On isometrical extension properties of function spaces

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Abstract. In this note, we prove that any "bounded" isometries of separable metric spaces can be represented as restrictions of linear isometries of function spaces $C(Q)$ and $C(\Delta)$, where Q and Δ denote the Hilbert cube $[0,1]^\infty$ and a Cantor set, respectively.

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1. Introduction

A well-known theorem of Banach and Mazur is the result that $C(I)$ ($I = [0,1]$) is a universal space of separable metric spaces up to isometry (see [1]). But it seems that there is no research of extension properties of isometries on separable metric spaces in function spaces $C(Z)$ because that $C(I)$ does not have such a property.

In this note, we prove that any "bounded" isometries of separable metric spaces can be represented as restrictions of linear isometries of function spaces $C(Q)$ and $C(\Delta)$, where Q and Δ denote the Hilbert cube $[0,1]^\infty$ and a Cantor set, respectively. Also, Urysohn [7] constructed a complete separable metric space $\mathbb{U}$ that is also universal up to isometry. In [8], Uspenskij proved that for any separable metric space X there is a natural isometrical embedding $i : X \to \mathbb{U}$ such that i induces a natural continuous monomorphism $i^\star : Iso(X) \to Iso(\mathbb{U})$ satisfying that $i^\star(g) \in Iso(\mathbb{U})$ is an extension of $g \in Iso(X)$. In fact, we show that the function spaces $C(Q)$ and $C(\Delta)$ satisfy similar extension properties.

In this note, unless stated otherwise, we assume that all maps are continuous functions. Let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{R}$ denote the set of integers, the set of natural numbers and the set of real numbers, respectively. If K is a subset of a space X, then $\mathrm{Cl}(K)$, $\mathrm{Bd}(K)$ and $\mathrm{Int}(K)$ denote the closure, the boundary and the interior of K in X, respectively. For any compact metric space Z, $C(Z)$ denotes the function space of all (continuous) maps from Z to $\mathbb{R}$ with the supremum metric $\tilde{d}$, i.e.,

$$\tilde{d}(f,g) = \sup\{|f(z) - g(z)| \mid z \in Z\}$$

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for $f, g \in C(Z)$. A map $i : (X, d_X) \to (Y, d_Y)$ between separable metric spaces is an *isometrical embedding* from (X, d_X) into (Y, d_Y) if i satisfies the condition $d_Y(i(x), i(x')) = d_X(x, x')$ for each $x, x' \in X$. A map $g : (X, d_X) \to (Y, d_Y)$ between separable metric spaces is an *isometry* if g is surjective and $d_Y(g(x), g(x')) = d_X(x, x')$ for each $x, x' \in X$. For a separable metric space (X, d), let $Iso(X)$ be the group of all isometries of X equipped with the pointwise convergent topology, i.e.,

$$Iso(X) = \{g : X \to X \mid g \text{ is an isometry}\}.$$

Let (X, d) be a separable metric space and $x_0 \in X$. A subgroup G of $Iso(X)$ is *bounded* if $\operatorname{diam} G(x_0) < \infty$, where $G(x_0) = \{g(x_0) | \ g \in G\}$ $(\subset X)$. The definition of "bounded subgroup" of $Iso(X)$ does not depend on the choice of the point $x_0 \in X$. Also, each $g \in Iso(X)$ is *bounded* if $\operatorname{diam}\{g^n(x_0) | \ n \in \mathbb{Z}\} < \infty$. Note that if (X, d) is bounded, i.e. $\operatorname{diam}_d X < \infty$, then $Iso(X)$ itself is bounded. In particular, if X is a compact metric space, then $Iso(X)$ is bounded. In [3], Mazur and Ulam proved that if B and B' are Banach spaces, then every isometry $T : B \to B'$ with $T(0) = 0$ is linearly isometric and moreover, Banach and Stone proved that if X and Y are compact Hausdorff spaces, then every isometry $T : C(X) \to C(Y)$ with $T(0) = 0$ is linearly isometric and moreover, T is induced by a homeomorphism $h : Y \to X$ (see [1], [6]).

Theorem 1.1 (Banach [1] and Stone [6]). *Let X and Y be compact Hausdorff spaces. Then the followings hold.*

(1) *$C(X)$ is isometric to $C(Y)$ if and only if X is homeomorphic to Y.*
(2) *If $T : C(X) \to C(Y)$ is a linear isometry, then there is a homeomorphism $h : Y \to X$ and a (continuous) map $\alpha : Y \to \mathbb{R}$ with $|\alpha(y)| = 1$ for $y \in Y$ such that*
$$(T(f))(y) = \alpha(y) \cdot (f \circ h)(y)$$
for $f \in C(X)$ and $y \in Y$. Moreover, if Y is connected, $T(f) = f \circ h$ or $T(f) = -(f \circ h)$.

For any Banach space B, let

$$LinIso(B) = \{f \in Iso(B) \mid f \text{ is linear}\}.$$

Note that $LinIso(B)$ is bounded, because $LinIso(B)(0) = \{0\}$.

2. Extensions of bounded isometries in $C(Q)$

In this section, we assume that (X, d) is a separable metric space and x_0 is a fixed point of X. In [5], Sierpiński considered the space

$$X' = \{f : X \to \mathbb{R} \mid f(x_0) = 0 \text{ and } |f(x) - f(y)| \le d(x, y) \text{ for } x, y \in X\}$$

which is a topological space equipped with the pointwise convergent topology and by use of the spaces X', he proved that $C(I)$ is a universal space of separable metric spaces up to isometry. We modify the Sierpiński's method of [5]. In this note, for any bounded subgroup G of $Iso(X)$, we consider the following more general space

$$\begin{aligned}\tilde{X}\ (=\tilde{X}_G) = \{f : X \to \mathbb{R} \mid f(z) \in [-\operatorname{diam}(G(x_0)), \operatorname{diam}(G(x_0))] \\ \text{for}\ \ z \in G(x_0)\ \ \text{and}\ \ |f(x) - f(y)| \le d(x,y) \text{ for } x, y \in X\}\end{aligned}$$

which is a topological space equipped with the pointwise convergent topology. Since $\tilde{X}$ is compact convex, we can easily see the following.

Lemma 2.1. *$\tilde{X}(=\tilde{X}_G)$ is a compact metric absolute retract* (= AR). *Moreover, if $g \in G$, then $\tilde{g} : \tilde{X} \to \tilde{X}$ is a homeomorphism, where $\tilde{g}$ is defined by $\tilde{g}(f) = f \circ g$ for $f \in \tilde{X}$.*

Lemma 2.2. *Suppose that $p_G : Z \to \tilde{X}(=\tilde{X}_G)$ is a map from a compact metric space Z onto $\tilde{X}$ such that for each $g \in G$ there is a* (*lift*) *homeomorphism $L_g : Z \to Z$ satisfying the following commutative diagram.*

$$\begin{array}{ccc} Z & \xrightarrow{L_g} & Z \\ {\scriptstyle p_G}\big\downarrow & & \big\downarrow{\scriptstyle p_G} \\ \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{X} \end{array}$$

Then there is an isometrical embedding $i_G : X \to C(Z)$ such that for each $g \in G$, the following commutative diagram holds.

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ {\scriptstyle i_G}\big\downarrow & & \big\downarrow{\scriptstyle i_G} \\ C(Z) & \xrightarrow{\tilde{L_g}} & C(Z) \end{array}$$

Here $\tilde{L_g} : C(Z) \to C(Z)$ is the isometry defined by $\tilde{L_g}(f) = f \circ L_g$ for $f \in C(Z)$. In particular, $\tilde{L_g} \in LinIso(C(Z))$ is an isometrical extension of $g \in G$.

Proof: Define $i_G : X \to C(Z)$ by $i_G(x)(z) = p_G(z)(x)$ for each $x \in X$ and $z \in Z$ (see [5]). We will show that i_G is an isometrical embedding. Let $x, y \in X$. Since $p_G(z) \in \tilde{X}$ $(z \in Z)$,

$$\begin{aligned} d(i_G(x), i_G(y)) &= \sup\{|i_G(x)(z) - i_G(y)(z)| \mid z \in Z\} \\ &= \sup\{|p_G(z)(x) - p_G(z)(y)| \mid z \in Z\} \le d(x,y). \end{aligned}$$

Let $x \in X$ be fixed. Define $h_x : X \to \mathbb{R}$ by $h_x(y) = d(x,y) - d(x,x_0)$. We will show that $h_x \in \tilde{X}$. Note that $h_x(x_0) = 0$ and $|h_x(y) - h_x(y')| \le d(y,y')$ for $y, y' \in X$. Let $z \in G(x_0)$. Since $d(x_0, z) \le \operatorname{diam}(G(x_0))$,

$$|h_x(z)| = |h_x(x_0) - h_x(z)| \le d(x_0, z) \le \operatorname{diam}(G(x_0)).$$

Then $h_x(z) \in [-\operatorname{diam}(G(x_0)), \operatorname{diam}(G(x_0))]$ and hence $h_x \in \tilde{X}$. Since p_G is surjective, there is $z \in Z$ such that $p_G(z) = h_x$. Then

$$|i_G(x)(z) - i_G(y)(z)| = |p_G(z)(x) - p_G(z)(y)| = |h_x(x) - h_x(y)| = d(x, y).$$

Therefore, we see that i_G is an isometrical embedding.

Finally, we will show that $\tilde{L_g} \circ i_G = i_G \circ g$. By the commutative diagram of the assumption, we see that for $z \in Z$,

$$(p_G \circ L_g)(z) = (\tilde{g} \circ p_G)(z).$$

For each $z \in Z$ and $x \in X$,

$$\begin{aligned}(\tilde{L_g} \circ i_G(x))(z) &= (i_G(x) \circ L_g)(z) = i_G(x)(L_g(z)) = p_G(L_g(z))(x) \\ &= (\tilde{g} \circ p_G(z))(x) = (p_G(z) \circ g)(x) = p_G(z)(g(x)) = i_G(g(x))(z).\end{aligned}$$

Consequently, we see that $\tilde{L_g} \circ i_G = i_G \circ g$. This completes the proof. $\square$

Here we have the following theorem which implies that $C(Q)$ is universal concerning isometrical extensions of bounded isometry groups of separable metric spaces.

Theorem 2.3. *Let (X, d) be a separable metric space and let G be any bounded subgroup of $Iso(X)$. Then there is an isometrical embedding $i_G : X \to C(Q)$ such that i_G induces a continuous monomorphism $i_G^\star : G \to LinIso(C(Q))$ such that $i_G^\star(g) \in LinIso(C(Q))$ is an extension of $g \in G$.*

Proof: By Lemma 2.1, $\tilde{X}(= \tilde{X}_G)$ is a compact metric AR and hence $\tilde{X} \times Q$ is homeomorphic to Q (see [4, Theorems 7.5.8 and 7.8.1]). We identify $\tilde{X} \times Q$ with Q, i.e. $\tilde{X} \times Q = Q$. Let $p = p_G : \tilde{X} \times Q \to \tilde{X}$ is the natural projection. For each $g \in G$, there is the natural (lift) homeomorphism $L_g = \tilde{g} \times \mathrm{id}_Q : \tilde{X} \times Q \to \tilde{X} \times Q$ satisfying the following commutative diagram.

$$\begin{array}{ccc} \tilde{X} \times Q & \xrightarrow{L_g} & \tilde{X} \times Q \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle p_G} \\ \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{X} \end{array}$$

By Lemma 2.2, we see that for each $g \in G$, the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ {\scriptstyle i_G}\downarrow & & \downarrow{\scriptstyle i_G} \\ C(\tilde{X} \times Q) & \xrightarrow{\tilde{L_g}} & C(\tilde{X} \times Q) \end{array}$$

Note that the map $i_G^\star : G \to LinIso(C(\tilde{X} \times Q)) = LinIso(C(Q))$ defined by $i_G^\star(g) = \tilde{L_g}$ is the desired continuous monomorphism. $\square$

Corollary 2.4. *Suppose that (X,d) is a bounded separable metric space. Then there is an isometrical embedding $i : X \to C(Q)$ such that i induces a continuous monomorphism $i^\star : Iso(X) \to LinIso(C(Q))$ such that $i^\star(g) \in LinIso(C(Q))$ is an extension of $g \in Iso(X)$.*

Remark 1. Note that for any Banach space B, $LinIso(B)$ is a bounded group. Hence in this paper, we cannot omit the condition that G is bounded.

If we observe the proof of Lemma 2.2, we see that some converse assertions of Lemma 2.2 are also true. In fact, we show that "isometrical extension properties" are equivalent to "lifting properties" of homeomorphisms.

Proposition 2.5. *Suppose that $p_G : Z \to \tilde{X}(= \tilde{X}_G)$ is a map from a compact metric space Z onto $\tilde{X}$, $i_G : X \to C(Z)$ is the isometrical embedding as in the proof of Lemma 2.2 and $g \in G$. Let $L_g : Z \to Z$ be a homeomorphism. Then the followings hold.*

(1) *The following diagram is commutative:*

$$\begin{array}{ccc} Z & \xrightarrow{L_g} & Z \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle p_G} \\ \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{X} \end{array}$$

if and only if the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ {\scriptstyle i_G}\downarrow & & \downarrow{\scriptstyle i_G} \\ C(Z) & \xrightarrow{\tilde{L_g}} & C(Z) \end{array}$$

(2) *The following diagram is commutative:*

$$\begin{array}{ccc} Z & \xrightarrow{L_g} & Z \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle p_G} \\ \tilde{X} & \xrightarrow{-\tilde{g}} & \tilde{X} \end{array}$$

if and only if the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ {\scriptstyle i_G}\downarrow & & \downarrow{\scriptstyle i_G} \\ C(Z) & \xrightarrow{-\tilde{L_g}} & C(Z) \end{array}$$

PROOF: We will prove only the converse assertion of (2). We assume that $i_G \circ g = -\tilde{L_g} \circ i_G$. Hence we have $i_G(g(x))(z) = (-\tilde{L_g} \circ i_G(x))(z) = -i_G(x)(L_g(z))$ for each $x \in X$ and $z \in Z$. Then

$$\begin{aligned}(p_G \circ L_g(z))(x) &= p_G(L_g(z))(x) = i_G(x)(L_g(z)) \\ &= -i_G(g(x))(z) = -p_G(z)(g(x)) = ((-\tilde{g}) \circ p_G(z))(x).\end{aligned}$$

Hence $p_G \circ L_g = (-\tilde{g}) \circ p_G$. □

Example. Let $X = \{x_i |\ i = 0, 1, 2\}$ be the set of three elements and let d be the metric on X defined by $d(x_i, x_j) = r > 0$ $(i \neq j)$. Define the isometry $g : X \to X$ by $g(x_0) = x_0, g(x_1) = x_2$ and $g(x_2) = x_1$. Let $G = \{\mathrm{id}_X, g\}$. Note that $G(x_0) = \{x_0\}$. We will show that there is an isometrical embedding $i_G : X \to C(Q)$ such that there is no isometrical extension of g on $C(Q)$. Note that

$$\tilde{X}_G = \{f : X \to \mathbb{R} \mid f(x_0) = 0 \ \text{ and } \ |f(x) - f(y)| \leq d(x, y) \ \text{ for } \ x, y \in X\}.$$

Then $\tilde{X}_G$ is homeomorphic to the compact subset

$$K = \{(x, y) \in \mathbb{R}^2 |\ |x - y| \leq r, -r \leq x, y \leq r\}$$

of $\mathbb{R}^2$ and we may assume $\tilde{X}_G = K$. Also, we may assume that $\tilde{g} : \tilde{X}_G \to \tilde{X}_G$ is equal to the map $\tilde{g} : K \to K$ defined by $\tilde{g}((x, y)) = (y, x)$ for $(x, y) \in K$. Also, we see that $-\tilde{g} : K \to K$ is the map defined by $-\tilde{g}((x, y)) = (-y, -x)$ for $(x, y) \in K$. Consider the copy K' of K and subsets

$$K_+ = \{(x, y) \in K |\ y \leq x\} \bigcup \{(x, y) \in K |\ y \leq -x\} \subset K$$

and

$$K'_+ = \{(x', y') \in K' |\ y' \leq x'\} \bigcup \{(x', y') \in K' |\ y' \leq -x'\} \subset K'.$$

Define the homeomorphism $h : K_+ \to K'_+$ by $h(x, y) = (x', y')$. Let $K \cup_h K'$ be the adjunction space by h and let $\alpha : K \cup_{top} K' \to K \cup_h K'$ be the quotient map. Put $Z = (K \cup_h K') \times Q$. Since $K \cup_h K'$ is an AR, we see that Z is homeomorphic to Q (see [4]). Let $p_G : Z \to K = \tilde{X}_G$ be the map defined by $p_G(\alpha(x, y), q) = (x, y)$ for $(x, y) \in K$ and $p_G(\alpha(x', y'), q) = (x, y)$ for $(x', y') \in K'$ $(q \in Q)$. Define $i_G : X \to C(Z)$ by $i_G(x)(z) = p_G(z)(x)$ for each $x \in X$ and $z \in Z$. Note that $p_G^{-1}(x, y)$ is connected for $(x, y) \in K_+$ and $p_G^{-1}(x, y)$ is not connected for $(x, y) \notin K_+$. Hence we see that there is no homeomorphism $L_g : Z \to Z$ so that $\tilde{g} \circ p_G = p_G \circ L_g$. Also, there is no homeomorphism $L_g : Z \to Z$ so that $-\tilde{g} \circ p_G = p_G \circ L_g$. Suppose, on the contrary, that there is an isometry $T : C(Z) \to C(Z)$ such that $T \circ i_G = i_G \circ g$. Note that for any $z \in Z$, $i_G(x_0)(z) = p_G(z)(x_0) = 0$ and hence $i_G(x_0) = 0 \in C(Z)(= C(Q))$. Since $g(x_0) = x_0$, we see that $T(0) = 0$ and hence by [3], the isometry T is a linear isometry of $C(Z)$. Since $Z = Q$ is a continuum, Theorem 1.1 implies that there is a homeomorphism $L_g : Z \to Z$ such that

$\tilde{L_g} = T$ or $-\tilde{L_g} = T$. By use of Proposition 2.5, we see that $\tilde{g} \circ p_G = p_G \circ L_g$ or $-\tilde{g} \circ p_G = p_G \circ L_g$. This is a contradiction.

3. Extensions of bounded isometries in $C(\Delta)$

In this section, we need the following notions. For a compact metric space X, let $H(X)$ be the space of all homeomorphisms of X with the supremum metric. A closed set K in X is *regular closed* in X if $\mathrm{Cl}(\mathrm{Int}(K)) = K$. A collection $\mathcal{C}$ of regular closed sets in X is called a *regular closed partition* of X provided that $\bigcup \mathcal{C} = X$ and $C \cap C' = \mathrm{Bd}(C) \cap \mathrm{Bd}(C')$ for each $C, C' \in \mathcal{C}$ with $C \neq C'$. Let $\mathcal{A}$ and $\mathcal{B}$ be regular closed partitions of X. Then $\mathcal{A}$ *refines* $\mathcal{B}$ ($\mathcal{A} \leq \mathcal{B}$) if for each $A \in \mathcal{A}$ there is (a unique) $B \in \mathcal{B}$ such that $A \subset B$. Also, $\mathcal{A}@\mathcal{B}$ denotes the regular closed partition

$$\{\mathrm{Cl}[\mathrm{Int}(A) \cap \mathrm{Int}(B)] \mid A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$$

of X.

Then we have the following proposition (cf. [2]).

Proposition 3.1. *Let X be a compact metric space and let G be a countable subset of $H(X)$. Then there is an onto map $p_G : \Delta \to X$ such that for any $g \in G$ there is a (lift) homeomorphism $L_g : \Delta \to \Delta$ of Δ such that the following diagram is commutative.*

$$\begin{array}{ccc} \Delta & \xrightarrow{L_g} & \Delta \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle p_G} \\ X & \xrightarrow{g} & X \end{array}$$

Proof: Since G is a countable subset of $H(X)$, we put $G = \{g_n \mid n \in \mathbb{N}\}$. By induction on $n \in \mathbb{N}$, we can construct a sequence $\{\mathcal{C}(n) \mid n \in \mathbb{N}\}$ of finite regular closed partitions of X such that

(1) $\mathcal{C}(2) \leq g_1^{-1}(\mathcal{C}(1))@\mathcal{C}(1)@g_1(\mathcal{C}(1))$, and
(2) for each $n \geq 2$,

$$\mathcal{C}(n) \leq @_{j \in \{1,2,\dots,n-1\}}[g_j^{-1}(\mathcal{C}(n-1))@\mathcal{C}(n-1)@g_j(\mathcal{C}(n-1))],$$

(3) $\lim_{n\to\infty} \mathrm{mesh}\, \mathcal{C}(n) = 0$.

We may assume that any element of each $\mathcal{C}(n)$ is nonempty. Note that

$$\mathcal{C}(1) \geq \mathcal{C}(2) \geq \dots$$

Put

$$C = \{(c_1, c_2, \dots) \in \prod_{n\in\mathbb{N}} \mathcal{C}(n) \mid c_1 \supset c_2 \supset \cdots\}$$

We may assume that each $\mathcal{C}(n)$ is a discrete finite space, i.e., each one point set $\{c\}$ is open in $\mathcal{C}(n)$ for each $c \in \mathcal{C}(n)$, and hence $\mathcal{C}(n)$ is a compact metric space. Note that any decreasing sequence of nonempty compact sets has a nonempty intersection. Then by (3), we see that $\bigcap_{n\in\mathbb{N}} c_n$ is a one point set for each $(c_1, c_2, \dots) \in C$. Since C is the limit of the inverse sequence

$$\mathcal{C}(1) \leftarrow \mathcal{C}(2) \leftarrow \mathcal{C}(3) \leftarrow \cdots,$$

C is also a compact metric space. Define $p : C \to X$ by

$$p(c_1, c_2, \dots) = \bigcap_{n\in\mathbb{N}} c_n.$$

It is easy to see that p is continuous and an onto map. Since C is a zero-dimensional compact metric space, we see that $C \times \Delta$ is homeomorphic to the Cantor set Δ. We identify $C \times \Delta$ with Δ. Put $p_G = p \circ q : C \times \Delta \to X$, where $q : C \times \Delta \to C$ is the natural projection.

Next, for each $g \in G$, we will construct a homeomorphism $h = h_g : C \to C$ as follows. Note that there is some $j \in \mathbb{N}$ such that $g = g_j$. By (2), note that both $g(\mathcal{C}(n+1))$ and $g^{-1}(\mathcal{C}(n+1))$ are refinements of $\mathcal{C}(n)$ for each $n \geq j$. Thus, by regular closedness, for each $n \geq j$ there are unique maps $h_n, k_n : \mathcal{C}(n+1) \to \mathcal{C}(n)$ given by $h_n(c_{n+1}) = c_{n,g}$ if $g(c_{n+1}) \subset c_{n,g}$, and $k_n(c_{n+1}) = c_{n,g^{-1}}$ if $g^{-1}(c_{n+1}) \subset c_{n,g^{-1}}$. Now define $h, k : C \to C$ by

$$\begin{aligned} h(c_1, c_2, \dots) &= (c'_1, c'_2, \dots, c'_{j-1}, h_j(c_{j+1}), h_{j+1}(c_{j+2}), \dots) \\ &\in \varprojlim\, (\mathcal{C}(1) \leftarrow \mathcal{C}(2) \leftarrow \mathcal{C}(3) \leftarrow \cdots), \end{aligned}$$

and

$$\begin{aligned} k(c_1, c_2, \dots) &= (c''_1, c''_2, \dots, c''_{j-1}, k_j(c_{j+1}), k_{j+1}(c_{j+2}), \dots) \\ &\in \varprojlim\, (\mathcal{C}(1) \leftarrow \mathcal{C}(2) \leftarrow \mathcal{C}(3) \leftarrow \cdots). \end{aligned}$$

We will show that the following conditions (a), (b) and (c) are satisfied.

(a) h is continuous.

This is obvious since each h_n is continuous.

(b) h is bijective.

Let $(c_1, c_2, \dots) \in C$. Then for $n \geq j$,

$$\begin{aligned} h_n \circ k_{n+1}(c_{n+2}) &\supset g \circ g^{-1}(c_{n+2}) = c_{n+2} \\ k_n \circ h_{n+1}(c_{n+2}) &\supset g^{-1} \circ g(c_{n+2}) = c_{n+2}. \end{aligned}$$

Note that for $n \geq j$, $h_n \circ k_{n+1}(c_{n+2}) = k_n \circ h_{n+1}(c_{n+2}) = c_n$. Then $h \circ k = k \circ h = \mathrm{id}_C$. Therefore h is bijective and $h^{-1} = k$.

(c) $p \circ h = g \circ p$.
Let $(c_1, c_2, \dots) \in C$. Then

$$p \circ h(c_1, c_2, \dots) = \bigcap_{n \geq j} h_n(c_{n+1}) \supset \bigcap_{n \geq j} g(c_{n+1}),$$
$$g \circ p(c_1, c_2, \dots) = g(\bigcap_{n \in \mathbb{N}} c_n) = \bigcap_{n \in \mathbb{N}} g(c_n) = \bigcap_{n \in \mathbb{N}} g(c_{n+1}).$$

Therefore $p \circ h(c_1, c_2, \dots) \supset g \circ p(c_1, c_2, \dots)$. Note that $p \circ h(c_1, c_2, \dots)$ and $g \circ p(c_1, c_2, \dots)$ are one point sets in X. Thus $p \circ h = g \circ p$.

$$\begin{array}{ccc} C & \xrightarrow{h\ (=h_g)} & C \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{g} & X \end{array}$$

Finally, we define a homeomorphism $L_g : C \times \Delta \to C \times \Delta$ by $L_g = h_g \times \mathrm{id}_\Delta$. Then we see that for any $g \in G$, the following diagram is commutative.

$$\begin{array}{ccc} C \times \Delta & \xrightarrow{L_g} & C \times \Delta \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle p_G} \\ X & \xrightarrow{g} & X \end{array}$$

This completes the proof. □

Then we have the following theorem.

Theorem 3.2. *Let (X, d) be any separable metric space and let G be a countable bounded subgroup of $Iso(X)$. Then there is an isometrical embedding $i_G : X \to C(\Delta)$ such that there exist a countable subgroup $G^\star$ of $LinIso(C(\Delta))$ and a continuous epimorphism $r^\star : G^\star \to G$ such that each $g^\star \in G^\star$ is an extension of $r^\star(g^\star) \in G$. In particular, if $g \in G$, then there is an extension $g^\star \in LinIso(C(\Delta))$ of g.*

Proof: Recall that $\tilde{X} (= \tilde{X}_G)$ is a compact metric space. Since G is a countable bounded subgroup of $Iso(X)$, $\tilde{G} = \{\tilde{g} \mid g \in G\}$ is a countable subset of $H(\tilde{X})$. By Proposition 3.1, we have an onto map $p_G : \Delta \to \tilde{X}$ such that for any $g \in G$, there is a (lift) homeomorphism $L_g : \Delta \to \Delta$ such that the following diagram is commutative.

$$\begin{array}{ccc} \Delta & \xrightarrow{L_g} & \Delta \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle p_G} \\ \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{X} \end{array}$$

By Lemma 2.2, we have the following commutative diagram.

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ {\scriptstyle i_G}\downarrow & & \downarrow{\scriptstyle i_G} \\ C(\Delta) & \xrightarrow{\tilde{L_g}} & C(\Delta) \end{array}$$

Let $G^\star$ be the subgroup of $LinIso(C(\Delta))$ generated by $\{\tilde{L_g} \mid g \in G\}$. Then we can easily see that there is the desired epimorphism $r^\star : G^\star \to G$ such that each $g^\star \in G^\star$ is an extension of $r^\star(g^\star) \in G$. □

Remark 2. Note that the space $H(\Delta)$ of all homeomorphisms of Δ is homeomorphic to the space P of irrationals, and hence $H(\Delta)$ is zero-dimensional. If G is any bounded subgroup of $Iso(X)$ with $\dim G \geq 1$, there is no embedding from G to $H(\Delta)$.

Corollary 3.3. *Let (X, d) be any separable metric space. If $g \in Iso(X)$ is periodic, i.e. $g^n = \mathrm{id}_X$ for some $n \in \mathbb{N}$, then there is an isometrical embedding $i_g : X \to C(\Delta)$ such that there is an extension $g^\star \in LinIso(C(\Delta))$ of g with $(g^\star)^n = \mathrm{id}_{C(\Delta)}$.*

Proof: Since $g^n = \mathrm{id}_X$, g is bounded. Then $\tilde{g}^n = \mathrm{id}_{\tilde{X}}$. By the proof of Lemma 2.2, we have an onto map $p : C \to \tilde{X}$ and a homeomorphism $h_g : C \to C$ of a zero-dimensional compact metric space C such that $p \circ h_g = \tilde{g} \circ p$ and $h_g^n = \mathrm{id}_C$. By use of this fact, we can obtain an extension $g^\star \in LinIso(C(\Delta))$ of g with $(g^\star)^n = \mathrm{id}_{C(\Delta)}$.

Remark 3. Let (X, d) be any separable metric space and let $g \in Iso(X)$ such that g has a periodic point x_0 with period $n \in \mathbb{N}$. We see that if $n \geq 3$, there is no isometrical embedding i from X to $C(I)$ such that g has an extension in $LinIso(C(I))$. In fact, suppose, on the contrary, that there is an isometrical embedding $i : X \to C(I)$ such that there is an extension $T \in LinIso(C(I))$ of g. Since I is a continuum, Theorem 1.1 implies that there is a homeomorphism $h : I \to I$ such that $T = \pm\tilde{h}$. Note that $T^2 = \tilde{h}^2$. We have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{g^2} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ C(I) & \xrightarrow{\tilde{h}^2} & C(I) \end{array}$$

Since $T^n(i(x_0)) = i(x_0)$, we see that $\tilde{h}^{2n}(i(x_0)) = i(x_0)$. Note that $h^2(0) = 0$, $h^2(1) = 1$. Then there are countable open and disjoint subintervals $I_j = (a_j, b_j)$ $(j \in \mathbb{N})$ of I such that

(1) $h^2(a_j) = a_j,\ h^2(b_j) = b_j$,
(2) $h^2(t) \neq t$ for any $t \in I_j$ $(j \in \mathbb{N})$, and

(3) $h^2|(I - \cup_{j\in\mathbb{N}} I_j) = \mathrm{id}|(I - \cup_{j\in\mathbb{N}} I_j)$.

Put $f_0 = i(x_0)$. Note that $f_0 \circ h^{2n} = f_0$. Hence $f_0 \circ (h^2)^{nk} = f_0$ for each $k \in \mathbb{N}$. If $t \in I_j$, then $\lim_{k\to\infty}(h^2)^{nk}(t) = a_j$ or b_j. Since f_0 is continuous, we see that $f_0|\mathrm{Cl}(I_j)$ is a constant map. By use of these facts and (3), we see that $f_0 \circ h^2 = f_0$. This implies that $g^2(x_0) = x_0$. Hence $n \leq 2$. This is a contradiction. $\square$

Problem 3.4. *Let (X, d) be any separable metric space. Is it true that there is an isometrical embedding i from X to $C(Q)$ such that each $g \in Iso(X)$ has an extension which is an affine isometry of $C(Q)$?*

References

[1] Banach S., *Théories des Opérations Linéaires*, Hafner, New York, 1932, p. 185.

[2] Ikegami Y., Kato H., Ueda A., *Dynamical systems of finite-dimensional metric spaces and zero-dimensional covers*, Topology Appl. **160** (2013), 564–574.

[3] Mazur S., Ulam S., *Sur les transformation isométriques d'espace vectoriel normés*, C.R. Acad. Sci. Paris **194** (1932), 946–948.

[4] van Mill J., *Infinite-dimensional Topology: Prerequisites and Introduction*, North-Holland, Amsterdam, 1989.

[5] Sierpiński W., *Sur un espace métrique séparable universel*, Fund. Math. **33** (1945), 115–122.

[6] Stone M.H., *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc. **41** (1937), 375–381.

[7] Urysohn P., *Sur un espace métrique universel*, Bull. Sci. Math. **51** (1927), 43–64.

[8] Uspenskij V.V., *On the group of isometries of the Urysohn universal metric space*, Comment. Math. Univ. Carolin. **31** (1990), 181–182.

[9] Uspenskij V.V., *A universal topological group with a countable base*, Funct. Anal. Appl. **20** (1986), 86–87.

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