On the metric reflection of a pseudometric space in ZF

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Abstract. We show:
(i) The countable axiom of choice **CAC** is equivalent to each one of the statements:
(a) a pseudometric space is sequentially compact iff its metric reflection is sequentially compact,
(b) a pseudometric space is complete iff its metric reflection is complete.
(ii) The countable multiple choice axiom **CMC** is equivalent to the statement:
(a) a pseudometric space is Weierstrass-compact iff its metric reflection is Weierstrass-compact.
(iii) The axiom of choice **AC** is equivalent to each one of the statements:
(a) a pseudometric space is Alexandroff-Urysohn compact iff its metric reflection is Alexandroff-Urysohn compact,
(b) a pseudometric space **X** is Alexandroff-Urysohn compact iff its metric reflection is ultrafilter compact.
(iv) We show that the statement "The preimage of an ultrafilter extends to an ultrafilter" is not a theorem of **ZFA**.

Keywords: weak axioms of choice; pseudometric spaces; metric reflections; complete metric and pseudometric spaces; limit point compact; Alexandroff-Urysohn compact; ultrafilter compact; sequentially compact

Classification: 54E35, 54E45

1. Notation and terminology

Let $\mathbf{X} = (X, T)$ be a topological space. As usual, we denote topological spaces by fat letters and underlying sets by non-fat letters.

$\mathbf{X}$ is said to be *compact* iff every open cover $\mathcal{U}$ of $\mathbf{X}$ has a finite subcover $\mathcal{V}$.

$\mathbf{X}$ is said to be *countably compact* iff every countable open cover $\mathcal{U}$ of $\mathbf{X}$ has a finite subcover $\mathcal{V}$.

$\mathbf{X}$ is said to be *sequentially compact* iff every sequence $(x_n)_{n\in\mathbb{N}}$ of points of X has a convergent subsequence.

$\mathbf{X}$ is called *Alexandroff-Urysohn compact* iff every infinite subset A of X has a *complete accumulation point* x (for every neighborhood V of x, $|A \cap V| = |A|$).

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$\mathbf{X}$ is called *limit point compact* iff every infinite subset A of X has a *limit point* x (for every neighborhood V of x, $V \cap A \backslash \{x\} \neq \varnothing$).

$\mathbf{X}$ is called *Weierstrass-compact* iff every infinite subset A of X has an *accumulation point* x (for every neighborhood V of x, $V \cap A$ is infinite).

$\mathbf{X}$ is said to be *ultrafilter compact* iff every ultrafilter $\mathcal{F}$ of X *converges* to some point x in $\mathbf{X}$ (for every neighborhood V of x, there exists $F \in \mathcal{F}$ with $V \supseteq F$).

Let X be a non-empty set. A function $\rho : X \times X \to \mathbb{R}$ is called *pseudometric* in case ρ satisfies all the requirements of a metric except possibly the requirement $\rho(x, y) = 0$ implies $x = y$. If (X, ρ) is a pseudometric space, then the metric reflection (X^*, ρ^*) of (X, ρ) is the set X^* of all equivalence classes in X of the equivalence relation $\sim$ given by:

$$x \sim y \text{ iff } \rho(x, y) = 0$$

and $\rho^* : X^* \times X^* \to \mathbb{R}$ is given by

$$\rho^*([x], [y]) = \rho(x, y), \tag{1}$$

where $[x]$ denotes the equivalence class of the element x.

Let $\mathbf{X} = (X, d)$ be a pseudometric space, $x \in X$ and $\varepsilon > 0$.

$$D(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$$

denotes the open disc in $\mathbf{X}$ with center x and radius ε. If $B \subseteq X$, then $\delta(B) = \sup\{d(x, y) : x, y \in B\}$ is the *diameter* of B.

$\mathbf{X}$ is *totally bounded* iff for every real number $\varepsilon > 0$, there exists an ε-net, i.e., a finite subset $\{x_i : i \leq n\}$ of X such that $\bigcup\{D(d_i, \varepsilon) : i \leq n\} = X$.

$\mathbf{X}$ is *Cantor complete* iff $\bigcap\{G_n : n \in \omega\} \neq \varnothing$ for every descending set $\{G_n : n \in \omega\}$ of non-empty closed subsets of $\mathbf{X}$ with $\lim_{n\to\infty} \delta(G_n) = 0$.

$\mathbf{X}$ is said to be *sequentially bounded* if each sequence of points of $\mathbf{X}$ has a Cauchy-subsequence.

Let $h : \mathbf{X} \to \mathbf{X}^*$ be the mapping given by $h(x) = [x]$. Clearly, a set A in $\mathbf{X}$ is closed (open) iff it is saturated (i.e., contains with any element a each element b with $a \sim b$) and $h(A)$ is closed (open) in $\mathbf{X}^*$. Also, in view of (1) we see that for every $\varepsilon > 0$ and every $y, x \in X$,

$$y \in D(x, \varepsilon) \text{ iff } [y] \in D([x], \varepsilon),$$

i.e.,

$$h(D(x, \varepsilon)) = D([x], \varepsilon). \tag{2}$$

Therefore, we have the following straightforward result:

Proposition 1. *Let* $\mathbf{X}$ *be a pseudometric space. Then:*

(i) $\mathbf{X}$ *is compact (resp. countably compact) iff* $\mathbf{X}^*$ *is compact (resp. countably compact).*
(ii) $\mathbf{X}$ *is totally bounded iff* $\mathbf{X}^*$ *is totally bounded.*
(iii) $\mathbf{X}$ *is Cantor complete iff* $\mathbf{X}^*$ *is Cantor complete.*

Below we list the choice principles we shall be dealing with in the sequel.

1. **AC** (**Form 1**, in [6]): Every family $\mathcal{A} = (A_i)_{i \in I}$ of non-empty sets has a choice function.
2. **CAC** (**Form 8**, in [6]): Every family $\mathcal{A} = (A_i)_{i \in \mathbb{N}}$ of non-empty sets has a choice function. Equivalently, for every family $\mathcal{A} = (A_i)_{i \in \mathbb{N}}$ of non-empty sets has a partial choice function. i.e., there exists an infinite subfamily $\mathcal{B}$ of $\mathcal{A}$ with a choice function.
3. $\mathbf{CAC}_{fin}$ (**Form 10**, in [6]): Every family $\mathcal{A} = (A_i)_{i \in \mathbb{N}}$ of non-empty finite sets has a choice function.
4. **IDI** (**Form 9** in [6]): Every infinite set is Dedekind infinite (has a countably infinite subset).
5. **IWDI** (**Form 82** in [6]): Every infinite set is weakly Dedekind infinite (its powerset has a countably infinite subset). Equivalently, for every infinite set X there is a function from X onto ω, (**Form 82** [A] in [6]).
6. **CMC** (**Form 126** in [6]): For every family $\mathcal{A} = (A_i)_{i \in \mathbb{N}}$ of non-empty sets there exists a family $\mathcal{B} = (B_i)_{i \in \mathbb{N}}$ of non-empty finite sets such that for every $i \in \mathbb{N}$, $B_i \subseteq A_i$. Equivalently, for every family $\mathcal{A} = (A_i)_{i \in \mathbb{N}}$ of non-empty sets there exists an infinite subfamily $\mathcal{C} = (A_{i_n})_{n \in \mathbb{N}}$ and a family $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ of non-empty finite sets such that for every $n \in \mathbb{N}$, $B_n \subseteq A_{i_n}$.
7. **KW** (Kinna-Wagner selection principle, Form 15 in [6]): Every family $\mathcal{A} = (A_i)_{i \in I}$ of non-empty sets has a Kinna-Wagner selection, i.e., a family $\mathcal{B} = (B_i)_{i \in I}$ of non-empty sets such that for every $i \in I$, $B_i \subseteq A_i$ and if $|A_i| > 1$ then $B_i \neq A_i$.
8. **BPI** (Boolean Prime Ideal Theorem, **Form 14** in [6]): Every Boolean algebra has a prime ideal.
9. **SPI** (Weak Ultrafilter Principle, **Form 63** in [6]): Every infinite set has a non-trivial ultrafilter.
10. **UF**(ω) (**Form 70** in [6]): There is a non-trivial ultrafilter on ω.
11. **PUU** : The preimage of an ultrafilter extends to an ultrafilter. Equivalently, for every set X, for every partition P of X, if $\mathcal{F}$ is an ultrafilter of P then the filterbase $\{\bigcup F : F \in \mathcal{F}\}$ of X extends to an ultrafilter.

2. Introduction and some preliminary results

The set theoretic setting in this paper is the Zermelo-Fraenkel set theory **ZF** without the axiom of choice **AC**. In **ZFC** (= **ZF** and **AC**), there are several equivalent notions of compactness for pseudometric, as well as, for metric spaces.

See, e.g., [1] and [7]. The following theorem is by no means a complete list of these equivalent forms.

Theorem 2 ([1], [7], [8] (**ZFC**)). *Let* $\mathbf{X}$ *be a pseudometric space. Then the following are equivalent:*

(i) $\mathbf{X}$ *is compact;*
(ii) $\mathbf{X}$ *is Weierstrass-compact;*
(iii) $\mathbf{X}$ *is sequentially compact;*
(iv) $\mathbf{X}$ *is Cantor complete and totally bounded;*
(v) $\mathbf{X}$ *is complete and totally bounded;*
(vi) $\mathbf{X}$ *is countably compact;*
(vii) $\mathbf{X}$ *is Alexandroff-Urysohn compact;*
(viii) $\mathbf{X}$ *is ultrafilter compact;*
(ix) $\mathbf{X}$ *is complete and sequentially bounded.*

In the present project, we study those compactness forms which are shared in **ZF** (resp. **ZF** + **WAC**, where **WAC** is some weak axiom of choice, **ZFC**) by pseudometric spaces and their metric reflections.

Let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ be a disjoint family of sets such that for every $n \in \mathbb{N}$, $1 < |A_n| < \aleph_0$. Define a pseudometric d of $X = \bigcup\{A_n : n \in \mathbb{N}\}$ by requiring:

$$d(x,y) = \begin{cases} 0 & \text{is } \ x, y \in A_n \text{ for some } n \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases} . \tag{3}$$

It is easy to see that every non-empty subset A of $\mathbf{X}$ has trivially a limit point and, no infinite subset of $\mathbf{X}$ has an accumulation point. In particular, $\mathbf{X}$ is limit point compact but not Weierstrass-compact. Thus, in **ZFC**, limit point compact pseudometric spaces need not be Weierstrass-compact. In addition, the metric reflection $\mathbf{X}^*$ of $\mathbf{X}$ being a discrete space is not limit point compact. So, the statement:

(a) If a pseudometric space $\mathbf{X}$ is limit point compact then so is $\mathbf{X}^*$

is a false statement in **ZFC**. However, the statement:

(b) If a pseudometric space $\mathbf{X}$ is Weierstrass-compact then $\mathbf{X}^*$ is limit point compact,

as is shown in Theorem 6, is equivalent to the countable multiple choice axiom **CMC**.

In Theorem 4 we show that for every pseudometric space $\mathbf{X}$ if its metric reflection $\mathbf{X}^*$ is sequentially compact then $\mathbf{X}$ is sequentially compact. Moreover, the converse holds iff the countable axiom of choice **CAC** holds true.

Likewise, in Theorem 5 we show that for every pseudometric space $\mathbf{X}$ if the metric reflection $\mathbf{X}^*$ is complete then $\mathbf{X}$ is complete and, in addition, the converse holds iff **CAC** holds true.

Finally, in Theorem 8 we show that **AC** is equivalent to each one of the statements:

(c) A pseudometric space $\mathbf{X}$ is Alexandroff-Urysohn compact iff $\mathbf{X}^*$ is Alexandroff-Urysohn compact,

and

(d) A pseudometric space $\mathbf{X}$ is Alexandroff-Urysohn compact iff $\mathbf{X}^*$ is ultrafilter compact.

Theorem 3. **CMC** *iff for every family $\mathcal{A} = (A_i)_{i\in I}$ of disjoint non-empty sets there exists an infinite subfamily $\mathcal{B}$ of $\mathcal{A}$ with a multiple choice function.*

PROOF: It suffices to show ($\rightarrow$) as the other implication is straightforward. Fix a disjoint family $\mathcal{A} = (A_i)_{i\in I}$ of non-empty sets. For every $n \in \mathbb{N}$, let $B_n = [I]^n$ denote the set of all n-element subsets of I. Fix, by **CMC**, a multiple choice set $\mathcal{C} = \{C_n : n \in \mathbb{N}\}$ of the set $\{B_n : n \in \mathbb{N}\}$ and let for every $n \in \mathbb{N}$, $I_n = \bigcup C_n$. Clearly, $I_n \in [I]^{<\omega}$ where $[I]^{<\omega}$ denotes the set of all finite subsets of I. Without loss of generality we may assume that for all $n, m \in \mathbb{N}$, $I_n \cap I_m = \varnothing$. Put $\mathcal{E} = \{E_n : n \in \mathbb{N}\}$ where, for all $n \in \mathbb{N}$, $E_n = \bigcup\{A_i : i \in I_n\}$ and let, by **CMC** again, $\mathcal{H} = \{H_n : n \in \mathbb{N}\}$ be a multiple choice set of $\mathcal{E}$. For every $n \in \mathbb{N}$, let $I_n^{'} = \{i \in I_n : H_n \cap A_i \neq \varnothing\}$. Clearly, the subfamily $\mathcal{F} = (A_i)_{i\in I'}$ where $I^{'} = \bigcup\{I_n^{'} : n \in \mathbb{N}\}$ is infinite and has a multiple choice set, finishing the proof of the theorem. □

3. Main results

It is known, in the realm of pseudometric spaces, that total boundedness implies sequential boundedness, and that both concepts are equivalent iff **CAC** holds. See [1] Section 2. We show next that **CAC** is equivalent to each one of the statements: "A pseudometric space $\mathbf{X}$ is sequentially compact iff $\mathbf{X}^*$ is sequentially compact" and "a pseudometric space $\mathbf{X}$ is sequentially bounded iff $\mathbf{X}^*$ is sequentially bounded".

Theorem 4. *The following statements are equivalent:*

(i) **CAC**;
(ii) *a pseudometric space $\mathbf{X}$ is sequentially compact iff $\mathbf{X}^*$ is sequentially compact;*
(iii) *a pseudometric space $\mathbf{X}$ is sequentially bounded iff $\mathbf{X}^*$ is sequentially bounded.*

PROOF: (i)$\rightarrow$(ii) ($\rightarrow$) Fix a sequence $(c_n)_{n\in\mathbb{N}}$ of points of X^* and fix, by **CAC**, $x_n \in c_n$ for every $n \in \mathbb{N}$. Since $\mathbf{X}$ is sequentially compact, it follows that some subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ converges to some point $x \in \mathbf{X}$. We claim that the subsequence $(c_{n_k})_{k\in\mathbb{N}}$ of $(c_n)_{n\in\mathbb{N}}$ converges to $c = [x]$. Indeed, fix $\varepsilon > 0$ and let $n_0 \in \mathbb{N}$ satisfy: $\forall k \geq n_0, d(x_{n_k}, x) < \varepsilon$. Then,

$$\forall k \geq n_0, d^*(c_{n_k}, c) = d^*([x_{n_k}], [x]) = d(x_{n_k}, x) < \varepsilon$$

and $(c_{n_k})_{k\in\mathbb{N}}$ converges to c as required.

($\leftarrow$) We show that this direction holds true in **ZF**. Fix a sequence $(x_n)_{n\in\mathbb{N}}$ of points of X and let for every $n \in \mathbb{N}$, $c_n = [x_n] \in X^*$. Since $\mathbf{X}^*$ is sequentially compact, it follows that $(c_n)_{n\in\mathbb{N}}$ has a limit point $c = [x]$. Let $(c_{n_k})_{k\in\mathbb{N}}$ be a subsequence of $(c_n)_{n\in\mathbb{N}}$ converging to c. Fix $\varepsilon > 0$ and let $n_0 \in N$ satisfy: $\forall k \geq n_0, d^*(c_{n_k}, c) < \varepsilon$. Then, $\forall n \geq n_0, d(x_{n_k}, x) = d^*(c_{n_k}, c) < \varepsilon$. Thus, the subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ converges to x and $\mathbf{X}$ is limit point compact as required.

(ii)$\to$(i) Fix $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ a disjoint family of non-empty sets. Assume, aiming for a contradiction, that $\mathcal{A}$ has no infinite subfamily with a choice set. Let d be the pseudometric on $X = \bigcup\{A_n : n \in \mathbb{N}\}$ given by (3).

We claim that $\mathbf{X}$ is sequentially compact. To see this, fix $(x_n)_{n\in\mathbb{N}}$ a sequence of points of X. Then for some $n \in \mathbb{N}$,

$$E_n = \{m \in \mathbb{N} : x_m \in A_n\}$$

is infinite as otherwise a partial choice for the family $\mathcal{A}$ can be easily derived. Let k_m denote the m-th element of E_n. Clearly, $(x_{k_m})_{m\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$ and for every $x \in A_n$, $d(x_{k_m}, x) = 0$. Thus, $(x_{k_m})_{m\in\mathbb{N}}$ converges to x and $\mathbf{X}$ is sequentially compact as required. Therefore, by our hypothesis, $\mathbf{X}^*$ is sequentially compact. Since for every $n \in \mathbb{N}$, $A_n \in X^*$ and for every $n, m \in \mathbb{N}$ with $n \neq m$, $d^*(A_n, A_m) = 1$, it follows that the sequence $(A_n)_{n\in\mathbb{N}}$ of points of $\mathbf{X}^*$ has no convergent subsequence. Contradiction! Thus, $\mathcal{A}$ has a choice set and **CAC** holds as required.

(i)$\to$(iii) This is straightforward.

(iii)$\to$(i) Fix $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$, d and $X = \bigcup\{A_n : n \in \mathbb{N}\}$ as in the proof of (ii)$\to$(i). We claim, assuming that $\mathcal{A}$ has no infinite subfamily with a choice function, that $\mathbf{X}$ is sequentially compact. To see this, fix $(x_n)_{n\in\mathbb{N}}$ a sequence of points of X. Clearly, the subsequence $(x_{k_m})_{m\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ given in the proof of (ii)$\to$(i) is Cauchy. Thus, $\mathbf{X}$ is sequentially bounded. Hence, by our hypothesis, $\mathbf{X}^*$ is sequentially bounded. Contradiction! (The sequence $(A_n)_{n\in\mathbb{N}}$ of points of $\mathbf{X}^*$ has clearly no Cauchy subsequence). $\square$

Theorem 5. *The following statements are equivalent:*

(i) **CAC**;
(ii) *a pseudometric space* $\mathbf{X}$ *is complete iff* $\mathbf{X}^*$ *is complete.*

Proof: (i)$\to$(ii) ($\to$) Fix a Cauchy sequence $(c_n)_{n\in\mathbb{N}}$ of points of X^* and fix, by **CAC**, $x_n \in c_n$ for every $n \in \mathbb{N}$. Since, by (1), for all $n, m \in \mathbb{N}$,

$$d(x_n, x_m) = d^*(c_n, c_m) \tag{4}$$

it follows that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of points of $\mathbf{X}$. Since $\mathbf{X}$ is complete, it follows that $(x_n)_{n\in\mathbb{N}}$ converges to some point $x \in \mathbf{X}$. In view of (4) it follows that the sequence $(c_n)_{n\in\mathbb{N}}$ converges to $c = [x]$ and $\mathbf{X}^*$ is complete as required.

($\leftarrow$) We show that this direction holds true in **ZF**. Fix a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ of points of X and let for every $n \in \mathbb{N}$, $c_n = [x_n] \in X^*$. By (4), $(c_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of $\mathbf{X}^*$. Since $\mathbf{X}^*$ is complete, it follows that $(c_n)_{n\in\mathbb{N}}$ converges to some point $c = [x]$ of $\mathbf{X}^*$. In view of (1) we see that $(x_n)_{n\in\mathbb{N}}$ converges to x. Hence, $\mathbf{X}$ is complete as required.

(ii)$\rightarrow$(i) Fix $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ and X as in the proof of (ii)$\rightarrow$(i) of Theorem 4 and let $d : X \times X \to \mathbb{R}$ be given by the rule:

$$(5) \qquad d(x,y) = \begin{cases} 0 \text{ is } x, y \in A_n \text{ for some } n \in \mathbb{N} \\ 1/n \text{ if } x \in A_n, y \in A_m \text{ and } n < m \end{cases}.$$

Assume, aiming for a contradiction, that $\mathcal{A}$ has no partial choice. We claim that (X,d) is complete. Indeed, if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of points of X then, as in the proof of (ii)$\rightarrow$(i), $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{k_n})_{n\in\mathbb{N}}$ say to the point $x \in X$. Since, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, it follows that $(x_n)_{n\in\mathbb{N}}$ converges to x. Hence, (X,d) is complete as required.

We claim that $(A_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of $\mathbf{X}^*$. Indeed, for every $\varepsilon > 0$ pick $n_0 \in \mathbb{N}$ such that $1/n_0 < \varepsilon$. Then, for every $n, m \geq n_0$, $d^*(A_n, A_m) \leq 1/n_0 < \varepsilon$. However, $(A_n)_{n\in\mathbb{N}}$ converges to no point of $\mathbf{X}^*$. Contradiction! Thus, $\mathcal{A}$ has a choice set and **CAC** holds as required. $\square$

Theorem 6. *The following statements are equivalent:*

(i) **CMC**;
(ii) *a pseudometric space* $\mathbf{X}$ *is Weierstrass-compact iff* $\mathbf{X}^*$ *is Weierstrass-compact.*

Proof: Assume that **CMC** holds and show that "a pseudometric space $\mathbf{X}$ is Weierstrass-compact iff its metric reflection $\mathbf{X}^*$ is limit point compact".

($\leftarrow$) We show that this direction holds true in **ZF**. Fix an infinite subset A of X. If for some $x \in X$, $[x] \cap A$ is infinite then x is clearly an accumulation point of A. Otherwise, the set $B = \{[a] : a \in A\}$ is an infinite subset of $\mathbf{X}^*$. Hence, by our hypothesis, B has a limit point $b = [x]$ for some $x \in X$. Thus, for every $\varepsilon > 0$, $D(b,\varepsilon) \cap B$ is an infinite subset of B. Since $y \in D(x,\varepsilon) \iff [y] \in D(b,\varepsilon)$ we see that for every $\varepsilon > 0$, $D(x,\varepsilon) \cap A$ is an infinite subset of A. Thus, x is a limit point of A and $\mathbf{X}$ is Weierstrass-compact.

($\rightarrow$) Fix an infinite subset A of X^*. Let, by Theorem 3, $\{h^{-1}(a_i) : i \in I\}$ be an infinite subfamily of $\{h^{-1}(a) : a \in A\}$ with a multiple choice set $\mathcal{G} = \{G_i : i \in I\}$. Clearly, $G = \bigcup\{G_i : i \in I\}$ is an infinite subset of X. Hence, by our hypothesis G has an accumulation point g, i.e., for every $\varepsilon > 0$, $D(g,\varepsilon) \cap G$ is an infinite subset of G. Since, for every $a \in A$, $G \cap h^{-1}(a)$ is a finite set, it follows that for every $\varepsilon > 0$, $D([g],\varepsilon) \cap A$ is an infinite subset of A. Hence, $[g]$ is an accumulation point of A and $\mathbf{X}^*$ is Weierstrass-compact as required.

We assume that for every pseudometric space (X,d), $\mathbf{X}$ is Weierstrass-compact iff $\mathbf{X}^*$ is Weierstrass-compact and show that **CMC** holds true.

Assume on the contrary and let $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ be a family of non-empty sets having no infinite subfamily $\mathcal{B} = \{B_{i_n} : n \in \mathbb{N}\}$ with a multiple choice set. Put $X = \bigcup \mathcal{A}$ and let $d : X \times X \to \mathbb{R}$ be the pseudometric given by (3). We show that $\mathbf{X}$ is Weierstrass-compact. Fix K an infinite subset of X. If $K \cap A_i$ is infinite for some $i \in \mathbb{N}$, then every member of A_i is an accumulation point of K. So, we assume that for every $i \in \mathbb{N}$, $K_i = K \cap A_i$ is a finite subset of A_i. Since K is infinite, it follows that $K = \{K_i : i \in \mathbb{N}\} \backslash \{\varnothing\}$ is a multiple choice of an infinite subfamily of $\mathcal{A}$. Contradiction! Thus, $\mathbf{X}$ is Weierstrass-compact. Hence, by our hypothesis, $\mathbf{X}^*$ is Weierstrass-compact contradicting the fact that $\mathbf{X}^*$ is an infinite discrete space. Thus, **CMC** holds true as required. □

Proposition 7. *The following statements are equivalent:*

(i) **AC**;
(ii) *a topological space is compact iff it is Alexandroff-Urysohn compact* ([4]);
(iii) *a topological space is ultrafilter compact iff it is Alexandroff-Urysohn compact* ([4]);
(iv) *a pseudometric space is compact iff it is Alexandroff-Urysohn compact;*
(v) *a pseudometric space is ultrafilter compact iff it is Alexandroff-Urysohn compact.*

PROOF: (i)↔(ii)↔(iii) These have been established in [4].

The implications (ii)→(iv) and (iii)→(v) are straightforward.

(iv)→(i) We mimic the proof of Theorem 3.22 from [4]. It suffices to show that for any two non-empty disjoint sets A, B either $|A| \leq |B|$ or $|B| \leq |A|$. Let $X = A \cup B$ and define a pseudometric $d : X \times X \to \mathbb{R}$ by requiring:

$$d(x,y) = \begin{cases} 1 \text{ if } x \in A \text{ and } y \in B \text{ or, } x \in B \text{ and } y \in A \\ 0 \text{ otherwise} \end{cases} .$$

Clearly, $\mathbf{X}$ is a compact pseudometric space. Thus, by our hypothesis, $\mathbf{X}$ is Alexandroff-Urysohn compact and consequently X has a complete accumulation point x. If $x \in A$ then $A = D(x,1)$ is a neighborhood of x and $|A| = |A \cup B|$ meaning that $|B| \leq |A|$. Similarly, if $x \in B$ then (v)→(i) This can be proved as in (iv)→(i). $|A| \leq |B|$, finishing the proof of the proposition.

□

Theorem 8. *The following statements are equivalent:*

(i) **AC**;
(ii) *a pseudometric space* $\mathbf{X}$ *is Alexandroff-Urysohn compact iff* $\mathbf{X}^*$ *is Alexandroff-Urysohn compact;*
(iii) *a pseudometric space* $\mathbf{X}$ *is Alexandroff-Urysohn compact iff* $\mathbf{X}^*$ *is ultrafilter compact.*

PROOF: (i)→(ii) (→) This follows at once from Proposition 7. If $\mathbf{X}$ is Alexandroff-Urysohn compact then by Proposition 7, $\mathbf{X}$ is compact. Hence, by Proposition 1, $\mathbf{X}^*$ is compact. By Proposition 7 again, $\mathbf{X}^*$ is Alexandroff-Urysohn compact.

Similarly, if $\mathbf{X}^*$ is Alexandroff-Urysohn compact then $\mathbf{X}$ is Alexandroff-Urysohn compact.

(ii)→(i) This follows from the observation that in the proof of Proposition 7 the metric reflection $\mathbf{X}^*$ of $\mathbf{X}$ is a two point discrete space which is trivially Alexandroff-Urysohn compact.

(i)→(iii) (→) This follows at once from the proof of Proposition 7, the fact that the pseudometric space $\mathbf{X}$ is clearly ultrafilter compact and the proof of (i)→(ii) of the present theorem.

(iii)→(i) Note that in the proof of Proposition 7 the metric reflection $\mathbf{X}^*$ of $\mathbf{X}$ is a two point discrete space which is trivially ultrafilter compact. □

Clearly, the image of a filterbase under a function $f : X \to Y$ is a filterbase. In contrast with the image of a filterbase, the preimage of a filterbase need not be a filterbase. Indeed, if f is not onto then $f^{-1}(F)$ might be empty for some non-empty set A. Likewise, even in case where f is onto, the preimage of a filter $\mathcal{F}$ need not be a filter.

Remark 9. We remark here that **PUU** is strictly weaker than **BPI**. Indeed, in any **ZF** model without free ultrafilters, such as the Feferman/Blass Model, model $\mathcal{M}15$ in [6], **PUU** holds. Indeed, fix an onto function $f : X \to Y$ and let $\mathcal{F}$ be an ultrafilter of Y. Clearly, $\mathcal{F} = \{F \subset Y : f(x) \in F\}$ for some $x \in X$. Then, it is easy to see that $\mathcal{F}^* = \{A \subset X : x \in A\}$ is the required ultrafilter of X extending the filterbase $\mathcal{W} = \{f^{-1}(F) : F \in F\}$. However, **BPI** fails in $\mathcal{M}15$ because the filter of all cofinite subsets of ω does not extend to an ultrafilter (such an ultrafilter is clearly a free ultrafilter of ω). The last observation also shows that $\mathcal{M}15$ witnesses the fact that **PUU** does not imply $\mathbf{UF}(\omega)$ in **ZF**.

Clearly, **AC** implies the statement:

(h) *A pseudometric space* $\mathbf{X}$ *is ultrafilter compact iff* $\mathbf{X}^*$ *is ultrafilter compact.*
However, (h) does not imply **AC**. Indeed, in $\mathcal{M}15$ every space is ultrafilter compact, thus (h) holds but **AC** fails.

Next, we show that **PUU** implies (h).

Theorem 10. (i) *For every pseudometric space* $\mathbf{X}$*, if* $\mathbf{X}^*$ *is ultrafilter compact then so is* $\mathbf{X}$*.*

(ii) **PUU** *implies "for every pseudometric space* $\mathbf{X}$*, if* $\mathbf{X}$ *is ultrafilter compact then so is* $\mathbf{X}^*$*".*

(iii) $\mathbf{UF}(\omega)$ *and* **IWDI** *and "for every pseudometric space* $\mathbf{X}$*, if* $\mathbf{X}$ *is ultrafilter compact then so is* $\mathbf{X}^*$*" together imply* **SPI***.*

(iv) *The negation of* **PUU** *implies "there exists an infinite set* X *and a free ultrafilter* $\mathcal{F}$ *on* X*" (***Form** 206 *in* [6]*).*

Proof: Fix $\mathbf{X}$, a pseudometric space, and let $h : \mathbf{X} \to \mathbf{X}^*$ be the mapping given by $h(x) = [x]$.

(i) Fix an ultrafilter $\mathcal{F}$ of $\mathbf{X}$. We show that $\mathcal{F}$ converges to some point $y \in X$. For every $A \in \mathcal{P}(X)$ let $A^* = h(A) = \{[a] : a \in A\}$. Clearly, $\mathcal{F}^* = \{F^* : F \in \mathcal{F}\}$

a filterbase of $\mathbf{X}^*$. To see that $\mathcal{F}^*$ is an ultrafilter of $\mathbf{X}^*$, fix $H \subseteq X^*$, such that for all $F \in \mathcal{F}$, $H \cap F^* \neq \varnothing$. Since $H = (\bigcup H)^*$ it follows that for all $F \in \mathcal{F}$, $(\bigcup H) \cap F \neq \varnothing$ (if $(\bigcup H) \cap F = \varnothing$ for some $F \in \mathcal{F}$, then $(\bigcup H)^* \cap F^* = \varnothing$). Thus, by the fact that $\mathcal{F}$ is an ultrafilter, $\bigcup H \in \mathcal{F}$ and consequently $H \in \mathcal{F}^*$ and $\mathcal{F}^*$ is an ultrafilter as required. By our hypothesis, it follows that for some $[y] \in \mathbf{X}^*$, $D([y], \varepsilon) \in \mathcal{F}^*$ for every $\varepsilon > 0$. Hence, for every $\varepsilon > 0$, $\bigcup D([y], \varepsilon) \in \mathcal{F}$. Since, $\bigcup D([y], \varepsilon) = D(y, \varepsilon)$ it follows that for every $\varepsilon > 0$, $D(y, \varepsilon) \in \mathcal{F}$ and consequently $\mathcal{F}$ converges to y as required.

(ii) Fix $\mathcal{F}$ an ultrafilter of $\mathbf{X}^*$ and let $\mathcal{H} = \{h^{-1}(F)(= \bigcup F) : F \in \mathcal{F}\}$. By **PUU**, there exists an ultrafilter $\mathcal{G}$ of $\mathbf{X}$ extending $\mathcal{H}$. By the ultrafilter compactness of $\mathbf{X}$, $\mathcal{G}$ converges to some point $x \in \mathbf{X}$. Thus, $\{D(x, \varepsilon) : \varepsilon > 0\} \subseteq \mathcal{G}$. Hence, $\{D(x, \varepsilon) : \varepsilon > 0\} \cup \mathcal{H}$ has the fip. Since $\mathcal{F}$ is an ultrafilter, $\{D([x], \varepsilon) : \varepsilon > 0\} \subseteq \mathcal{F}$ meaning that $\mathcal{F}$ converges to $[x]$. Hence, $\mathbf{X}^*$ is ultrafilter compact finishing the proof of (ii).

(iii) Assume on the contrary that **SPI** fails and fix X an infinite set without a free ultrafilter. Fix, by **IWDI**, an onto function $f : X \to \omega$ and define a pseudometric d on X by requiring:

$$d(x, y) = \begin{cases} 0 \text{ if } x, y \in f^{-1}(n) \text{ for some } n \in \omega \\ 1 \text{ otherwise} \end{cases}.$$

Clearly, $\mathbf{X}$ is ultrafilter compact. Hence, by our hypothesis $\mathbf{X}^*$ is ultrafilter compact. Without loss of generality we may identify X^* with ω and the topology T_{d^*} with the discrete topology on ω. Fix, by $\mathbf{UF}(\omega)$, a free ultrafilter $\mathcal{F}$ of ω and let $\mathcal{F}$ converge to a point, say n, of ω. Then $\{n\} \in \mathcal{F}$ contradicting the fact that $\mathcal{F}$ is free. Thus, **SPI** holds as required.

(iv) This, in view of Remark 9, is straightforward. □

It is easy to see that:

(A) $\mathbf{UF}(\omega) + \mathbf{IDI} \to \mathbf{SPI}$

and,

(B) $\mathbf{UF}(\omega) + \mathbf{IWDI} \to$ "for every infinite set X, $\wp(X)$ has a free ultrafilter".

In [3] it has been shown in **ZF** that:

> For every well-ordered cardinal number k, k has a free ultrafilter iff $\wp(k)$ has a free ultrafilter.

Hence, the statement: "For every infinite set X, $\wp(X)$ has a free ultrafilter" implies $\mathbf{UF}(\omega)$. Combining the latter implication with (A) we get:

Proposition 11. *The conjunction* **IDI** *and "for every infinite set X, $\wp(X)$ has a free ultrafilter" implies* **SPI**.

Remark 12. A. Blass has shown in [2] that in the model $\mathcal{M}15$ in [6], $\mathbf{UF}(\omega)$ fails but there is a free ultrafilter on the set of equivalence classes of reals modulo finite difference. Hence in $\mathcal{M}15$, $\wp(\mathbb{R})$ has a free ultrafilter but $\mathbb{R}$ has no free ultrafilter.

Question 1. Can **IDI** be replaced by **IWDI** in (A)?

If the answer to Question 1 is in the negative, then the statement "if the pseudometric space $\mathbf{X}$ is ultrafilter compact then so is $\mathbf{X}^*$" is unprovable in **ZF**.

Clearly, the statement:

(e) "Cantor complete pseudometric spaces are complete"

is a theorem of **ZF**. The standard proof that Cantor complete pseudometric spaces are complete goes through in **ZF**. In [7] it is shown that the statement:

(f) "Every complete metric space (X, d) is Cantor complete"

implies $\mathbf{CAC}_{fin}$. Hence, the statement

(g) "every complete pseudometric space (X, d) is Cantor complete"

also implies $\mathbf{CAC}_{fin}$. We show next that (g) implies something stronger than $\mathbf{CAC}_{fin}$.

Theorem 13. *The following statements are equivalent:*

(i) **CAC**;
(ii) *every pseudometric space* $\mathbf{X}$ *is complete iff it is Cantor complete.*

Proof: (i)→(ii) The standard **ZFC** proof that a pseudometric space is Cantor complete iff it is complete goes through if we only assume **CAC**.

(ii)→(i) Let $\mathcal{A} = (A_n)_{n\in\mathbb{N}}$ be a disjoint family of non-empty sets such that no infinite subfamily of $\mathcal{A}$ has a choice function and consider the pseudometric d on $X = \bigcup\{A_n : n \in \mathbb{N}\}$ given by (5). Clearly, $\mathbf{X}$ is complete. For every $n \in \mathbb{N}$ let

$$G_n = \bigcup\{A_m : m \geq n\}.$$

It can be readily verified that each G_n is a closed subset of $\mathbf{X}$, $\lim_{n\to\infty} \delta(G_n) = 0$ and $\bigcap\{G_n : n \in \mathbb{N}\} = \varnothing$. Thus, $\mathbf{X}$ is not Cantor complete. Contradiction! Hence, $\mathcal{A}$ has a choice function. □

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