \cdot)$, the symbol A^∞ denotes the Płonka sum of $A_1 = A$ and $A_0 = \{\infty\}$ over the semilattice $\mathbf{2}_s$.

The two following lemmas may also be deduced from results of [1] and the characterization of subdirectly irreducible $\mathcal{C}_m$-groupoids.

Lemma 3.2. *Let $m = p_1^{j_1} \dots p_r^{j_r}$,where all p_i are odd prime numbers.*

(a) $\widetilde{\mathcal{C}}_m^q = \mathsf{SP}(\mathbb{Z}_{p_1^{j_1}}, \dots, \mathbb{Z}_{p_r^{j_r}}, \mathbf{2}_s)$.
(b) *For $0 < k_i \leq j_i$, the groupoid $\mathbb{Z}^\infty_{p_i^{k_i}}$ does not belong to $\widetilde{\mathcal{C}}_m^q$.*

Proof: The first statement is obvious. For the second, note that the quasi-identity (α_m) does not hold in a groupoid A of $\mathcal{CBM}$ precisely when there are elements $x, y, z \in A$ such that the premises of (α_m) hold, but $x \neq y$. Let $a \neq b \in A$. Let $\infty = a\varphi_{1,0} = b\varphi_{1,0}$ in A^∞, where $\varphi_{1,0} : A_1 \to A_0$ is a Płonka homomorphism. Then $a \star \infty = \infty \star a = b \star \infty = \infty \star b = \infty$. It follows, with a, b, ∞ as witnesses for x, y, z, that the quasi-identity (α_m) fails in A. □

Lemma 3.3. [1, §3] *There are one-to-one correspondences between the poset of irregular subvarieties of $\mathcal{CBM}$, the poset of their quasi-regularizations, and the poset of their regularizations. In fact, the three posets form three isomorphic lattices.*

The following theorem will describe the lattice $\mathcal{L}_q(\widetilde{\mathcal{C}}_m^q)$ of subquasivarieties of the quasi-regularization $\widetilde{\mathcal{C}}_m^q$.

Theorem 3.4. *For any variety $\mathcal{C}_m = \mathcal{C}_{2k+1}$, the lattice $\mathcal{L}_q(\widetilde{\mathcal{C}}_m^q)$ of subquasivarieties of the quasi-regularization $\widetilde{\mathcal{C}}_m^q$ is isomorphic to $\mathcal{L}(\mathcal{C}_m) \times \mathbf{2}_l$, the direct product of the lattice of subvarieties of $\mathcal{C}_m$ and the 2-element lattice $\mathbf{2}_l$.*

Proof: Recall that each variety $\mathcal{C}_m$ has no subquasivarieties that are not varieties, whence $\mathcal{L}(\mathcal{C}_m) \cong \mathcal{L}_q(\mathcal{C}_m)$. This lattice is isomorphic to the lattice of divisors of m. By Lemma 3.3, the quasi-regularizations of the subvarieties of $\mathcal{C}_m$ form a lattice isomorphic to $\mathcal{L}(\mathcal{C}_m)$. Our aim is to show that there are no other subquasivarieties of the quasivariety $\widetilde{\mathcal{C}}_m^q$.

The proof is by induction on the height of elements in the lattice of irregular subvarieties of the variety $\mathcal{CBM}$. The elements of height 1 are the varieties $\mathcal{C}_p$, where p is an odd prime number. In this case, our result follows directly by Theorem 4.3 of [1], since $\mathcal{C}_p$ is locally finite and minimal as a variety and as a quasivariety. In particular, the lattice $\mathcal{L}_q(\widetilde{\mathcal{C}}_p)$ has the diagram presented in Figure 1.

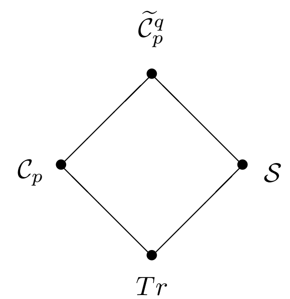

FIGURE 1. Subquasivarieties of $\widetilde{\mathcal{C}}_p^q$

Assume now that m is not prime, and that our result holds for all varieties $\mathcal{C}_j$ with height less than the height of $\mathcal{C}_m$. Since $\mathcal{C}_m$ is deductive, one has

$$\mathcal{C}_m = \mathsf{V}(\mathbb{Z}_m) = \mathsf{Q}(\mathbb{Z}_m) = \mathsf{SP}(\mathbb{Z}_m).$$

Recall that $\mathbb{Z}_{p^j}$, where p is a prime number and p^j divides m, are all the subdirectly irreducible members of $\mathcal{C}_m$. Note that all $\mathbb{Z}_{p^j}$ are (isomorphic to) subalgebras of $\mathbb{Z}_m$, and at the same time $\mathbb{Z}_m$ is a subdirect product of such subdirectly irreducible algebras. Then one also has the following:

$$\mathsf{SP}(\mathbb{Z}_m) = \mathsf{P_s}(\{\mathbb{Z}_{p^j} \mid p^j | m\}).$$

Recall as well that by Proposition 3.1, $\widetilde{\mathcal{C}}_m^q$ is generated under the operator $\mathsf{P_s}$ by the subdirectly irreducible members of $\mathcal{C}_m$ and the 2-element semilattice. Now let $\mathcal{V}$ be a subquasivariety of $\widetilde{\mathcal{C}}_m^q$ that is neither contained in $\mathcal{C}_m$, nor in any of the $\widetilde{\mathcal{C}}_j^q$ such that the height of $\mathcal{C}_j$ is smaller than the height of $\mathcal{C}_m$. Consider groupoids in $\mathcal{V} - \mathcal{C}_m$ that are not members of any of these quasivarieties $\widetilde{\mathcal{C}}_j^q$. Then by Proposition 3.1, each such groupoid A is a non-trivial Płonka sum of $\mathcal{C}_m$-groupoids A_i over a semilattice I with injective Płonka homomorphisms, and with at least some fibres A_i not in proper subvarieties of $\mathcal{C}_m$. Since in any $\mathcal{C}_m$-groupoid any two distinct elements generate a subdirectly irreducible algebra (see [12, §§5.5, 7.5]), the fibres should contain as subalgebras subdirectly irreducible members of $\mathcal{C}_m$, in particular subdirectly irreducibles of $\mathcal{C}_m$ which are not members of its proper subvarieties. (Otherwise, A would be a member of a quasivariety of smaller height.) It follows that

$$\mathcal{C}_m = \mathsf{SP}(\mathbb{Z}_{p^j} \mid p^j | m) \subseteq \mathcal{V}.$$

On the other hand, for any $i > j$ in I, and an element $a_i \in A_i$, the elements a_i and its image $a_i\varphi_{i,j}$ under the Płonka homomorphism $\varphi_{i,j}$ form a 2-element semilattice $\mathbf{2}_s$. This implies that

$$\mathcal{S} = \mathsf{SP}(\mathbf{2}_s) \subseteq \mathcal{V}.$$

Consequently, we obtain

$$\widetilde{\mathcal{C}}^q_m = \mathsf{Q}(\mathcal{C}_m, \mathcal{S}) \subseteq \mathcal{V} \subseteq \widetilde{\mathcal{C}}^q_m,$$

completing the proof of the theorem. □

Let us note that the lattice $\mathcal{L}(\mathcal{C}_m)$ and the lattice $\mathcal{L}^{>s}(\widetilde{\mathcal{C}}^q_m)$ of the quasi-regularizations of subvarieties of $\mathcal{C}_m$ are isomorphic.

4. The regularization and its subquasivarieties

In this section, we will describe the lattice of subquasivarieties of the regularization $\widetilde{\mathcal{C}}_m$ of any variety $\mathcal{C}_m$. The following proposition follows directly by Proposition 3.1, with a proof similar to the proof of Theorem 5.1 in [1].

Proposition 4.1. *Let $p \geq 3$ be a prime number. The lattice $\mathcal{L}_q(\widetilde{\mathcal{C}}_p)$ of subquasivarieties of the regularization $\widetilde{\mathcal{C}}_p$ consists of the five members displayed in Figure 2.*

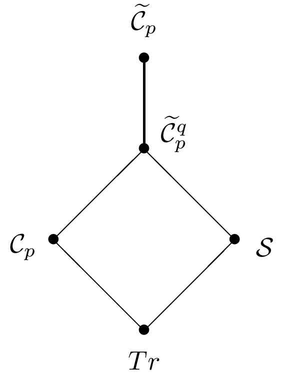

FIGURE 2. Subquasivarieties of $\widetilde{\mathcal{C}}_p$

To describe the lattice $\mathcal{L}_q(\widetilde{\mathcal{C}}_m)$ of subquasivarieties of the regularization $\widetilde{\mathcal{C}}_m$ for any odd natural number m, we will need some further definitions and lemmas.

Recall that subdirectly irreducible members of $\widetilde{\mathcal{C}}_m$ are subgroupoids of $\mathbb{Z}_m^\infty$. In particular, these are groupoids $\mathbb{Z}_{p^j}$ and $\mathbb{Z}_{p^j}^\infty$ for p^j dividing m, and the 2-element semilattice $\mathbf{2}_s$. Moreover

$$\mathsf{Q}(\mathbb{Z}_m^\infty) = \mathsf{SP}(\mathbb{Z}_m^\infty) = \mathsf{V}(\mathbb{Z}_m^\infty) = \widetilde{\mathcal{C}}_m.$$

Recall that, when we write $m = p_1^{j_1} \dots p_r^{j_r}$, this means that m is the product of powers of pairwise distinct odd prime numbers p_i. We will not repeat this assumption later on. Let us first note one more well-known property of $\widetilde{\mathcal{C}}_m$.

Lemma 4.2. *Let $m = p_1^{j_1} \dots p_r^{j_r}$. Then*

$$\widetilde{\mathcal{C}}_m = \mathsf{SP}(\mathbb{Z}^\infty_{p_1^{j_1}}, \dots, \mathbb{Z}^\infty_{p_r^{j_r}}).$$

Note that the varieties $\mathcal{C}_m$ are determined by their maximal subdirectly irreducible members. We will show that the quasivarieties of $\mathcal{CBM}$-groupoids have a similar property.

Lemma 4.3. *Let $m = p_1^{j_1} \dots p_r^{j_r}$. Let $\mathcal{Q}$ be a subquasivariety of the regularization $\widetilde{\mathcal{C}}_m$ not contained in $\widetilde{\mathcal{C}}_n$ for a proper divisor n of m. Then $\mathcal{Q}$ contains $\widetilde{\mathcal{C}}_m^q$. Moreover, $\mathcal{Q}$ is generated by the subdirectly irreducible $\mathcal{C}_m$-groupoids $\mathbb{Z}_{{p_i}^{j_i}}$ for $i = 1, \dots, r$ and a subset of subdirectly irreducible $\widetilde{\mathcal{C}}_m$-groupoids of the form $\mathbb{Z}^\infty_{{p_i}^{k_i}}$, where $k_i \leq j_i$.*

Proof: By Lemma 3.2, $\widetilde{\mathcal{C}}_m^q = \mathsf{SP}(\mathbb{Z}_{p_1^{j_1}}, \dots, \mathbb{Z}_{p_r^{j_r}}, \mathbf{2}_s)$, and by Lemma 4.2, $\widetilde{\mathcal{C}}_m = \mathsf{SP}(\mathbb{Z}^\infty_{p_1^{j_1}}, \dots, \mathbb{Z}^\infty_{p_r^{j_r}})$. Since $\mathcal{Q}$ is not contained in $\widetilde{\mathcal{C}}_n$ for divisors n of m, it follows that $\mathcal{Q}$ contains all subdirectly irreducibles $\mathbb{Z}_{p_1^{j_1}}, \dots, \mathbb{Z}_{p_r^{j_r}}, \mathbf{2}_s$, and hence also the quasivariety $\widetilde{\mathcal{C}}_m^q$.

Now assume that $\widetilde{\mathcal{C}}_m^q < \mathcal{Q} < \widetilde{\mathcal{C}}_m$. Let A belong to $\mathcal{Q}$. As a member of $\widetilde{\mathcal{C}}_m$, the groupoid A is a Płonka sum $\sum_{s\in S} A_s$ of $\mathcal{C}_m$-groupoids A_s. The Płonka sum A embeds into the product of some A_s^∞ and some A_t for $s,t \in S$ (cf. [12, Theorem 4.3.5]). On the other hand, each summand of A embeds into a product of subdirectly irreducible members of $\mathcal{C}_m$. It follows that the groupoid A embeds into a product of some $\mathbb{Z}_{{p_i}^{j_i}}$ and some $\mathbb{Z}^\infty_{{p_i}^{k_i}}$, where $i \in \{1, \dots, r\}$. Our assumptions require that the summands of all such groupoids A run over all subdirectly irreducibles $\mathbb{Z}_{p_i^{j_i}}$ for $i = 1, \dots, r$. Consequently, the quasivariety $\mathcal{Q}$ coincides with the quasivariety generated by all $\mathbb{Z}_{{p_i}^{j_i}}$ and a subset of $\mathbb{Z}^\infty_{{p_i}^{k_i}}$, where $k_i \leq j_i$. $\square$

For a prime number $p \geq 3$ and a positive integer j, consider the following quasi-identity

$$(\beta_{p,j}) \qquad (x \star_{p^{j+1}} y = x \,\&\, y \star_{p^{j+1}} x = y \,\&\, x \cdot z = z \,\&\, y \cdot z = z) \\ \to (x \star_{p^j} y = x).$$

Additionally, define

$$(\beta_{p,0}) \qquad (x \star_p y = x \,\&\, y \star_p x = y \,\&\, x \cdot z = z \,\&\, y \cdot z = z) \\ \to (y = x).$$

Note that if elements x, y, z of a $\mathcal{CBM}$-groupoid A satisfy the premises of $(\beta_{p,j})$, then x, y, z belong to a subgroupoid of A isomorphic to a subgroupoid of $\mathbb{Z}^{\infty}_{p^{j+1}}$ with x, y in a subgroupoid isomorphic to $\mathbb{Z}_{p^{j+1}}$ and $z = \infty$. Then the satisfaction of $(\beta_{p,j})$ means in fact that the latter subgroupoid must be isomorphic to $\mathbb{Z}_{p^j}$. Let us also note that $\mathbb{Z}^{\infty}_{p_i{}^0} = \mathbf{2}_s$. The following result is readily verified.

Lemma 4.4. *Let i, j be positive integers. Then $\mathbb{Z}_{p^j} \models (\beta_{p,i})$, $\mathbb{Z}^{\infty}_{p^i} \models (\beta_{p,i})$ and $\mathbb{Z}^{\infty}_{p^{i+1}} \not\models (\beta_{p,i})$. In particular, for $i = 0, 1, \dots, j$,*

$$\mathcal{Q}(\mathbb{Z}_{p^j}, \mathbb{Z}^{\infty}_{p^i}) \models (\beta_{p,i}),$$

and

$$\mathcal{Q}(\mathbb{Z}_{p^j}, \mathbb{Z}^{\infty}_{p^{i+1}}) \not\models (\beta_{p,i}).$$

Moreover (α_{p^j}) implies all $(\beta_{p,i})$, and $(\beta_{p,i})$ implies $(\beta_{p,i+1})$.

In what follows, if a subquasivariety of $\widetilde{\mathcal{C}}_m$ is defined by quasi-identities $\sigma_1, \dots, \sigma_n$, then we will denote it by $\mathsf{Q}(\sigma_1, \dots, \sigma_n)$.

Lemma 4.5. *Let $p \geq 3$ be a prime number and $j \geq 2$ an integer. Then for $i = 1, \dots, j$,*

$$\begin{aligned} \widetilde{\mathcal{C}}^q_{p^j} &= \mathsf{Q}(\mathbb{Z}_{p^j}, \mathbf{2}_s) = \mathsf{Q}(\mathbb{Z}_{p^j}, \mathbb{Z}^{\infty}_{p^0}) < \mathsf{Q}(\mathbb{Z}_{p^j}, \mathbb{Z}^{\infty}_{p^1}) < \dots \\ &\dots < \mathsf{Q}(\mathbb{Z}_{p^j}, \mathbb{Z}^{\infty}_{p^i}) < \dots < \mathsf{Q}(\mathbb{Z}_{p^j}, \mathbb{Z}^{\infty}_{p^j}) = \widetilde{\mathcal{C}}_{p^j}. \end{aligned}$$

Each quasivariety $\mathsf{Q}(\mathbb{Z}_{p^j}, \mathbb{Z}^{\infty}_{p^i})$ is defined by the quasi-identity $(\beta_{p,i})$. Moreover, the quasivarieties $\mathsf{Q}(\mathbb{Z}_{p^j}, \mathbb{Z}^{\infty}_{p^i})$, for $i = 1, \dots, j$, form a strictly increasing chain of pairwise distinct subquasivarieties of the regularization $\widetilde{\mathcal{C}}_{p^j}$ properly containing the quasiregularization $\widetilde{\mathcal{C}}^q_{p^j}$.

Proof: Lemma 4.3 implies that the quasivarieties $\mathsf{Q}(\mathbb{Z}_{p^j}, \mathbb{Z}^{\infty}_{p^i})$ form a chain, and Lemma 4.4 shows that the chain is strictly increasing. Clearly, $\mathsf{Q}(\mathbb{Z}_{p^j}, \mathbb{Z}^{\infty}_{p^i}) \subseteq \mathsf{Q}(\beta_{p,i})$. Now, if a groupoid A belongs to $\mathsf{Q}(\beta_{p,i})$, then A, as a member of $\widetilde{\mathcal{C}}_{p^j}$, is a Płonka sum $\sum_{s \in S} A_s$ of $\mathcal{C}_{p^j}$-groupoids A_s. Similarly as in the proof of Lemma 4.3, one shows that A embeds into the product of some $\mathbb{Z}_{p^l}$ for $l = 1, \dots, j$ and some $\mathbb{Z}^{\infty}_{p^k}$, where $k \leq j$. Since A satisfies $(\beta_{p,i})$, it follows by Lemma 4.4 that there are such A that embed into a product containing $\mathbb{Z}_{p^j}$ and such that embed into a product containing $\mathbb{Z}^{\infty}_{p^i}$, but not $\mathbb{Z}^{\infty}_{p^{i+1}}$. Hence $\mathsf{Q}(\beta_{p,i}) \subseteq \mathsf{SP}(\mathbb{Z}_{p^j}, \mathbb{Z}^{\infty}_{p^i}) = \mathsf{Q}(\mathbb{Z}_{p^j}, \mathbb{Z}^{\infty}_{p^i})$. $\square$

Let us observe that, unlike in the case $j = 1$, in the case $j \geq 2$, there are quasivarieties properly contained between the quasi-regularization and the regularization of a given variety $\mathcal{C}_{p^j}$. If m is a product of distinct prime numbers, the situation is more complicated.

Lemma 4.6. *Let $m = p_1^{j_1} \dots p_r^{j_r}$. Let $p = p_i, j = j_i$ and $q = p_{i'}, l = j_{i'}$ for some $i, i' = 1, \dots, r$. Then for $s = 1, \dots, j$ and $t = 1, \dots l$, we have*

$$\mathbb{Z}_{q^l}, \mathbb{Z}_{q^t}^{\infty} \models (\beta_{p,s}) \quad \text{and} \quad \mathbb{Z}_{p^j}, \mathbb{Z}_{p^s}^{\infty} \models (\beta_{q,t}).$$

Proof: To show the lemma, it is enough to note that no elements of $\mathbb{Z}_{q^l}, \mathbb{Z}_{q^t}^{\infty}$ satisfy the premises of $(\beta_{p,s})$, and similarly no elements of $\mathbb{Z}_{p^j}, \mathbb{Z}_{p^s}^{\infty}$ satisfy the premises of $(\beta_{q,t})$. □

Let $m = p_1^{j_1} \dots p_r^{j_r}$. Let $\mathcal{Q}$ be a subquasivariety of $\widetilde{\mathcal{C}}_m$ containing $\widetilde{\mathcal{C}}_m^q$. Then $\mathcal{Q}$ is determined by its maximal subdirectly irreducible algebras of the form $\mathbb{Z}_{p_i{}^{s_i}}^{\infty}$, where $s_i \in \{0, 1, \dots, j_i\}$. We will sometimes refer to such algebras as *extended subdirectly irreducibles*. Recall that $\mathbb{Z}_{p_i{}^0}^{\infty} = \mathbf{2}_s$. To each quasivariety $\mathcal{Q}$ we will assign the sequence $(s_1, \dots, s_r)$ of the powers s_i of p_i in maximal subdirectly irreducible groupoids of $\mathcal{Q}$. We will denote such a quasivariety $\mathcal{Q}$ by $\mathsf{Q}(s_1, \dots, s_r)$. Note that

$$\mathsf{Q}(0, \dots, 0) = \widetilde{\mathcal{C}}_m^q,$$

and

$$\mathsf{Q}(j_1, \dots, j_r) = \widetilde{\mathcal{C}}_m.$$

Lemma 4.7. *Let $m = p_1^{j_1} \dots p_r^{j_r}$. Then for a fixed $i \in \{1, \dots, r\}$, one has*

$$\begin{aligned} &\mathsf{Q}(j_1, \dots, j_{i-1}, 0, j_{i+1}, \dots, j_r) < \\ &\mathsf{Q}(j_1, \dots, j_{i-1}, 1, j_{i+1}, \dots, j_r) < \dots < \mathsf{Q}(j_1, \dots, j_{i-1}, s_i, j_{i+1}, \dots, j_r) < \dots \\ &\dots < \mathsf{Q}(j_1, \dots, j_{i-1}, j_i, j_{i+1}, \dots, j_r) = \widetilde{\mathcal{C}}_m. \end{aligned}$$

Each quasivariety $\mathsf{Q}(j_1, \dots, j_{i-1}, s_i, j_{i+1}, \dots, j_r)$, where $s_i = 0, 1, \dots, j_i$, is defined by (β_{p_i,s_i}). The quasivarieties $\mathsf{Q}(j_1, \dots, j_{i-1}, s_i, j_{i+1}, \dots, j_r)$ form a strictly increasing chain of subquasivarieties of the regularization $\widetilde{\mathcal{C}}_m$ containing the quasivariety $\mathsf{Q}(j_1, \dots, j_{i-1}, 0, j_{i+1}, \dots, j_r)$.

The proof follows by Lemmas 4.3, 4.4 and 4.6, and is similar to the proof of Lemma 4.5.

Lemma 4.8. *Let $m = p_1^{j_1} \dots p_r^{j_r}$. Then each proper subquasivariety $\mathsf{Q}(s_1, \dots, s_r)$ of the regularization $\widetilde{\mathcal{C}}_m$ is defined by the set of quasi-identities (β_{p_i,s_i}) for $i = 1, \dots, r$. Moreover, the quasi-identity (α_m) is equivalent to the set of quasi-identities $(\beta_{p_i,0})$ for $i = 1, \dots, r$.*

Proof: It is clear that the $(\beta_{p_i,0})$ imply (α_m). On the other hand, by Lemma 3.2, $\widetilde{\mathcal{C}}_m^q$ does not contain extended subdirectly irreducibles. □

Example 4.9. Let us consider the variety $\widetilde{\mathcal{C}}_{15}$. Analysing the subdirectly irreducible members of this variety, one can check that the lattice of subquasivarieties of $\widetilde{\mathcal{C}}_{15}$ has 13 members ordered as in Figure 3. There are two quasivarieties properly contained between $\widetilde{\mathcal{C}}^q_{15}$ and $\widetilde{\mathcal{C}}_{15}$, the quasivariety $\mathcal{Q}(\mathbb{Z}_3^\infty, \mathbb{Z}_5) = \mathsf{Q}(1,0)$ defined by $(\beta_{5,0})$ and the quasivariety $\mathcal{Q}(\mathbb{Z}_3, \mathbb{Z}_5^\infty) = \mathsf{Q}(0,1)$ defined by $(\beta_{3,0})$.

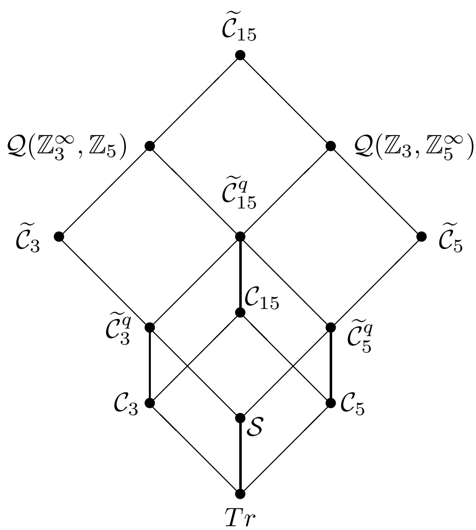

FIGURE 3. Subquasivarieties of $\widetilde{\mathcal{C}}_{15}$

Example 4.9 may easily be extended to the case where m is a product of pairwise distinct prime numbers. Let $\mathcal{L}^{>q}(\widetilde{\mathcal{C}}_m)$ denote the lattice of subquasivarieties of $\widetilde{\mathcal{C}}_m$ containing $\widetilde{\mathcal{C}}^q_m$.

Proposition 4.10. *Let $m = p_1 \dots p_r$. Then a typical quasivariety contained in $\widetilde{\mathcal{C}}_m$ and containing $\widetilde{\mathcal{C}}^q_m$ has the form $\mathsf{Q}(s_1, \dots, s_r)$, where each s_i equals 0 or 1. Such a quasivariety is defined by the set of quasi-identities $(\beta_{p_i,0})$ for all i such that $s_i = 0$. Then the lattice $\mathcal{L}^{>q}(\widetilde{\mathcal{C}}_m)$ is isomorphic to the Boolean lattice of divisors of m.*

Proposition 4.11. *Let $m = p_1^{j_1} \dots p_r^{j_r}$. Then the lattice $\mathcal{L}^{>q}(\widetilde{\mathcal{C}}_m)$ is isomorphic to the lattice of all divisors of m.*

PROOF: Note that each member of $\mathcal{L}^{>q}(\widetilde{\mathcal{C}}_m)$ has the form $\mathsf{Q}(s_1, \dots, s_r)$, where $0 \leq s_i \leq j_r$. The mapping $\mathsf{Q}(s_1, \dots, s_r) \mapsto p_1^{s_1} \dots p_r^{s_r}$ provides the required isomorphism. $\square$

The two following lemmas provide the final missing component of our description of the lattice $\mathcal{L}(\widetilde{\mathcal{C}}_m)$.

Lemma 4.12. *Let* $m = {p_1}^{j_1} \dots {p_i}^{j_i} \dots {p_r}^{j_r}$ *and* $m' = {p_1}^{j_1} \dots {p_i}^{j_i+1} \dots {p_r}^{j_r}$*. Then the lattice* $\mathcal{L}^{>q}(\widetilde{\mathcal{C}}_m)$ *embeds into the lattice* $\mathcal{L}^{>q}(\widetilde{\mathcal{C}}_{m'})$*.*

Proof: Let $\mathsf{Q}^m(i_1, \dots, i_r)$, where $i_k \leq j_k$, denote a typical member of the lattice $\mathcal{L}^{>q}(\widetilde{\mathcal{C}}_m)$, and $\mathsf{Q}^{m'}(i_1, \dots, i_r)$ a typical member of $\mathcal{L}^{>q}(\widetilde{\mathcal{C}}_{m'})$. Then the embedding is defined by $\mathsf{Q}^m(i_1, \dots, i_r) \mapsto \mathsf{Q}^{m'}(i_1, \dots, i_r)$. □

Lemma 4.13. *Let* $m = {p_1}^{j_1} \dots {p_r}^{j_r}$ *and* $m' = {p_1}^{j_1} \dots {p_r}^{j_r} {p_{r+1}}^{j_{r+1}}$*. Then the lattice* $\mathcal{L}^{>q}(\widetilde{\mathcal{C}}_m)$ *embeds into the lattice* $\mathcal{L}^{>q}(\widetilde{\mathcal{C}}_{m'})$*.*

Proof: As before, let $\mathsf{Q}^m(i_1, \dots, i_r)$, where $i_k \leq j_k$, denote a typical member of the lattice $\mathcal{L}^{>q}(\widetilde{\mathcal{C}}_m)$, and $\mathsf{Q}^{m'}(i_1, \dots, i_r, i_{r+1})$ a typical member of $\mathcal{L}^{>q}(\widetilde{\mathcal{C}}_{m'})$. Then the required embedding is defined by $\mathsf{Q}^m(i_1, \dots, i_r) \mapsto \mathsf{Q}^{m'}(i_1, \dots, i_r, 0)$. □

Now we are ready to describe the lattice $\mathcal{L}_q(\widetilde{\mathcal{C}}_m)$ of subquasivarieties of $\widetilde{\mathcal{C}}_m$. First let us define a certain new lattice $K(m)$. For $m = {p_1}^{m_1} \dots {p_r}^{m_r}$, the set $K(m)$ is the set of functions

$$f : \{0, 1, \dots, 2r\} \to \mathbb{N}$$

satisfying the following conditions:

- $f(0) \in \{0, 1\}$,
- for all $i = 1, \dots, r$, one has $f(i) = j_i$, where $0 \leq j_i \leq m_i$,
- if $f(0) = 0$, then for all $i = r+1, \dots, 2r$, one has $f(i) = 0$,
- if $f(0) = 1$, then for all $i = r+1, \dots, 2r$, one has $f(i) = s_{i-r}$, where $0 \leq s_{i-r} \leq j_i$.

It is easy to see that $K(m)$ is an ordered set with bounds $(0, \dots, 0)$ and $(1, m_1, \dots, m_r, m_1, \dots$

The following lemma is immediate.

Lemma 4.14. *The set* $K(m)$ *is a distributive lattice with respect to the following operations:*

$$(f \vee g)(i) = \max\{f(i), g(i)\},\ (f \wedge g)(i) = \min\{f(i), g(i)\},$$

where $i \in \{0, 1, \dots, 2r\}$*.*

Theorem 4.15. *The lattice* $\mathcal{L}_q(\widetilde{\mathcal{C}}_m)$ *of subquasivarieties of the regularization* $\widetilde{\mathcal{C}}_m$ *is isomorphic to the lattice* $K(m)$*.*

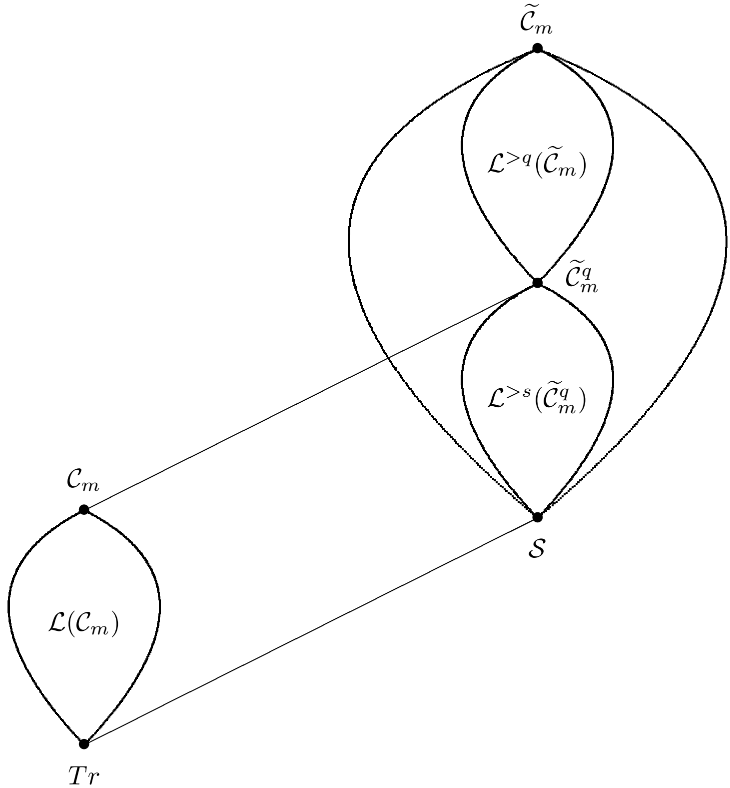

FIGURE 4. Subquasivarieties of $\widetilde{\mathcal{C}}_m$

PROOF: The required isomorphism h is described as follows. Let $n = {p_1}^{j_1} \dots {p_r}^{j_r}$ be a divisor of m. Let $0 \leq s_i \leq j_i$, for $i = 1, \dots, r$. Then

$$\begin{aligned}
\mathcal{C}_n &\mapsto (0, j_1, \dots, j_r, 0, \dots, 0),\\
\widetilde{\mathcal{C}}_n^q &\mapsto (1, j_1, \dots, j_r, 0, \dots, 0),\\
\widetilde{\mathcal{C}_n} &\mapsto (1, j_1, \dots, j_r, j_1, \dots, j_r),\\
\mathsf{Q}(s_1, \dots, s_r) &\mapsto (1, j_1, \dots, j_r, s_1, \dots s_r).
\end{aligned}$$

The proof follows by the Propositions 4.10 and 4.11 and Lemmas 4.12, 4.13 and 4.14. □

Figure 4 provides a large scale view of the lattice $\mathcal{L}_q(\widetilde{\mathcal{C}}_m)$ of subquasivarieties of $\widetilde{\mathcal{C}}_m$.

To conclude the paper, let us note that the method we used to describe the lattice of subquasivarieties of the regularization of a variety $\mathcal{C}_m$ is in fact quite general, and can be applied to any irregular deductive variety generated by a finite number of finite subdirectly irreducible algebras.

The next step of our investigations should be a description of the lattice of subquasivarieties of the quasi-regularization of $\mathcal{CBM}_{cl}$. This would require different methods.

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(*Received* October 26, 2013)