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The Cayley graph and the growth of Steiner loops

P. Plaumann, L. Sabinina, I. Stuhl

Abstract. We study properties of Steiner loops which are of fundamental importance to develop a combinatorial theory of loops along the lines given by Combinatorial Group Theory. In a summary we describe our findings.

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In this note we define a *Steiner loop* as a loop that is satisfying the relations

$$xy = yx, \quad (xy)y = x \tag{1}$$

(see e.g. [10, Theorem V.1.1 and Definition V.1.9]). The variety of all diassociative loops of exponent 2 is precisely the variety of all Steiner loops. These loops are in a one-to-one correspondence with Steiner triple systems (see [5, p. 310]).

A *Steiner triple system* is an incidence structure consisting of points and blocks such that every two distinct points are contained in precisely one block, and any block consists of precisely three points. Due to this well-known connection between Steiner loops and Steiner triple systems the number of elements of a finite Steiner loop is congruent $2\, mod\, 6$ or $4\, mod\, 6$ (see [2]).

Steiner loops form a variety $\mathfrak{S}$ which even is a Schreier variety (see e.g., [3], [8]). Denote by $\mathfrak{M}_1$ the variety of all magmas with a neutral element. One can identify the elements of $\mathfrak{S}$ with elements of $\mathfrak{M}_1$ satisfying the relations (1).

Consider a free magma $\mathsf{M} = \mathsf{M}(X) \in \mathfrak{M}_1$ with a countable base X. The elements of M are defined recursively by their length. Put

$$\begin{aligned}
&\mathsf{M}(0) = 1,\\
&\mathsf{M}(i) = \{w \in \mathsf{M} \mid \exists r, s \in \mathbb{N}, w_L \in \mathsf{M}(r), w_R \in \mathsf{M}(s) : r + s = i, w = w_L w_R\}.
\end{aligned}$$

We call w_L the left part of w and w_R the right part of w. Inductively, we define

$$\begin{aligned}
\mathsf{T}_0 &= \mathsf{M}(0),\\
\mathsf{T}_{i+1} &= \{w \in \mathsf{M} \mid \exists (x \in X, v \in \mathsf{T}_i) : w = xv\}
\end{aligned}$$

and call the elements of $\mathsf{T} = \bigcup_{i\in\mathbb{N}} \mathsf{T}_i$ *right normed words*. Denote them by $\rho(u_1, u_2, \ldots, u_i)$. Note, that the elements of T are in a one-to-one correspondence with the elements of the free monoid over X.

For a finitely generated free magma with a neutral element one has the following

Proposition 1. *Let* $X = \{x_1, x_2, \ldots, x_d\}$ *be such that* $3 \leq d < \infty$ *and let* $\mathsf{M} = \mathsf{M}(X)$ *be a free magma over* X*. Then for* $n > 0$ *the following statements hold:*

(i) $|\mathsf{M}(n)| = \sum_{i=0}^{n} |\mathsf{M}(i)||\mathsf{M}(n-i)|$,
(ii) $|\mathsf{T}_{n+1}(\mathsf{M})| = d \cdot |\mathsf{T}_n(\mathsf{M})|$.

In particular, the sequences $(|\mathsf{M}(n)|)_{n\in\mathbb{N}}$ *and* $(|\mathsf{T}_n(\mathsf{M})|)_{n\in\mathbb{N}}$ *both have exponential growth.*

Proof: The recursion formulas (i) and (ii) follow immediately from our definitions. Putting $\gamma_k = \frac{1}{d^k}|\mathsf{M}(k)|$ it follows from (i) that the sequence $\left(\gamma_n\right)_{n\in\mathbb{N}}$ satisfies the well-known recursion formula for the Catalan numbers. But this sequence has exponential growth $4^n/\sqrt{\pi n^3}$ (see [4, p. 39]). Hence the sequence $(|\mathsf{M}(n)|)_{n\in\mathbb{N}}$ has exponential growth, too. The fact that the sequence in (ii) has exponential growth is trivial. $\square$

We denote the free Steiner loop of finite rank d by S_d. For an arbitrary Steiner loop S and a finite subset $E \subseteq S$ one can define the Cayley graph $\mathsf{Cay}(S, E)$ without using in detail the construction given for loops by G. Sabidoussi [11]. We shall prove the following

Theorem 1. *For* $d \geq 3$ *the Cayley graph of the free Steiner loop* S_d *with respect to a basis is not connected.*

The approach we use to prove Theorem 1 will help us to prove

Theorem 2. *For* $d \geq 3$ *the free Steiner loop* S_d *has exponential growth.*

Now we begin to prove Theorem 1. The multiplication group $\mathsf{Mult}(\mathsf{S}_d)$ and the inner mapping group $\mathcal{J}(\mathsf{S}_d)$ of a finitely generated free Steiner loop S_d are determined in [6, Proposition 2] and for some finite Steiner loops in [12]. First one notes, that for every Steiner loop S the group $\mathsf{Mult}(S)$ is generated by the set $\{L_v \mid v \in S\}$ of (left) translations, where $L_v : S \to S$ is the mapping defined by $L_v(x) = vx$. Due to the relations which are valid in Steiner loops the inner mapping group $\mathcal{J}(S)$ (see [1, p. 61]) is generated by the mappings $\ell_{u,v} = L_{uv}L_uL_v \in \mathsf{Mult}(S)$. For elements $u \neq v$ of a free Steiner loop S_d the inner mappings $\ell_{u,v}$ have infinite order. We denote by $\mathfrak{L}(\mathsf{S}_d)$ the set of cyclic subgroups of $\mathcal{J}(\mathsf{S}_d)$ generated by the elements $\ell_{u,v}$ for $u \neq v \in \mathsf{S}_d$.

In [6] it is shown that for $d > 2$ the multiplication group $\mathsf{Mult}(\mathsf{S}_d)$ of the free Steiner loop S_d is the free product of cyclic groups of order 2

$$\mathsf{Mult}(\mathsf{S}_d) = \bigstar_{v\in\mathsf{S}_d} C_v,$$

where $C_v = \langle L_v \rangle$. Furthermore, it is shown there that the inner mapping group $\mathcal{J}(\mathsf{S}_d)$ is the free product of the set $\mathfrak{L}(\mathsf{S}_d)$ of subloops.

For a given loop Q and a subset $E \subseteq Q$ we consider the subgroup

$$\mathsf{Mult}_E(Q) = \left\langle L_a \mid a \in E \right\rangle$$

of the group $\mathsf{Mult}(Q)$.

In [11] the construction of the Cayley graph $\mathsf{Cay}(Q,E)$ of a quasigroup Q with respect to a finite subset $E \subseteq Q$ was introduced (see also [9]). Considering an arbitrary loop for this purpose the following properties of E are needed

$$(\mathbf{C1})\,\forall x \in Q : x \notin xE, \quad (\mathbf{C2})\,\forall x \in Q, u \in E : x \in x(uE).$$

A subset satisfying $(\mathbf{C1})$ and $(\mathbf{C2})$ is called a *Cayley set*. Dealing with an (IP)–loop these conditions are equivalent to

$$(\mathbf{C1})^*\, 1 \notin E, \quad (\mathbf{C2})^*\, E = E^{-1}.$$

Note, that for a Steiner loop the condition $(\mathbf{C2})^*$ is satisfied for every subset. Hence every finite subset not containing the identity of a Steiner loop is a Cayley set.

For a Steiner loop S with a Cayley set E one defines the Cayley graph $\mathsf{Cay}(S,E)$ as the graph with the vertex set $\mathcal{V}_{S,E} = S$ calling $u, v \in \mathcal{V}_{S,E}$ adjacent if there is $x \in E$ such that $v = xu$. For $u \in \mathcal{V}_{S,E}$ we denote the connected component of u in $\mathsf{Cay}(S,E)$ by $\mathcal{C}_{S,E}(u)$.

Immediately from the definitions follows

Proposition 2. *For a Steiner loop S with a finite Cayley set E which generates S and for $a \in S$ the component of a in $\mathsf{Cay}(S,E)$ is the orbit $\mathsf{Mult}_E(S)(a)$.*

For the *associator* of elements u, v, w of a Steiner loop one has

$$(u,v,w) = (uv)w \cdot u(vw) = w(uv) \cdot u(vw). \tag{2}$$

Proposition 3. *For a free Steiner loop S_d with a basis B the set $\mathcal{C}_{\mathsf{S}_d,B}(1)$ consists of the elements of S which are an irreducible right normed word over B. In particular, for $d > 2$ not every element of S_d belongs to $\mathcal{C}_{\mathsf{S}_d,B}(1)$.*

PROOF: The vertices in $\mathcal{C}_{\mathsf{S}_d,B}(1)$ are of the form $L_{a_n} \dots L_{a_2} L_{a_1} 1$ for a finite sequence $a_1, a_2, \dots, a_n$ in B. If a, b, c are 3 different elements of B then $(ab)(bc)$ and (a,b,c) do not belong to $\mathcal{C}_{\mathsf{S}_d,B}(1)$. $\square$

Proposition 3 proves Theorem 1. Furthermore, Propositions 2 and 3 can be used to treat algorithmic questions in Steiner loops which are known to have positive answers in this category by Evans [3]:

(a) Determine a normal form of the elements in S_d.
(b) Give an algorithm deciding the word problem for S_d.

If a Cayley set satisfies the condition

$$(\mathbf{C3})\,\forall x, y \in Q : (xy)E = x(yE),$$

it is called *quasi–associative*. In [9, Proposition 8], it is shown that for a quasi–associative Cayley set E of a loop Q the graph $\mathsf{Cay}(Q,E)$ is vertex transitive.

Proposition 4. *In a free Steiner loop S_d, $d>2$, there is no finite quasi-associative Cayley set.*

PROOF: In an arbitrary loop S the condition (**C3**) is equivalent to

$$(\mathbf{C3})^* \ \forall x,y\in S: \ell_{x,y}E=E.$$

Above we had seen that for $d>2$ and $1\neq u\in\mathsf{S}_d$ the orbit $\mathcal{J}(\mathsf{S}_d)u$ is infinite. □

PROOF OF THEOREM 2: To discuss the growth of a Steiner loop S with respect to a finite generating set $B=\{b_1,\dots,b_d\}$ we recall that S is a magma with a neutral element satisfying the relations $XY=YX$, $X(XY)=Y$. For a free magma M_d with a neutral element over the basis $\{x_1,\dots,x_d\}$ one has an epimorphism $\psi:\mathsf{M}_d\twoheadrightarrow S$ such that $\psi(x_i)=b_i$ for $1\le i\le d$. For $s\in S$ define

$$\lambda(s)=\lambda_{S,B}(s)=\min_i\{i\mid\exists w\in\mathsf{M}(i): s=\psi(w)\}. \tag{3}$$

and

$$a(n)=a_{S,B}(n)=|\{s\in S\mid\lambda(s)=n\}|. \tag{4}$$

It follows from Proposition 1(i) that $a_{S,B}(n)\le d^n\gamma_n$ for all $n\in\mathbb{N}$ where γ_n are the Catalan numbers.

Conversely, denote by M_d a free magma with a neutral element over the basis $X=\{x_1,x_2,\dots,x_d\}$. Let S_d be a free Steiner loop with a basis $B=\{b_1,b_2,\dots,b_d\}$ and let $\psi:\mathsf{M}_d\twoheadrightarrow\mathsf{S}_d$ be the epimorphism given by $\psi(x_i)=b_i$ for all $1\le i\le d$.

Then ψ maps $\mathsf{T}_k(\mathsf{M}_d)$ onto $\mathsf{T}_k(\mathsf{S}_d)$ for all $k\in\mathbb{N}$. By construction all elements of $\mathsf{T}(\mathsf{S}_d)$ have a unique presentation $\rho(c_1,c_2,\dots,c_n)$ as a right normed word such that $\forall 1\le i\le n: c_i\in B$ and $\forall 1\le i\le n-1: c_i\neq c_{i+1}$. It follows from Proposition 1 (ii) that

$$|\mathsf{T}_0(\mathsf{S}_d)|=1, |\mathsf{T}_1(\mathsf{S}_d)|=d,$$
$$|\mathsf{T}_k(\mathsf{S}_d)|=d(d-1)^{k-1} \text{ for } k\ge 2.$$

Thus we obtain

$$d(d-1)^{k-1}=|\mathsf{T}_k(\mathsf{S}_d)|<a_{\mathsf{S}_d,B}(k)\le d^k\gamma_k$$

for all $k\ge 3$: S_d has exponential growth. □

Of course, the growth function $\big(a_{S,B}(n)\big)_{n\in\mathbb{N}}$ depends on the generating set B. But the usual equivalence relation for sequences of numbers (see [7, Proposition 1.4]) yields the result that all growth functions of a loop belong to the same equivalence class.

Summary. The variety of Steiner loops shows that growth functions can be well treated for loops. However, so far we have no convincing examples showing how the growth of a loop L is connected with algebraic properties of L. One obstacle for this task lies in the fact that many tools used in groups for this purpose are not available for loops. Deeper questions like the existence of loops of intermediate growth remain open even for Steiner loops. We have to admit that the only finitely generated Steiner loops that we know are finite or have exponential growth. The following question stays open.

Open question: *Are there Steiner loops with polynomial growth of arbitrary degree?*

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Escuela de Ciencias, Universidad Autónoma Benito Juárez de Oaxaca, Av. Universidad S/N, Ex - Hacienda de 5 Señores, Oaxaca, México. C.P. 68120, México
and

Universität Erlangen-Nürnberg, Department Matematik, Cauerstrasse 11, 91058 Erlangen, Germany

E-mail: peter.plaumann@mi.uni-erlangen.de

Facultad de Ciencias, Universidad Autónoma del Estado de Morelos, Av. Universidad 1001, Col. Chamilpa, Cuernavaca Morelos, 62209, México

E-mail: liudmila@uaem.mx

Institute of Mathematics and Statistics, University of São Paulo, R. do Matao 1010, São Paulo, SP, 05508-090, Brazil
and
University of Debrecen, Egyetem tér 1, H-4010 Debrecen, Hungary

E-mail: izabella@ime.usp.br

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