Nonconvex Lipschitz function in plane which is locally convex outside a discontinuum

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Abstract. We construct a Lipschitz function on $\mathbb{R}^2$ which is locally convex on the complement of some totally disconnected compact set but not convex. Existence of such function disproves a theorem that appeared in a paper by L. Pasqualini and was also cited by other authors.

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1. Introduction

In his work from 1938 L. Pasqualini presents a theorem (see [4, Theorem 51, p. 43]) of which the following statement is a reformulation:

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function and $M \subset \mathbb{R}^d$ a set not containing any continuum of topological dimension $(d-1)$. Suppose that f is locally convex on the complement of M. Then f is convex on $\mathbb{R}^d$.

The proof however contains a gap. This result also appeared in the survey paper [1], where the (incorrect) proof was shortly repeated. Also V.G. Dmitriev mentions this result in [2], although he provides a wrong reference.

As a counterexample to the theorem of Pasqualini we present the following theorem:

Theorem 1.1. *There is a Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$ and $M \subset \mathbb{R}^2$ such that*

- *f is locally convex on $\mathbb{R}^2 \setminus M$,*
- *f is not convex on $\mathbb{R}^2$,*
- *M is compact and totally disconnected,*
- *f has compact support.*

Note that it is a simple observation that the set M from Theorem 1.1 cannot be of one dimensional Hausdorff measure 0.

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2. Preliminaries

In the paper we will use the following more or less standard notation and definitions. For $a, b \in \mathbb{R}^d$ and $r > 0$ we will denote by $B(a, r)$ the closed ball with center a and radius r and $[a, b]$ will denote the closed line segment with endpoints a and b. For $A \subset \mathbb{R}^d$ the symbol $\operatorname{co} A$ will mean the convex hull of A and A^c will mean the complement of A. If $l \subset \mathbb{R}^2$ is a line and $\varepsilon > 0$ then we define $l(\varepsilon) = \{x \in \mathbb{R}^2 : \operatorname{dist}(x, l) < \varepsilon\}$.

A function f defined on a set $A \subset \mathbb{R}^2$ is called L-Lipschitz, if for every $x, y \in A$, $x \neq y$, we have $|f(x) - f(y)| \leq L|x - y|$.

We will call f locally convex on A if for every x, y such that $[x, y] \subset A$ and $\alpha \in [0, 1]$ we have $f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$.

Finally, f will be called piecewise affine on A if there is a locally finite triangulation Δ of A such that f is affine on every triangle from Δ.

3. Construction of the function

Definition 3.1. Let $\mathcal{Q}$ be the system of all unions of finite systems of (closed) polytopes in $\mathbb{R}^2$. Let $L > 0$, $f : \mathbb{R}^2 \to \mathbb{R}$ and $P \in \mathcal{Q}$. We say that a pair (P, f) is L-good if

(1) f is L-Lipschitz,
(2) f is piecewise affine on P^c,
(3) f is locally convex on P^c.

The key technical result is the following:

Lemma 3.2. *Let $\delta, \varepsilon, L > 0$ and let l be a line in $\mathbb{R}^2$. Let (P, g) be an L-good pair. Then there is an $(L + \varepsilon)$-good pair (Q, h) such that*

(1) $Q \subset P$,
(2) $h = g$ on P^c,
(3) if $x, y \in Q$ belong to different components of $\mathbb{R}^2 \setminus l(\delta)$ then they belong to different components of Q.

We first prove Theorem 1.1 using Lemma 3.2

Proof of Theorem 1.1: Choose a sequence $\{x_n\}_{n=1}^{\infty}$ dense in the plane and consider any sequence of lines $\{l_n\}_{n=1}^{\infty}$ with the property that for any $i, j \in \mathbb{N}$ there is some $k \in \mathbb{N}$ such that $x_i, x_j \in l_k$. Choose a sequence $\{\varepsilon_n\}_{n=1}^{\infty} \subset (0, \infty)$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Then the sequence $\{l_n(\varepsilon_n)\}_{n=1}^{\infty}$ has the property that for every $x, y \in \mathbb{R}^2$, $x \neq y$, there is some $k \in \mathbb{N}$ such that x and y belong to the different component of $\mathbb{R}^2 \setminus l_k(\varepsilon_k)$.

In the proof we will proceed by induction and construct a sequence of functions $f_i : \mathbb{R}^2 \to \mathbb{R}$ and a sequence $\{P_i\} \subset \mathcal{Q}$, $i = 0, 1, \ldots$, such that for every i the following conditions hold:

(1) pair (P_i, f_i) is $(1 + \sum_{n=1}^{i} \varepsilon_n)$-good,
(2) if $i > 0$ then $P_i \subset P_{i-1}$,
(3) if $i > 0$ then $f_i = f_{i-1}$ on $(P_{i-1})^c$,

(4) if $i > 0$ and if $x, y \in P_i$ belong to the different component of $\mathbb{R}^2 \setminus l_i(\varepsilon_i)$ then they belong to the different component of P_i.

To do this let f_0 be an arbitrary 1-Lipschitz function on $\mathbb{R}^2$ which is equal to 0 on $((-3,3)^2)^c$ and equal to 1 on $[-1,1]^2$ and put $P_0 := [-3,3]^2 \setminus (-1,1)^2$. Validity of conditions (1)–(4) is obvious.

Now, if we have constructed f_{i-1} and P_{i-1} we obtain f_i and P_i simply by applying Lemma 3.2 with $\varepsilon = \delta = \varepsilon_i$, $L = (1 + \sum_{n=1}^{i-1} \varepsilon_n)$, $l = l_i$, $P = P_{i-1}$ and $g = f_{i-1}$. The function f_i will be then equal to h from the statement of Lemma 3.2 and P_i will be equal to the corresponding Q. Validity of conditions (1)–(4) follows directly from Lemma 3.2.

Put $M := \bigcap P_i$. Due to property (2) M is compact and nonempty. To prove that M is totally disconnected consider $x, y \in M$, $x \neq y$. By the choice of the sequences $\{l_n\}_{n=1}^{\infty}$ and $\{\varepsilon_n\}_{n=1}^{\infty} \subset \mathbb{R}^+$ there is some i such that x and y belong to the different component of $\mathbb{R}^2 \setminus l_i(\varepsilon_i)$. By property (3) we have that x and y belong to the different component of P_i. Using property (2) again we then obtain that x and y belong to the different component of M as well.

Define $\tilde{f} : M^c \to \mathbb{R}$ in such a way that $\tilde{f}(x) = f_i(x)$ whenever $x \in (P_i)^c$. It is easy to see that the definition of $\tilde{f}$ is correct due to properties (2) and (3) and the definition of M, and also that by property (1) the function $\tilde{f}$ is $(1 + \sum_{n=1}^{\infty} \varepsilon_n)$-Lipschitz and locally convex on M^c. By Kirszbraun's theorem (see [3]) there is a $(1 + \sum_{n=1}^{\infty} \varepsilon_n)$-Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f = \tilde{f}$ on M^c. Therefore f is locally convex on M^c as well. Also, f has compact support due to properties (2) and (3), the fact that P_0 is compact and that f_0 is supported in P_0.

It remains to show that f is not convex on $\mathbb{R}^2$, but this is easy since

$$\frac{f(-3,0) + f(3,0)}{2} = 0 < 1 = f(0,0). \qquad \square$$

The proof of Lemma 3.2 is divided into several lemmas.

Lemma 3.3. *Let $H \subset \mathbb{R}^2$ be a closed halfplane, $x \in \mathbb{R}^2 \setminus H$, $w \in \partial H$ and $L > 0$. If $f : H \cup \{x\} \to \mathbb{R}$ is L-Lipschitz and affine on H, then the function*

$$g_w(u) = \begin{cases} f(u), & \text{if} \quad u \in H, \\ \alpha f(x) + (1-\alpha) f(w), & \text{for} \quad u = \alpha x + (1-\alpha) w, \alpha \in [0,1], \end{cases}$$

is L-Lipschitz as well.

Proof: Without any loss of generality we can suppose that $f(w) = 0$ and $w = (0,0)$. This means that g_w is in fact linear on both H and $[x, w]$. Choose $a \in H$

and $b = \alpha x$ for some $\alpha \in [0,1]$. Now,

$$
\begin{aligned}
|g_w(a) - g_w(b)| &= \alpha \left| g_w\left(\frac{1}{\alpha}a\right) - g_w\left(\frac{1}{\alpha}b\right)\right| = \alpha \left| g_w\left(\frac{1}{\alpha}a\right) - g_w\left(\frac{1}{\alpha}\alpha x\right)\right| \\
&= \alpha \left| g_w\left(\frac{1}{\alpha}a\right) - g_w(x)\right| \le \alpha L \left|\frac{1}{\alpha}a - x\right| = \alpha L \left|\frac{1}{\alpha}a - \frac{1}{\alpha}\alpha x\right| \\
&= L|a - \alpha x| = L|a-b|.
\end{aligned}
$$

Similarly, if $a = \alpha x$ and $b = \beta x$ for some $\alpha, \beta \in [0,1]$, $\alpha \neq \beta$ we have

$$|g_w(a) - g_w(b)| = |\alpha f(x) - \beta f(x)| = |f(x)| \cdot |\alpha - \beta| \le L|x| \cdot |\alpha - \beta| = L|a-b|.$$

$\square$

Lemma 3.4. *Let $\varepsilon, L, K > 0$. Let f be an L-Lipschitz function on $[-K,K]^2$, which is equal to an affine function f_1 on $[-K,0] \times [-K,K]$, and $z \in (0,K) \times (-K,K)$. Then there is an $x \in [(0,0),z]$ and $\gamma > 0$ such that for every $y \in B(x,\gamma)$ and every $w \in B((0,0),\gamma) \cap (\{0\} \times (-K,K))$ the function*

$$g_{y,w}(u) = \begin{cases} f(u), & \text{if} \quad u \in [-K,0] \times [-K,K], \\ \alpha f(w) + (1-\alpha) f(x), & \text{for} \quad u = \alpha w + (1-\alpha)y, \alpha \in [0,1], \end{cases}$$

is $(L+\varepsilon)$-Lipschitz and $|g_{y,w} - f| < \varepsilon$ on $[-K,0] \times [-K,K] \cup [w,y]$.

Proof: Without any loss of generality we can suppose that $\varepsilon < 1$, $L = 1$ and that $f(0,0) = 0$. Indeed, if $f(0,0) \neq 0$ we can just consider the function $u \mapsto f(u) - f(0,0)$ in the place of f and then add $f(0,0)$ to the resulting function $g_{y,w}$. If $L \neq 1$ then we can just consider the function $u \mapsto \frac{f(u)}{L}$ in the place of f and $\frac{\varepsilon}{L}$ in the place of ε and multiply the resulting function $g_{y,w}$ by L.

Since f is 1-Lipschitz we can find a sequence $\{x_i\}_{i=1}^{\infty} \subset [(0,0),z]$ converging to $(0,0)$ such that for some $s \in [-1,1]$

$$s_i := \frac{f(x_i)}{|x_i|} \to s \quad \text{as} \quad i \to \infty. \tag{3.1}$$

Denote $\tilde{z} := \frac{z}{|z|}$. Consider now the sequence of functions $h_i : [-\frac{K}{|x_i|}, 0] \times [-\frac{K}{|x_i|}, \frac{K}{|x_i|}] \cup \{\tilde{z}\} \to \mathbb{R}$ defined as

$$h_i(u) := \frac{1}{|x_i|} f(|x_i| \cdot u).$$

Then h_i is 1-Lipschitz for every i. Since f is equal to an affine function f_1 on $[-K,0] \times [-K,K]$ and $f(0,0) = 0$ we have $h_i = f_1$ on $[-\frac{K}{|x_i|}, 0] \times [-\frac{K}{|x_i|}, \frac{K}{|x_i|}]$. Also $h_i(\tilde{z}) = s_i$, because $\tilde{z} = \frac{z}{|z|} = \frac{x_i}{|x_i|}$. Therefore by (3.1) the function $h := \lim h_i : H \cup \{\tilde{z}\} \to \mathbb{R}$ which is equal to f_1 on $H := (-\infty, 0] \times (-\infty, \infty)$ and such that $h(\tilde{z}) = s$, is also 1-Lipschitz.

Consider $\tilde{\gamma} > 0$ such that $\tilde{\gamma} < \frac{\varepsilon \tilde{z}_1}{4}$ (here by $\tilde{z}_1$ we mean the first coordinate of $\tilde{z}$). This choice then implies

$$\frac{|v-\tilde{z}|}{|v-\tilde{z}|-\tilde{\gamma}} = 1 + \frac{\tilde{\gamma}}{|v-\tilde{z}|-\tilde{\gamma}} < 1 + \frac{\frac{\varepsilon \tilde{z}_1}{4}}{\tilde{z}_1 - \frac{\varepsilon \tilde{z}_1}{4}} = 1 + \frac{\varepsilon}{4-\varepsilon}$$

for $v \in H$, which gives us inequality

$$\frac{|v-\tilde{z}|}{|v-\tilde{z}|-\tilde{\gamma}} < 1 + \frac{\varepsilon}{2},$$

as $\varepsilon < 1$. Now, for every $\tilde{s} \in [s-\tilde{\gamma}, s+\tilde{\gamma}]$, $v \in H$ and $t \in B(\tilde{z}, \tilde{\gamma})$

$$\begin{aligned} \frac{f_1(v)-\tilde{s}}{|v-t|} &\leq \frac{|f_1(v)-s|}{|v-t|} + \frac{|s-\tilde{s}|}{|v-t|} \leq \frac{|f_1(v)-s|}{|v-\tilde{z}|-\tilde{\gamma}} + \frac{\tilde{\gamma}}{|v-\tilde{z}|-\tilde{\gamma}} \\ &\leq \frac{|f_1(v)-s|}{|v-\tilde{z}|} \cdot \frac{|v-\tilde{z}|}{|v-\tilde{z}|-\tilde{\gamma}} + \frac{2\tilde{\gamma}}{\tilde{z}_1} \leq \left(1+\frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} = 1+\varepsilon. \end{aligned}$$

Therefore, by Lemma 3.3 for every $\tilde{s} \in [s-\tilde{\gamma}, s+\tilde{\gamma}]$, $w \in \{0\} \times (-\infty, \infty)$ and $t \in B(\tilde{z}, \tilde{\gamma})$ the function

$$\tilde{h}_{w,t,\tilde{s}}(u) = \begin{cases} f_1(u), & \text{if} \quad u \in H, \\ (1-\alpha)\tilde{s} + \alpha f_1(w), & \text{for} \quad u = (1-\alpha)t + \alpha w, \alpha \in [0,1], \end{cases}$$

is $(1+\varepsilon)$-Lipschitz as well.

Choose i such that $s_i \in [s-\tilde{\gamma}, s+\tilde{\gamma}]$ and put $x = x_i$ and $\gamma = \frac{|x|\tilde{\gamma}}{2}$. Now, consider some $y \in B(x, \gamma)$ and some $w \in B((0,0), \gamma) \cap \{0\} \times (-K, K)$ and let $g_{y,w}$ be as in the statement of the lemma. First we will prove that $g_{y,w}$ is $(1+\varepsilon)$-Lipschitz. To do this we first observe that $\frac{1}{|x|} g_{y,w}(|x| \cdot \xi)$ is equal to $\tilde{h}_{\frac{w}{|x|}, \frac{y}{|x|}, \frac{f(x)}{|x|}}(\xi)$, whenever the first function (as a function of ξ) is defined. Now, we have $\frac{w}{|x|} \in \{0\} \times (-\infty, \infty)$,

$$\left| \frac{y}{|x|} - \tilde{z} \right| = \left| \frac{y}{|x|} - \frac{x}{|x|} \right| = \frac{|y-x|}{|x|} \leq \frac{|x|\tilde{\gamma}}{2|x|} \leq \tilde{\gamma},$$

which means $\frac{y}{|x|} \in B(\tilde{z}, \tilde{\gamma})$ and finally $\frac{f(x)}{|x|} = s_i \in [s-\tilde{\gamma}, s+\tilde{\gamma}]$ and we are done since $\frac{1}{|x|} g_{y,w}(|x| \cdot \xi)$ (as a function of ξ) and $g_{y,w}$ have the same Lipschitz constant.

To finish the proof it is now sufficient to observe that if we additionally choose x_i small enough we obtain also $|g_{y,w} - f| < \varepsilon$ on $[-K, 0] \times [-K, K] \cup [w, y]$. □

Lemma 3.5. *Let $L, \varepsilon, \delta > 0$, $a < b$ and $c < d$ be given. Let*

$$P = \mathrm{co}\{(-1,a), (-1,b), (1,c), (1,d)\}$$

and

$$P^{\varepsilon} = \mathrm{co}\{(-1, a-\varepsilon), (-1, b+\varepsilon), (1, c-\varepsilon), (1, d+\varepsilon)\}.$$

Suppose that f is an L-Lipschitz function defined on $\mathbb{R}^2$ which is locally affine on $P^\varepsilon \setminus P$. Then there are

$$\frac{a+c}{2} =: a_0 < a_1 < \cdots < a_{n-1} < a_n := \frac{b+d}{2}$$

and $\frac{1}{2} > \kappa > 0$ such that, using the notation introduced below, the function $g_\kappa : \overline{P^\varepsilon \setminus (P \setminus [-\kappa,\kappa] \times \mathbb{R})} \to \mathbb{R}$ defined as $g_\kappa(z_i^\pm) = f(z_i^\pm)$ for $i = 0, n$, $g_\kappa(z_i^\pm) = f(z_i)$ for $i = 1, \ldots, n-1$ and

$$g_\kappa(u) = \begin{cases} f(u), & \text{if} \quad u \in P^\varepsilon \setminus P, \\ \alpha g(z_i^+) + \beta g(z_i^-) + \gamma g(z_{i+1}^+), & \text{for} \quad u = \alpha z_i^+ + \beta z_i^- + \gamma z_{i+1}^+, \\ & \alpha, \beta, \gamma \geq 0, \alpha + \beta + \gamma = 1, \\ \alpha g(z_i^-) + \beta g(z_{i+1}^-) + \gamma g(z_{i+1}^+), & \text{for} \quad u = \alpha z_i^- + \beta z_{i+1}^- + \gamma z_{i+1}^+, \\ & \alpha, \beta, \gamma \geq 0, \alpha + \beta + \gamma = 1 \end{cases}$$

is $(L+\delta)$-Lipschitz and such that $|f - g_\kappa| < \delta$ on $\mathbb{R}^2$. Here we denoted $z_0^\pm := \left(\pm\kappa, \frac{a+c}{2} \pm \frac{\kappa(c-a)}{2}\right)$, $z_n^\pm := \left(\pm\kappa, \frac{b+d}{2} \pm \frac{\kappa(d-b)}{2}\right)$, $z_i^\pm := (\pm\kappa, a_i)$ for $i = 1, \ldots, n-1$ and $z_i := (0, a_i)$ for $i = 0, \ldots, n$.

Proof: Without any loss of generality we can suppose $L = 1$. Denote P_i^ε the connectivity component of $\overline{P^\varepsilon \setminus P}$ containing z_i, $i = 0, n$. When we have found a_i we denote $P_i = \operatorname{co}\{z_i^\pm, z_{i+1}^\pm\}$ for $i = 0, \ldots, n-1$. Put $S = \operatorname{co}\{z_1^\pm, z_{n-1}^\pm\}$ and $\alpha = \operatorname{dist}(S, P^\varepsilon \setminus P)$. We always assume κ to be small enough that $1 > \alpha > 0$.

First, we will use Lemma 3.4 twice to find points $a_1 \in B(a_0, \frac{\min(|a_0 - a_n|, 1)}{2})$, $a_{n-1} \in B(a_n, \frac{\min(|a_0 - a_n|, 1)}{2})$ and $\kappa_1 > 0$ such that for every $\kappa_1 > \kappa > 0$ the functions $g_\kappa|_{P_0^\varepsilon \cup P_0}$ and $g_\kappa|_{P_n^\varepsilon \cup P_{n-1}}$ are both $(1+\delta)$-Lipschitz and such that $|f - g_\kappa| < \delta$ on $P_0^\varepsilon \cup P_n^\varepsilon \cup P_0 \cup P_{n-1}$. Here, in the notation of the points z_i, the point z_1 corresponds to the point x guaranteed by Lemma 3.4 (when we identify z_0 with the origin) and similarly the point z_{n-1} corresponds to x in the case when we apply Lemma 3.4 centred in z_n. Note that although Lemma 3.4 guarantees $(1+\delta)$-Lipschitzness on P_0 (or on P_{n-1}) only on line segments with one endpoint in P_0^ε (or in P_n^ε), this is enough for our purposes. Indeed, if for instance $a, b \in \operatorname{co}\{z_0^-, z_0^+, z_1^+\}$, we can always find $\tilde{a}, \tilde{b}$ with $\tilde{a} \in P_0^\varepsilon$ and such that the vector $a - b$ is parallel to the vector $\tilde{a} - \tilde{b}$. In such situation of course

$$\frac{|g_\kappa(a) - g_\kappa(b)|}{|a-b|} = \frac{|g_\kappa(\tilde{a}) - g_\kappa(\tilde{b})|}{|\tilde{a} - \tilde{b}|}.$$

Also, if $a, b \in \operatorname{co}\{z_0^-, z_1^-, z_1^+\}$ one can always consider $\tilde{a} = z_1^-$ or $\tilde{a} = z_1^+$ such that

$$\frac{|g_\kappa(a) - g_\kappa(b)|}{|a-b|} \leq \frac{|g_\kappa(\tilde{a}) - g_\kappa(z_0^-)|}{|\tilde{a} - z_0^-|}.$$

Similarly for P_{n-1}.

Observe that for every $u_0 \in P_0^\varepsilon \cup P_0$ and every $u_n \in P_n^\varepsilon \cup P_{n-1}$ we have

$$\begin{aligned}\frac{|g_\kappa(u_0) - g_\kappa(u_n)|}{|u_0 - u_n|} &\leq \frac{|g_\kappa(u_0) - g_\kappa(z_0)|}{|u_0 - u_n|} + \frac{|g_\kappa(z_0) - g_\kappa(z_n)|}{|u_0 - u_n|} + \frac{|g_\kappa(z_n) - g_\kappa(u_n)|}{|u_0 - u_n|} \\ &\leq \frac{|u_0 - z_0|}{|u_0 - u_n|} + \frac{|z_0 - z_n|}{|u_0 - u_n|} + \frac{|z_n - u_n|}{|u_0 - u_n|}.\end{aligned}$$

and since the last expression can be smaller than $1 + \delta$ when we assume $|a_0 - a_1|$ and $|a_{n-1} - a_n|$ to be small enough, we can additionally assume that $g|_{P^\varepsilon \cup P_0 \cup P_{n-1}}$ is $(1 + \delta)$-Lipschitz.

Next, note that the function $g_\kappa|_{[z_1, z_{n-1}]}$ is actually independent on κ and that it is 1-Lipschitz for any choice of $a_2, \ldots, a_{n-2}$ (this is true because in one dimension the affine extension never increases the Lipschitz constant). This also means that for $S = \mathrm{co}\{z_1^\pm, z_{n-1}^\pm\}$ we have $g_\kappa|_S$ is 1-Lipschitz for any choice of $a_2, \ldots, a_{n-2}$ as well. Put $\alpha = \mathrm{dist}\,(S, P^\varepsilon \setminus P)$, we can assume κ_2 to be small enough that $1 > \alpha > 0$ (here we used the fact that $|a_0 - a_1|, |a_{n-1} - a_n| \leq \frac{1}{2}$). Consider n big enough such that $\frac{|a_1 - a_{n-1}|}{n-1} \leq \frac{\alpha\delta}{4}$, put $a_i = a_1 + \frac{i|a_1 - a_{n-1}|}{n-1}$ and pick $\kappa_3 < \min(\kappa_2, \frac{\alpha\delta}{4})$. Then for $\kappa < \kappa_3$ and $a \in S$

$$\begin{aligned}|g_\kappa(a) - f(a)| &\leq |g_\kappa(a) - g_\kappa(z_i)| + |g_\kappa(z_i) - f(z_i)| + |f(z_i) - f(a)| \\ &\leq |a - z_i| + 0 + |a - z_i| \leq \frac{\delta}{2} < \delta,\end{aligned} \tag{3.2}$$

where i is chosen such that $a \in P_i$.

To finish the proof we need to observe that for $\kappa < \kappa_3$ the function g_κ is $(1+\delta)$-Lipschitz. Since $S \cup P_0 \cup P_{n-1}$ is convex, the remaining case we have to consider is $a \in S$ and $b \in P^\varepsilon \setminus P$. Find i such that $a \in P_i$. With this choice we have $|a - z_i| \leq \frac{\alpha\delta}{2}$ and therefore

$$|b - z_i| \leq |a - b| + |a - z_i| \leq |a - b| + \frac{\alpha\delta}{2} \leq (1 + \delta)\,|a - b|.$$

Now, we have

$$\begin{aligned}|g_\kappa(a) - g_\kappa(b)| &\leq |g_\kappa(a) - g_\kappa(z_i)| + |g_\kappa(z_i) - g_\kappa(b)| \\ &\leq \frac{\delta\alpha}{2} + |f(z_i) - f(b)| \leq \frac{\delta}{2}|a - b| + |b - z_i| \\ &\leq \frac{\delta}{2}|a - b| + \left(1 + \frac{\delta}{2}\right) \cdot |a - b| \leq (1 + \delta)|a - b|.\end{aligned}$$

□

Lemma 3.6. *Let $1 > \varepsilon > 0$ and $\alpha, L > 0$. Let f be a L-Lipschitz function on $[-1, 1]^2$ which is affine on both $[-1, 1] \times [-1, 0]$ and $[-1, 1] \times [0, 1]$ (and equal to affine functions f_1 and f_2, respectively). Put*

$$A_1 = [-1, -1/2] \times [-1, 0], A_2 = [1/2, 1] \times [0, 1],$$

$$B_1^\varepsilon = [-1,\varepsilon] \times [0,\varepsilon], B_2^\varepsilon = [-\varepsilon,1] \times [-\varepsilon,0]$$

and

$$A = A_1 \cup A_2 \cup B_1^\varepsilon \cup B_2^\varepsilon.$$

Then either f is convex on $[-1,1]^2$ or the function $g_\varepsilon : A \to \mathbb{R}$ defined as

$$g(u) = \begin{cases} f_1(u), & \text{if} \quad u \in A_1 \cup B_1^\varepsilon, \\ f_2(u), & \text{if} \quad u \in A_2 \cup B_2^\varepsilon. \end{cases}$$

is locally convex on A. Moreover, if ε is small enough, g_ε is $(L+\alpha)$-Lipschitz and $|g_\varepsilon - f| < \alpha$ on A.

Proof: It follows from a direct computation. □

Lemma 3.7. *Let $L, \alpha > 0$ and $1 > \gamma > \varepsilon > 0$. Let f be a L-Lipschitz function on $[-4,4]^2 \cup [1,2] \times [4,5]$ which is affine on both $[-4,4] \times [-4,0]$ and $[-4,4] \times [0,4] \cup [1,2] \times [4,5]$ (and equal to affine functions f_1 and f_2, respectively). Put*

$$A_1 = [-3,-2] \times [0,\gamma], A_2 = [-3,0] \times [\gamma, \gamma+\varepsilon], A_3 = [-1,2] \times [\gamma-\varepsilon, \gamma],$$
$$A_4 = [1,2] \times [\gamma,4], B_1 = [-4,4] \times [-4,0], B_2 = [1,2] \times [4,5],$$

and

$$A = A_1 \cup A_2 \cup A_3 \cup A_4 \cup B_1 \cup B_2.$$

Then either f is locally convex on $[-4,4]^2 \cup [1,2] \times [4,5]$ or the function

$$g(u) = \begin{cases} f_1(u), & \text{if} \quad u \in A_1 \cup A_2 \cup B_1, \\ f_2(u) + \frac{f_1(0,\gamma) - f_2(0,\gamma)}{\gamma - 4}(u \cdot (0,1) - 4), & \text{if} \quad u \in A_3 \cup A_4, \\ f_2(u), & \text{if} \quad u \in B_2, \end{cases}$$

is $(L+\alpha)$-Lipschitz, locally convex on A and $|f - g| < \alpha$ on A, if ε and γ are small enough.

Proof: Without any loss of generality we can suppose $L = 1$. First we prove that g is continuous on A. To do this we need to prove that

$$f_1(a,\gamma) = f_2(a,\gamma) + \frac{f_1(0,\gamma) - f_2(0,\gamma)}{\gamma - 4}((a,\gamma) \cdot (0,1) - 4) \tag{3.3}$$

whenever $(\gamma, a) \in A_2 \cap A_3$ and that

$$f_2(a,4) = f_2(a,4) + \frac{f_1(0,\gamma) - f_2(0,\gamma)}{\gamma - 4}((a,4) \cdot (0,1) - 4) \tag{3.4}$$

whenever $(a,4) \in A$. Define an affine function f_3 on $\mathbb{R}^2$ as

$$f_3(u,v) = \frac{f_1(0,\gamma) - f_2(0,\gamma)}{\gamma - 4}((u,v) \cdot (0,1) - 4).$$

To prove (3.3) we can write

$$\begin{aligned}
g(a,\gamma) &= f_2(a,\gamma) + f_3(a,\gamma)\\
&= f_2(a,\gamma) + \frac{f_1(0,\gamma) - f_2(0,\gamma)}{\gamma - 4}\cdot(\gamma - 4)\\
&= f_2(a,\gamma) + f_1(0,\gamma) - f_1(0,0) - f_2(0,\gamma) + f_2(0,0)\\
&= f_2(a,\gamma) + f_1(a,\gamma) - f_1(a,0) - f_2(a,\gamma) + f_2(a,0)\\
&= f_2(a,\gamma) + f_1(a,\gamma) - f_1(a,0) - f_2(a,\gamma) + f_1(a,0) = f_1(a,\gamma).
\end{aligned}$$

To prove (3.4) we can write

$$\begin{aligned}
g(a,4) &= f_2(a,4) + f_3(a,4)\\
&= f_2(a,4) + \frac{f_1(0,\gamma) - f_1(0,0) - f_2(0,\gamma) + f_1(0,0)}{\gamma - 4}(4-4) = f_2(a,4).
\end{aligned}$$

Next note that since both f_1 and f_2 are 1-Lipschitz we have

$$g \text{ is 1-Lipschitz on } B_1 \cup A_1 \cup A_2, \tag{3.5}$$

and

$$g \text{ is 1-Lipschitz on } B_2. \tag{3.6}$$

Since additionally f_3 is constant on all lines parallel to x-axis and since

$$\frac{f_3(0,\gamma) - f_3(0,4)}{4-\gamma} \le \frac{f_1(0,\gamma) - f_1(0,0) - f_2(0,\gamma) + f_2(0,0)}{3} \le \frac{2\gamma}{3} \le \gamma.$$

we have

$$g \text{ is } (1+\gamma)\text{-Lipschitz on } A_4 \cup A_3 \tag{3.7}$$

and

$$|g - f_2| \le 4\gamma \text{ on } A_4 \cup A_3. \tag{3.8}$$

Now, if $x \in B_1$ and $y \in A_3$ then $g(x) = f_1(x)$, $|g(y) - f_1(y)| \le 3\varepsilon$ and $|x-y| \ge \gamma - \varepsilon$ and therefore

$$|g(x) - g(y)| \le |g(x) - f_1(y)| + |f_1(y) - g(y)| \le |x-y| + 3\varepsilon \le \frac{\gamma + 2\varepsilon}{\gamma - \varepsilon}.$$

So we have

$$g \text{ is } \frac{\gamma + 2\varepsilon}{\gamma - \varepsilon}\text{-Lipschitz on } B_1 \cup A_3. \tag{3.9}$$

If $x \in B_1$ and $y \in A_4$ then $g(x) = f_1(x)$, $f(y) \le g(y) \le f_1(y)$ and therefore

$$g \text{ is 1-Lipschitz on } B_1 \cup A_4. \tag{3.10}$$

Using (3.6) and (3.7) and continuity of g we obtain that

$$g \text{ is } (1+\gamma)\text{-Lipschitz on } A_2 \cup A_3 \text{ and on } B_2 \cup A_4. \tag{3.11}$$

Finally, if $x \in A_1 \cup A_2$ and $y \in A_4 \cup B_2$ or $x \in A_1$ and $y \in A_3 \cup A_4 \cup B_2$ we have

$$|g(x) - f_2(x)| \le 2(\gamma + \varepsilon) \le 4\gamma, \quad |g(y) - f_2(y)| \le 4\gamma \tag{3.12}$$

and $|x - y| \ge 1$. This implies

$$\begin{aligned} |g(x) - g(y)| &\le |g(x) - f_2(x)| + |f_2(x) - f_2(y)| + |f_2(y) - g(y)| \\ &\le 4\gamma + |x - y| + 4\gamma \le (1 + 8\gamma)|x - y|. \end{aligned} \tag{3.13}$$

Now, according to (3.5)–(3.12) it is sufficient to choose $\frac{\alpha}{4} > \gamma > \varepsilon > 0$ small enough such that

$$\max\left(1 + 8\gamma, \frac{\gamma + 2\varepsilon}{\gamma - \varepsilon}\right) < 1 + \alpha$$

to obtain that g is $(1+\alpha)$-Lipschitz on A and $|f - g| < \alpha$ on A. $\square$

Lemma 3.8. *Under the assumptions of Lemma 3.5 there is $\frac{1}{2} > \kappa > 0$, $R \subset P \cap (-\kappa, \kappa) \times \mathbb{R}$ and a function $h : \overline{P^\varepsilon \setminus P} \cup R \to \mathbb{R}$ such that:*

(a) *$R \in \mathcal{Q}$,*
(b) *$h = f$ on $\overline{P^\varepsilon \setminus P}$,*
(c) *h is locally convex on $\overline{P^\varepsilon \setminus P} \cup R$,*
(d) *$\overline{P^\varepsilon \setminus P} \cup R$ is connected,*
(e) *h is piecewise affine on $\overline{P^\varepsilon \setminus P} \cup R$,*
(f) *h is $(L + \delta)$-Lipschitz.*

Proof: Without any loss of generality we can suppose $L = 1$. Let κ, z_i and g_κ be as in Lemma 3.5, but with $\frac{\delta}{2}$ in the place of δ. Consider the sets

$$X = [-4, 4]^2 \cup [1, 2] \times [4, 5] \quad \text{and} \quad Y = [-1, 1]^2.$$

Find homotheties $\Psi_i : x \mapsto \rho_i x + v_i$, $\rho_i > 0$, $v_i \in \mathbb{R}^2$, $i = 1, \dots, n-1$ and orientation preserving similarities Ψ_0 and Ψ_n, with scaling ratios ρ_0 and ρ_n, such that if we put $M_i = \Psi_i(X)$, $i = 0, n$ and $M_i = \Psi_i(Y)$, $i = 1, \dots, n-1$ we have

(A) $M_i \cap M_j = \emptyset$ if $i \ne j$,
(B) $\Psi_0([-4, 4] \times [-4, 0]) \subset \overline{P^\varepsilon \setminus P}$,
(C) $\Psi_n([-4, 4] \times [-4, 0]) \subset \overline{P^\varepsilon \setminus P}$,
(D) $M_i \subset (-\kappa, \kappa) \times \mathbb{R}$,
(E) $[z_i^-, z_i^+] \subset \Psi_i(\mathbb{R} \times \{0\})$,

Put $\Omega = \min_{i \ne j} \operatorname{dist}(M_i, M_j)$ and note that $\Omega > 0$ due to property (A). Define

$$T_i := \operatorname{co}\{\Psi_i(\frac{1}{2}, 1), \Psi_i(1, 1), \Psi_{i+1}(-\frac{1}{2}, -1), \Psi_{i+1}(-1, -1)\},$$

for $i = 1, \dots, n-2$,

$$T_0 := \mathrm{co}\{\Psi_0(1,5), \Psi_0(2,5), \Psi_1(-\frac{1}{2}, -1), \Psi_1(-1,-1)\}$$

and

$$T_{n-1} := \mathrm{co}\{\Psi_n(1,5), \Psi_n(2,5), \Psi_{n-1}(\frac{1}{2}, 1), \Psi_{n-1}(1,1)\}.$$

Put

$$R := \left(\bigcup_{i=0}^{n-1} T_i\right) \cup \left(\bigcup_{i=0}^{n} M_i\right). \tag{3.14}$$

Let g_i, $i = 1, \dots, n-1$ be the function g from Lemma 3.6 with $\alpha = \frac{\Omega\delta\rho_i}{4}$ (and corresponding ε) and with $f_1(x) = \rho_i g_\kappa \circ \Psi_i$ and $f_2(x) = \rho_i g_\kappa \circ \Psi_i$ (with the exception when g_κ is already convex on M_i, in which case we put $g_i = g_\kappa|_{M_i}$). Let g_0 be the function g from Lemma 3.7 with $\gamma = \frac{\Omega\delta\rho_i}{4}$ (and corresponding ε and γ) and with $f_1 = \rho_0 g_\kappa \circ \Psi_0$ and $f_2 = \rho_0 g_\kappa \circ \Psi_0$ and finally, let g_n be the function g from Lemma 3.7 with $\gamma = \frac{\Omega\delta\rho_i}{4}$ (and corresponding ε and γ) and with $f_1 = \rho_n g_\kappa \circ \Psi_n$ and $f_2 = \rho_n g_\kappa \circ \Psi_n$.

Consider now the function h defined by the formula

$$h = \begin{cases} \frac{1}{\rho_i} g_i \circ \Psi_i^{-1} & \text{on} \quad M_i \\ g_\kappa & \text{otherwise.} \end{cases}$$

Property (a) follows from (3.14) and the fact that every M_i and every T_i is a polygon. Properties (b), (c) and (e) follow directly from the construction and corresponding properties of the functions g_i and property (d) is obvious. We will now finish the proof by proving property (f).

So suppose that $a, b \in (P^\varepsilon \setminus P) \cup R$. We need to prove that $|h(a) - h(b)| \leq (1+\delta)|a-b|$. We can additionally suppose that either a or b belongs to some M_i since otherwise there is nothing to prove. We will prove only the case $a \in M_i$, $b \in M_j$, $i \neq j$, the other cases can be proved following the same lines. By Lemma 3.6 (for $i = 1, \dots, n-1$) and Lemma 3.7 (for $i = 0, n$) we can now write

$$\begin{aligned} |h(a) - h(b)| &\leq |h(a) - g_\kappa(a)| + |g_\kappa(a) - g_\kappa(b)| + |g_\kappa(b) - h(b)| \\ &< \frac{1}{\rho_i} \cdot \frac{\Omega\delta\rho_i}{4} + \left(1 + \frac{\delta}{2}\right) \cdot |a-b| + \frac{1}{\rho_j} \cdot \frac{\Omega\delta\rho_j}{4} \\ &\leq \frac{\delta}{2}|a-b| + \left(1 + \frac{\delta}{2}\right) \cdot |a-b| = (1+\delta)|a-b|, \end{aligned}$$

which is what we need. □

Proof of Lemma 3.2: Without any loss of generality we can suppose $L = 1$. Let V be the set of all points $v \in \partial P$ with the property that there is some $\varepsilon_v > 0$ such that $P \cap B(v, \varepsilon_v)$ is similar to $\{(x,y) : x \geq 0\} \cap B(0,1)$ and that g is affine on $P \cap B(v, \varepsilon_v)$. Since $P \in \mathcal{Q}$, the set $\partial P \setminus V$ is finite and without any loss of generality we can assume that $l(\delta) \cap (\partial P \setminus V) = \emptyset$. We can also assume that $l = \{0\} \times \mathbb{R}$ and that $\delta = 1$.

This means that the closure of every component P_i of $P \cap l(\delta)$ is of the form

$$\mathrm{co}\{(-1, a_i), (-1, b_i), (1, c_i), (1, d_i)\}$$

for some $a_i < b_i$, $c_i < d_i$ and such that, for some $\varepsilon_i > 0$, g is locally affine on $P_i^{\varepsilon_i} \setminus P_i$, where

$$P_i^{\varepsilon_i} := \mathrm{co}\{(-1, a_i - \varepsilon_i), (-1, b_i + \varepsilon_i), (1, c_i - \varepsilon_i), (1, d_i + \varepsilon_i)\}.$$

Then we have

$$\alpha = \min_{i \neq j} \operatorname{dist}(P_i, P_j) > 0.$$

Let κ_i, R_i and h_i be equal to κ, R and h obtained from Lemma 3.8 for ε_i in the place of ε, P_i in the place of P, g in the place of f and $\frac{\min(\alpha, \varepsilon_i, 1)\varepsilon}{4}$ in the place of δ.

Put $Q = P \setminus (\bigcup R_i)$ and define $\tilde{h} : Q^c \to \mathbb{R}$ by

$$\tilde{h}(u) = \begin{cases} h_i(u) & \text{on } R_i \\ g(u) & \text{otherwise.} \end{cases}$$

Let K be the Lipschitz constant of $\tilde{h}$. Using the Kirszbraun theorem we can find a K-Lipschitz function h on $\mathbb{R}^2$ such that $h = \tilde{h}$ on P^c.

Now, property (1) follows directly from the definition of Q and (a) in Lemma 3.8, property (2) from the definition of h and (b) in Lemma 3.8 and property (3) from (d) in Lemma 3.8.

It remains to prove that the pair (Q, h) is $(1+\varepsilon)$-good. The local convexity and piecewise affinity of h on Q^c follow from (c) and (e) in Lemma 3.8 and the corresponding properties of g, so the proof will be finished, if we verify that $K \leq (1 + \varepsilon)$.

To do this pick $a, b \in \mathbb{R}^2$, we need to prove that $|h(a) - h(b)| \leq (1+\varepsilon)|a - b|$. We can additionally suppose that either a or b belongs to some R_i since otherwise there is nothing to prove. We will prove only the case $a \in R_i$, $b \in R_j$, $i \neq j$, the other cases can be proved following the same lines.

Using the definition of h, namely property (f) from Lemma 3.8 we can now write

$$\begin{aligned} |h(a) - h(b)| = |h_i(a) - h_j(b)| &\leq |h_i(a) - f(a)| + |f(a) - f(b)| + |f(b) - h_j(b)| \\ &\leq \frac{\min(\alpha, \varepsilon_i, 1)\varepsilon}{4} + \left(1 + \frac{\varepsilon}{4}\right) \cdot |a - b| + \frac{\min(\alpha, \varepsilon_j, 1)\varepsilon}{4} \\ &\leq \frac{2\varepsilon}{4}|a - b| + \left(1 + \frac{\varepsilon}{2}\right) \cdot |a - b| < (1+\varepsilon)|a - b|. \end{aligned}$$

□

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