Some properties of Eulerian lattices

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Abstract. In this paper, we prove that Eulerian lattices satisfying some weaker conditions for lattices or some weaker conditions for 0-distributive lattices become Boolean.

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1. Introduction

The aim of this paper is to prove some results on Eulerian lattices. We start with the concept of Möbius function [7] defined on a partially ordered set (Poset) which we need in this investigation for the definition of an Eulerian lattice.
All lattices in this chapter are assumed to be finite with 0 and 1.

Let P be a finite poset. The Möbius function μ is an integer valued function defined on $P \times P$ by the formulae:

$$\mu(x,y) = \begin{cases} 1 & \text{if } \ x = y \\ 0 & \text{if } \ x \not< y \\ -\sum_{x \le z < y} \mu(x,z) & \text{if } \ x < y \end{cases}$$

Let us recall the height of an element of a lattice L. For $a \in L$, the height of a, $ht(a)$ is the length of the longest maximal chain in $[0, a]$.

For Boolean lattices, the Möbius function assumes the value

$$\mu(x,y) = (-1)^{ht(y)-ht(x)}.$$

A natural question arises for what larger class of lattices this condition is true. A lattice L is said to be graded if all its maximal chains have same length. An Eulerian lattice is a finite graded lattice L in which $\mu(x,y) = (-1)^{ht(y)-ht(x)}$, for every $x \le y$ in L.

For the concept of Eulerian lattices we refer to [15]. Several results on Eulerian lattices are known. For instance, an interval of an Eulerian lattice is Eulerian, an Eulerian lattice is atomistic, finite product of Eulerian lattices is Eulerian, etc.

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Example. Every Boolean algebra of rank n is Eulerian and the lattice C_4 of Figure 1 is an example of a non-modular Eulerian lattice. Also, every C_n is Eulerian for $n \geq 4$.

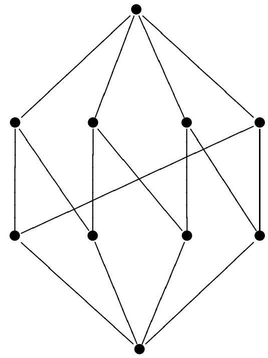

Figure 1

We note that in an Eulerian lattice L, any element of rank 2 is greater than exactly two atoms. Dually, any element of rank $r(L) - 2$ is less than exactly two co-atoms. We also know that dual of an Eulerian lattice is Eulerian. A modular Eulerian lattice turns out to be Boolean [12]. This result inspires one to ask what the resulting lattice would be if one takes weaker class of lattices than modular Eulerian lattices.

Our interest in this endeavour has originated from this result. In this paper, we are going to examine the classes of semimodular lattices, supersolvable lattices, consistent lattices, 0-distributive lattices and 0-modular lattices, 0-semi modular lattices and weaker conditions of 0-distributive lattices.

Let us first recall the definitions of these lattices.

Definition 1.1. Semimodular lattice.
A lattice L is called semimodular if whenever a covers $a \wedge b$ then $a \vee b$ covers b, for all $a, b \in L$.

Definition 1.2. Supersolvable lattice.
A lattice L is said to be supersolvable, if it contains a maximal chain called an M-chain in which every element is modular. An element m in a lattice L is modular, if $(m \vee x) \wedge y = (m \wedge y) \vee x$, whenever $x \leq y$ in L.

Definition 1.3. 0-distributive lattice.
A lattice L is said to be 0-distributive if for all $x, y, z \in L$, whenever $x \wedge y = 0$ and $x \wedge z = 0$, then $x \wedge (y \vee z) = 0$.

Definition 1.4. 0-modular lattice.
A lattice L is said to be 0-modular if whenever $x \leq y$ and $y \wedge z = 0$, then $x = (x \vee z) \wedge y$ for all $x, y, z \in L$.

Definition 1.5. 0-semimodular lattice.
A lattice L is 0-semimodular if whenever a is an atom of L and x is an element of L satisfying $a \wedge x = 0$ then $x \vee a$ covers x.

Definition 1.6. Consistent lattice.
A lattice L is said to be consistent if whenever j is a join-irreducible element in L, then for every $x \in L$, $x \vee j$ is join-irreducible in the upper interval $[x, 1]$.

Definition 1.7. Connectedness of a poset.
A finite poset P is said to be connected if for any two elements x and y in P, there is a path (in the undirected Hasse diagram) connecting x and y.

For example, C_4 which is given in Figure 1 is connected. We note that any interval of a semimodular, supersolvable and consistent lattice also has the corresponding property, see [7], [16] and [6]. So we can liberally apply inductive proof for the above lattices on the rank of the lattices.

The only Eulerian lattice of rank 2 is Boolean. So, hereafter we assume the rank of the lattices are greater than or equal to 3.

Lemma 1.8. *The meet of all co-atoms of an Eulerian lattice is* 0.

Proof: We prove this by induction on the rank of the lattice. The claim is obviously true in the only Eulerian lattice of rank 2, namely B_2.

Assume that $r(L) = d+1$ and that the result is true for any Eulerian lattice of rank $\leq d$. Now, let $a \in L$ be an atom. Then $[a, 1]$ is an Eulerian lattice of rank d. So, by induction hypothesis, a is the meet of all the co-atoms in $[a, 1]$. We have $0 = a \wedge b$, where $b \neq a$ is an atom of L. Therefore, 0 is the meet of the co-atoms containing a and b. So, 0 is the meet of all co-atoms of L. Hence we prove the result. □

Definition 1.9. Simplicial poset.
Let P be a poset with 0. P is said to be simplicial if for every element $t \in P$, $[0, t]$ is Boolean.

Dual simplicial poset is defined dually.

Lemma 1.10 ([12]). *An Eulerian lattice P of rank $d+1$ in which $P \setminus \{1\}$ is simplicial, $P \setminus \{0\}$ is dual simplicial and $P \setminus \{0, 1\}$ is connected is a Boolean algebra, for $d \geq 3$.*

The following lemma is not in print yet and the result is due to V.K. Santhi [12].

Lemma 1.11 ([12]). *Let L be an Eulerian lattice of rank $d+1$. Then $a_i \geq \binom{d+1}{i}$, $1 \leq i \leq d$, where a_i is the number of elements of height i in L. Also if $a_i = \binom{d+1}{i}$ for some $i, 1 \leq i \leq d$, then $L \approx B_{d+1}$, where B_{d+1} is a finite Boolean algebra of rank $d+1$.*

Proof: We prove this result by induction on d.

Claim: $a_i \geq \binom{d+1}{i}$.

For $d = 1$, the result is true. If $d = 2$, the only Eulerian lattice of rank 3 are either B_3 or C_n, $n \geq 4$ or their disjoint unions. Now let $d \geq 3$. Let us assume that the result is true for all the Eulerian lattices of rank d. Let $x_1, \cdots, x_{a_1}$ be the atoms and $y_1, \cdots, y_j$ be the elements of height d. Let y be a co-atom and x an atom, which is not contained below y. This is possible since L is atomistic. Let $1 \leq i \leq d$. Now consider the elements of height i in L. We know that an element of height i in $[0, y]$ is also of height i in L. Since $[0, y]$ is Eulerian of rank d, by induction hypothesis the number of elements of height i in $[0, y]$ is at least $\binom{d}{i}$. Similarly, an element of height $i-1$ in $[x, 1]$ is an element of height i in L. Since $[x, 1]$ is Eulerian of rank d, by induction hypothesis the number of elements of height $i-1$ in $[x, 1]$ is at least $\binom{d}{i-1}$. The set of all elements of height i contains the above mentioned elements in $[0, y]$ and $[x, 1]$ as a subset. Consequently, $a_i \geq \binom{d}{i} + \binom{d}{i-1} = \binom{d+1}{i}$.

Next we prove that if $a_i = \binom{d+1}{i}$ for some $i, 1 \leq i \leq d$, then L is Boolean.

Case (i): $a_1 = d + 1$ or $a_d = d + 1$.

If $a_1 = d+1$ then $[0, y_i]$ is an Eulerian lattice of rank d. Therefore, as claimed above $[0, y_i]$ should contain at least d atoms. Since L is atomistic, all the $d+1$ atoms cannot be inside $[0, y_i]$. Therefore $(d+1)$-th atom should be outside $[0, y_i]$. That is, $[0, y_i]$ contains exactly d atoms. Therefore, $[0, y_i]$ is Boolean, by induction hypothesis for all $1 \leq i \leq a_d$. Therefore $L \setminus \{1\}$ is simplicial. So it is clear that L is the lattice of all subsets of $x_i, \cdots, x_{d+1}$. Thus $L \approx B_{d+1}$. For $a_d = d+1$, by considering the dual lattice, we have $L \approx B_{d+1}$.

Case (ii): $a_i = \binom{d+1}{i}$ for some $i, 2 \leq i \leq d-1$.

Let y_1 be a coatom and x_1 an atom which is not contained below y_1. The number of height i elements in $[0, y_1]$ is at least $\binom{d}{i}$ and the number of elements of height $i-1$ in $[x_l, 1]$ is at least $\binom{d}{i-1}$. But since $a_i = \binom{d+1}{i}$, the number of height i elements in $[0, y_1]$ is $\binom{d}{i}$ and the number of height $i-1$ elements in $[x_l, 1]$ is $\binom{d}{i-1}$. So by induction, $[0, y_1]$ and $[x_l, 1]$ are Boolean. Similarly $[0, y_j]$ is Boolean for every $j, 1 \leq j \leq a_d$ and $[x_i, 1]$ is Boolean for every $i, 1 \leq i \leq a_1$. Hence, $L \setminus \{1\}$ is simplicial and $L \setminus \{0\}$ is dual simplicial. As $a_i = \binom{d+1}{i}$, $L \setminus \{1\}$ is simplicial and $L \setminus \{0\}$ is dual simplicial, $L \setminus \{0, 1\}$ is connected. Thus, by Lemma 1.10, $L \approx B_{d+1}$. $\square$

Lemma 1.12. *For an Eulerian lattice L of rank at least three satisfying any one of the following conditions:*

(i) *semimodularity,*
(ii) *supersolvability,*
(iii) *0-distributivity,*
(iv) *0-modularity,*
(v) *0-semimodularity and*
(vi) *consistency,*

$L \setminus \{0,1\}$ is connected.

Proof: Let L be an Eulerian lattice. Suppose $L \setminus \{0,1\}$ is not connected. Then there exist two elements $x, y \in L \setminus \{0,1\}$ such that $x \wedge y = 0$ and $x \vee y = 1$. Without loss of generality we can assume that x and y are atoms. We also note that any element of $L \setminus \{0,1\}$ which is connected with one of them is not connected with the other.

(i) If L is semimodular, x covers $0 = x \wedge y$ while y being an atom cannot be covered by $1 = x \vee y$. This is a contradiction with semimodularity. So, $L \setminus \{0,1\}$ is connected.

(ii) If L is supersolvable, no element of an M-chain of L is connected with both x and y. So, every element of an M-chain can be connected with at most one of x and y. Let m be an element in an M-chain of L. Let m be connected with x, say. Now take an element $z \in L \setminus \{0,1\}$ such that $y \leq z$. We have $(m \vee y) \wedge z = z \neq y = y \vee (m \wedge z)$, which contradicts the modularity of m. Similar argument works, if m is connected with y. Let m be connected with neither x nor y. Then the above argument also holds. So, $L \setminus \{0,1\}$ is connected.

(iii) Suppose L is 0-distributive. Since L is Eulerian, we can find one more atom, say, z not connected with y. Then we have that $x \wedge y = 0$ and $x \wedge z = 0$ imply that $x \wedge (y \vee z) = 0$, by 0-distributivity. That is, $x \wedge 1 = 0$ and $x = 0$, which is a contradiction. Therefore, $L \setminus \{0,1\}$ is connected.

(iv) Let L be 0-modular. Suppose $L \setminus \{0,1\}$ is not connected. Consider an element $z > x$, such that $z \neq 1$. Now $z \wedge y = 0$. But by 0-modularity, $x = (x \vee y) \wedge z = z$ which is impossible. So, $L \setminus \{0,1\}$ is connected.

(v) Let L be 0-semimodular. Now x is an atom, y being an atom is not covered by $1 = x \vee y$, which contradicts the 0-semimodularity of L. So, $L \setminus \{0,1\}$ is connected.

(vi) Let L be consistent. The atom x is join-irreducible and $[y,1]$ is an Eulerian lattice in its own right. So, we can find two co-atoms a, b in $[y,1]$ whose join is $1 = x \vee y$ in $[y,1]$. This says that $x \vee y$ is not join-irreducible in $[y,1]$ contradicting the consistency of L. Therefore, $L \setminus \{0,1\}$ is connected. □

Theorem 1.13. *The following conditions are equivalent in an Eulerian lattice L:*

(a) *L is Boolean,*
(b) *L is 0-distributive,*
(c) *L is 0-modular,*
(d) *L is 0-semimodular,*
(e) *L is consistent,*
(f) *L is supersolvable.*

Proof: Let $r(L) = d + 1$.

(a)⇒(b), (c), (d), (e), and (f) are trivial.

(b)⇒(a): Let L be a 0-distributive Eulerian lattice. Let a be an atom in L. We claim that there exists a unique co-atom c in L which is not comparable with a. First, note that there is a co-atom not greater than a. Indeed, if all the co-atoms of L are greater than a, then as $[a,1]$ is an Eulerian lattice, by Lemma 1.8, a is the meet of all the co-atoms of L. But the meet of all the co-atoms of L is 0, by Lemma 1.8. That is $a = 0$, which is a contradiction. Therefore, there is a co-atom c_1 such that $c_1 \not> a$. Suppose that there is one more co-atom $c_2 \not> a$ then we have $a \wedge c_1 = 0$ and $a \wedge c_2 = 0$ which is by 0-distributivity gives, $a \wedge (c_1 \vee c_2) = 0$, that is, $a \wedge 1 = 0$. That is, $a = 0$, which is absurd. So, there is a unique co-atom not greater than a. We call it c.

When $d = 1, 2$, L is Boolean, since C_n is not 0-distributive, for $n \geq 4$. By induction hypothesis, for the co-atom $c, [0, c]$ being an Eulerian 0-distributive lattice is Boolean of rank d. So, $[0, c]$ contains exactly d atoms. Now L being an Eulerian lattice of rank $d+1$, must have at least $d+1$ atoms, by Lemma 1.8. Let a be a $(d+1)$-th atom, outside $[0, c]$. If L has one more atom $b \neq a$ such that $b \notin [0, c]$, then since $[a, 1]$ and $[b, 1]$ are Eulerian and by the uniqueness of the co-atom $c \not> a,\ b$, we have that both a and b are less than every other co-atom. So by Lemma 1.8, a is the meet of all co-atoms except c and b is the meet of all co-atoms except c which means, $a = b$. That is, there is only a unique atom of L outside $[0, c]$. So L contains exactly $d+1$ atoms. Therefore, L is Boolean (by Lemma 1.10).

(c)⇒(a): Suppose, L is a 0-modular Eulerian lattice of rank $d+1$. When $d = 1, 2$, then since $L \setminus \{0, 1\}$ is connected, it is easy to see that the non-Boolean Eulerian lattices of rank 3 are not 0-modular. So L is Boolean. Assume all the Eulerian 0-modular lattices of rank $\leq d$ are Boolean. By induction hypothesis, $[0, x]$ is Boolean, for $x \in L$ of height d. We claim that L is Boolean of rank $d+1$.

Suppose, L is not Boolean. So, as L is of rank $d+1$, the number of atoms of L must be at least $d+2$, by Lemma 1.11. Let x and y be two co-atoms in L whose meet z is of rank $d-1$. So, $[0, x] \cup [0, y] \approx [0, z] \approx B_{d-1}$. So, $[0, x]$ and $[0, y]$ have $d-1$ atoms in common. So the number of atoms in $[0, x] \cup [0, y]$ is $d+1$. Since L has at least $d+2$ atoms, we can find one more atom a in L other than these such that $a \not< x$ and $a \not< y$. So, we have $(z \vee a) \wedge x = 1 \wedge x = x \neq z$, contradicting 0-modularity. Therefore, our assumption that L is not Boolean is not wrong. So, L is Boolean.

(d)⇒(a): When $d = 1, 2$, L is clearly Boolean, since $L \setminus \{0, 1\}$ is connected and the only non-Boolean Eulerian lattices C_n, $n \geq 4$ or rank 3 are not 0-semimodular. Indeed, we can find two atoms in C_n, $n \geq 4$, whose meet is 0 and whose join is 1, which does not cover the atom. The remaining part of the proof is the same as that for $(c) => (a)$, replacing 0-modularity with 0-semimodularity.

(e)⇒(a): When $d = 1, 2, L$ is clearly Boolean since $L \setminus \{0, 1\}$ is connected by Lemma 1.12 and the only non-Boolean Eulerian lattices L with $L \setminus \{0, 1\}$ being connected and of rank 3 are C_n, $n \geq 4$, which is not consistent. Assume the result is true for d. Now, let $x \in L \setminus \{1\}$ be such that $r(x) \leq d$. The interval $[0, x]$ is an Eulerian, consistent lattice of rank $\leq d$, which by induction hypothesis is Boolean. So, $L \setminus \{1\}$ is simplicial.

Similarly, for any atom $a \in L$, the interval $[a, 1]$ is an Eulerian, consistent lattice of rank d which is Boolean by induction hypothesis. Therefore, $L \setminus \{0\}$ is dual simplicial. By Lemma 1.12(vi), $L \setminus \{0,1\}$ is connected. So, by Lemma 1.10, L is Boolean. The proof for (f)$\Rightarrow$(a) is similar. $\square$

2. Weaker conditions for 0-distributive lattices

In this section, we examine the classes of pseudo-0-distributive lattices and super-0-distributive lattices, which have appeared in the paper [3].

Definition 2.1 ([3])**.** A lattice L is said to be pseudo-0-distributive if for all $a, b, c \in L$, $a \wedge b = 0$ and $a \wedge c = 0$ imply that $(a \vee b) \wedge c = b \wedge c$.

Definition 2.2 ([3])**.** A lattice L is said to be super-0-distributive if for $a, b, c \in L$, $a \wedge b = 0$ implies that $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$.

Remark 2.3 ([3])**.** If a lattice L is super-0-distributive then it is pseudo-0-distributive. Also if L is pseudo-0-distributive then it is 0-modular. Any pseudo-complemented 0-modular lattice is pseudo-0-distributive.

Lemma 2.4. *For an Eulerian lattice L of rank at least 3 satisfying any one of the following conditions:*

(i) *pseudo-0-distributive,*
(ii) *super-0-distributive,*

$L \setminus \{0,1\}$ *is connected.*

PROOF: **Case (i):** Let L be a pseudo-0-distributive lattice. Let us assume that the rank of L is greater than 2. Suppose $L \setminus \{0,1\}$ is not connected. Then there exist two elements $x, y \in L \setminus \{0,1\}$ such that $x \vee y = 1$ and $x \wedge y = 0$. Without loss of generality we can assume that x and y are atoms. Since L is Eulerian, we can find one more atom z in L. That is, $x \wedge y = 0$ and $x \wedge z = 0$ imply that $(x \vee y) \wedge z = z \neq y \wedge z = 0$, which is a contradiction to the pseudo-0-distributivity. Therefore, $L \setminus \{0,1\}$ is connected.

Case (ii) follows from Case (i), since every super-0-distributive is pseudo-0-distributive. $\square$

Theorem 2.5. *The following conditions are equivalent in an Eulerian lattice L:*

(a) *L is Boolean,*
(b) *L is Super-0-distributive,*
(c) *L is Pseudo-0-distributive.*

PROOF: (a)$\Rightarrow$(b) and (b)$\Rightarrow$(c) are trivial.

(c)$\Rightarrow$(a): Let the rank of L be $d+1$. We prove the result by induction on d. When $d = 1, 2$, L is clearly Boolean, since $L \setminus \{0,1\}$ is connected by Lemma 2.4. and the only non-Boolean Eulerian lattice of rank 3 with $L \setminus \{0,1\}$ connected are C_n, $n \geq 4$. However the lattices C_n, $n \geq 4$ are not pseudo-0-distributive.

Assume that the result is true for d. Let a be an atom of L. Then there exists a co-atom c such that $c \not\geq a$. Therefore, $a \wedge c = 0$ and $a \vee c = 1$. We show the uniqueness of c. If c_1 is another co-atom not greater than a then

$$\begin{aligned}(a \vee c) \wedge c_1 &= c \wedge c_1 \text{ (by pseudo-0-distributivity of } L),\\ 1 \wedge c_1 &= c \wedge c_1,\\ c_1 &= c \wedge c_1.\end{aligned}$$

That is, $c_1 \leq c$. Since both c and c_1 are co-atoms, $c_1 = c$. So there exists a unique co-atom c such that $c \not\geq a$.

Now, by induction hypothesis $[0, c]$ is Boolean. So, $[0, c]$ contains exactly d atoms and a is a $(d+1)$-th atom of L. Now, the rank of L is $d+1$. So L being Eulerian must contain at least $d+1$ atoms. If b is another atom such that $b \notin [0, c]$ and $b \neq a$ and $b \not< a$, by uniqueness of co-atom c, both a and b are less than every other co-atom. Therefore, a is the meet of all co-atoms other than c and b is the meet of all co-atoms other than c. So, $a = b$. Therefore, there is a unique atom of L outside $[0, c]$. So, L contains exactly $d+1$ atoms. Consequently, it is Boolean by Lemma 1.10. □

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