On graphs with maximum size in their switching classes

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Abstract. In his PhD thesis [*Structural aspects of switching classes*, Leiden Institute of Advanced Computer Science, 2001] Hage posed the following problem: "characterize the maximum size graphs in switching classes". These are called *s-maximal* graphs. In this paper, we study the properties of such graphs. In particular, we show that any graph with sufficiently large minimum degree is s-maximal, we prove that join of two s-maximal graphs is also an s-maximal graph, we give complete characterization of triangle-free s-maximal graphs and non-hamiltonian s-maximal graphs. We also obtain other interesting properties of s-maximal graphs.

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1. Introduction

Consider some group of people V endowed with symmetric binary relation "being a friend of" on it. Obviously, the set V can be viewed as a vertex set of a graph G, where $u, v \in V$ are adjacent if they are friends.

Now, what happens if some vertex $u \in V$ suddenly decides to switch its friends to non-friends and vice versa? This operation results in a graph $S(G, u)$ which is obtained from G by the deletion of the edges between u and $N_G(u)$ and the addition of new edges between u and $V - N_G[u]$. Such operation is called the *switching* of the vertex u.

Originally, the notion of graph switching was introduced in 1966 by Seidel and van Lint in their joint paper [11] on elliptic geometry. From there on, the concept of switching was developed in many interesting ways. One should mention switching reconstruction problems [9], [10], [15] and study of switching classes [2], [3], [4], [5], [6], [7], as well as of interplay between switching and the so-called two-graphs [1], [12], [14].

In 2001 in his PhD thesis [4], Hage posed two related problems:

(1) Characterize the maximum (or minimum) size graphs in switching classes.
(2) Characterize those switching classes that have a unique maximum (or minimum) size graph in it.

We mainly focus on the first problem. Graphs with maximum size in their switching classes will be called *s-maximal*. In this paper we study their properties.

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2. Preliminaries

In this paper all graphs are simple, finite and undirected. By $V(G)$ and $E(G)$ we denote the vertex set and the edge set of a graph G respectively. If two graphs G_1 and G_2 are isomorphic, we write $G_1 \simeq G_2$.

The *neighborhood* of a vertex $u \in V(G)$ is the set $N(u) = \{v \in V(G) : uv \in E(G)\}$. The *closed neighborhood* of u is $N[u] = N(u) \cup \{u\}$. The *degree* $d(u)$ of u is the number of its neighbors, i.e. $d(u) = |N(u)|$. By $\delta(G)$ and $\Delta(G)$ we denote the minimum and the maximum vertex degree in G, respectively.

As usual, by K_n we denote the *complete graph* with $n \geq 1$ vertices and by $K_{a,b}$ the *complete bipartite graph* with partitions of size $a \geq 1$ and $b \geq 1$. Also, the *null graph* K_0 is a graph with the empty set of vertices.

The *complement* $\overline{G}$ of a graph G is a graph with $V(\overline{G}) = V(G)$ and two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G. The *join* of two graphs G_1 and G_2 with disjoint vertex sets is the graph $G = G_1 + G_2$ with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$. Note that $G + K_0 = G$.

The set of vertices $U \subset V(G)$ is called *dominating* if each vertex $u \notin U$ is adjacent to some vertex from U. Further, the set U is *independent* if every two vertices $u, v \in U$ are nonadjacent in G. Dually, the set U is called *clique* if every two vertices $u, v \in U$ are adjacent in G. The *clique number* $\omega(G)$ is the number of vertices in a maximum clique of G.

Now let G be a graph and $A, B \subset V(G)$. By $e(A, B)$ we denote the number of edges between A and B. If $U \subset V(G)$, then we write $l(U)$ for $e(U, V(G) - U)$. Also, $e(U)$ denotes the number of edges whose endpoints are from U.

Definition 2.1. Let G be a graph and $U \subset V(G)$. The *switching* of U results in a graph $S = S(G, U)$ with $V(S) = V(G)$ and

$$E(S) = E_G(U) \cup E_G(V - U) \cup \{uv : u \in U, v \in V - U, uv \notin E(G)\}.$$

The following lemma describes some properties of switching operation.

Lemma 2.2 ([8]). *Let $G = (V, E)$ be a graph and $U, U_1, U_2 \subset V$. Then*

(1) $S(G, U) = S(G, V - U)$;
(2) $S(S(G, U_1), U_2) = S(G, U_1 \triangle U_2)$;
(3) *if* $U = \{u_1, \ldots, u_m\}$, *then* $G_m = S(G, U)$, *where* $G_0 = G$ *and* $G_i = S(G_{i-1}, u_i)$, $1 \leq i \leq m$;
(4) $\overline{S}(G, U) = S(\overline{G}, U)$.

Switching operation leads to natural equivalence relation on graphs. We say that two graphs G_1 and G_2 are *s-equivalent* if there exists $U \subset V(G_1)$ such that $S(G_1, U) \simeq G_2$. The corresponding equivalence class is called *s-class*. The s-equivalence of G_1 and G_2 will be denoted as $G_1 \sim_s G_2$. For example, every two complete bipartite graphs with the same number of vertices are s-equivalent and the s-class of $\overline{K}_n$ consists of $\overline{K}_n$ and all complete bipartite graphs with n vertices (see [4]).

It is trivial that $G_1 \sim_s G_2$ if and only if $\overline{G}_1 \sim_s \overline{G}_2$. An interesting result of Colbourn and Corneil [2] states that the problem of deciding s-equivalence of two graphs is polynomial-time equivalent to the problem of deciding isomorphism of graphs. To show this Colbourn and Corneil proposed the following construction. For any graph G take G and its copy G' and add a new edge between $u \in V(G)$ and $v' \in V(G')$ if u and v are not adjacent in G. The obtained graph is denoted by $Sw(G)$. Thus the nontrivial criterion of s-equivalence is the following: $G_1 \sim_s G_2$ if and only if $Sw(G_1) \simeq Sw(G_2)$.

Now we turn to the maximum size graphs in switching classes.

Definition 2.3. A graph G is called *s-maximal* if for every graph H with $G \sim_s H$ it holds that $|E(H)| \leq |E(G)|$.

It should be noted that null graph is s-maximal.

Remark 2.4. It is obvious that if G is the spanning subgraph of G' and G is s-maximal, then G' is also s-maximal.

Also, there exist non-isomorphic s-equivalent s-maximal graphs. For example, consider $G_1 = \overline{K}_2 + (K_1 \cup K_2)$ and $G_2 = \overline{K}_3 + K_2$ (note that G_1 is hamiltonian, but G_2 is not).

Dually, one can define *s-minimal* graphs. However it is easy to see that the graph is s-maximal if and only if its complement is s-minimal. Therefore, we study only s-maximal graphs.

We proceed with an obvious reformulation of the definition of s-maximal graphs. In the sequel this result will be used without any references.

Lemma 2.5. *A graph G is s-maximal if and only if for every $U \subset V(G)$ we have*

$$2l(U) \geq |U|(|V(G)| - |U|).$$

3. Results

We start with some easy properties of s-maximal graphs.

Theorem 3.1. *Let G be an s-maximal graph with n vertices. Then*

(1) $\delta(G) \geq \frac{n-1}{2}$;
(2) $\Delta(G) \geq \frac{n+\omega(G)}{2} - 1$;
(3) $|E(G)| \geq \frac{n(n-1)+\omega(G)(\omega(G)-1)}{4}$;
(4) *G is connected with $diam(G) \leq 2$;*
(5) *G has a hamiltonian path;*
(6) *the set $\{u \in V(G) : d(u) = \frac{n-1}{2}\}$ is independent;*
(7) *if $M \subset \{u \in V(G) : d(u) = n-1\}$ with $|M| < \frac{n}{3}$, then $G-M$ is connected.*

Proof: (1) For each $u \in V(G)$ apply Lemma 2.5 to $U = \{u\}$.

(2) Put $\omega = \omega(G)$ and let $U \subset V(G)$ induce a maximal clique in G. We have

$$\begin{aligned}\sum_{u\in U} d(u) &= l(U) + 2e(U) = l(U) + \omega(\omega-1)\\ &\geq \frac{\omega(n-\omega)}{2} + \omega(\omega-1)\\ &= \omega\cdot\left(\frac{n+\omega}{2}-1\right).\end{aligned}$$

Thus

$$\Delta(G) \geq \frac{1}{\omega}\sum_{u\in U} d(u) \geq \frac{n+\omega}{2} - 1.$$

(3) Again, let $U \subset V(G)$ induce a maximal clique in G. We have

$$\begin{aligned}2|E(G)| &= \sum_{u\in U} d(u) + \sum_{u\notin U} d(u)\\ &\geq \omega\cdot\left(\frac{n+\omega}{2}-1\right) + (n-\omega)\cdot\frac{n-1}{2}\\ &= \frac{n(n-1)+\omega(\omega-1)}{2}.\end{aligned}$$

(4) Let $u, v \in V(G)$ be two nonadjacent vertices. From (1) it follows that $d(u) + d(v) \geq n-1$. Therefore $N(u) \cap N(v)$ is nonempty and thus $diam(G) \leq 2$.

(5) Again, for every two vertices $u, v \in V(G)$ we have $d(u) + d(v) \geq n-1$. But it is well known [13] that in this case G has a hamiltonian path.

(6) Suppose that we have two adjacent vertices $u, v \in V(G)$ with $d(u) = d(v) = \frac{n-1}{2}$. Putting $U = \{u, v\}$ we obtain

$$2l(U) = 2\cdot\left(\frac{n-1}{2} + \frac{n-1}{2} - 2\right) = 2(n-3) < 2(n-2)$$

which is a contradiction.

(7) Assume to the contrary that $G - M$ is disconnected and let H_1 be its component. Put $H_2 = (G-M) - H_1$ and $a = |V(H_1)|$, $b = |V(H_2)|$, $m = |M|$.

Since G is s-maximal, then $2am = 2l(V(H_1)) \geq a(n-a) = a(m+b)$. It means that $m \geq b$. Similarly, $m \geq a$. Thus $2m \geq a + b = n - m$, which leads to $m \geq \frac{n}{3}$. But this is a contradiction. □

Now we show that every graph with sufficiently large minimum degree is necessarily s-maximal. It means that there is no "more structural" characterization of s-maximal graphs than provided by Lemma 2.5.

Proposition 3.2. *Let G be a graph with n vertices and $\delta(G) \geq \frac{3n}{4} - 1$. Then G is s-maximal.*

PROOF: Consider some set $U \subset V(G)$. Since $l(U) = l(V(G) - U)$, without loss of generality we can assume that $|U| \leq \frac{n}{2}$. We have

$$\begin{aligned} 2l(U) &= 2 \cdot \left(\sum_{u \in U} d(u) - 2e(U) \right) \geq 2(|U|\delta(G) - |U|(|U| - 1)) \\ &\geq |U| \cdot \left(\frac{3n}{2} - 2 \right) - 2|U|(|U| - 1) = |U| \cdot \left(\frac{3n}{2} - 2|U| \right) \\ &= |U| \cdot \left(n - |U| + \frac{n}{2} - |U| \right) \geq |U|(n - |U|) \end{aligned}$$

and thus G is s-maximal. □

The following result shows that the class of s-maximal graphs is closed under the join operation on graphs.

Proposition 3.3. *Let G_1 and G_2 be two s-maximal graphs. Then $G_1 + G_2$ is also s-maximal.*

PROOF: Let $G = G_1 + G_2$. Put $V = V(G)$ and $V_i = V(G_i)$ for $i = 1, 2$.

Also, let $n_i = |V_i|$, $i = 1, 2$.

Now consider nonempty set $U \subset V(G)$. We put $a = |U \cap V_1|$ and $b = |U \cap V_2|$. Note that $n_1 \geq a$ and $n_2 \geq b$.

It holds that

$$\begin{aligned} 2l_G(U) &= 2(e_G(U \cap V_1, V_1 - U) + e_G(U \cap V_1, V_2 - U) \\ &+ e_G(U \cap V_2, V_1 - U) + e_G(U \cap V_2, V_2 - U)) \\ &= 2(l_{G_1}(U \cap V_1) + a(n_2 - b) + b(n_1 - a) + l_{G_2}(U \cap V_2)) \\ &\geq a(n_1 - a) + 2a(n_2 - b) + 2b(n_1 - a) + b(n_2 - b) \\ &= a(n_1 - a + 2(n_2 - b)) + b(n_2 - b + 2(n_1 - a)) \\ &\geq a(n_1 - a + n_2 - b) + b(n_2 - b + n_1 - a) \\ &= (a + b)(n_1 + n_2 - a - b) = |U|(|V| - |U|) \end{aligned}$$

which completes the proof. □

When is the join of arbitrary graphs s-maximal? To answer this question we need the following lemma.

Lemma 3.4. *Let G be an s-maximal graph and H be a graph with $|V(H)| \leq |V(G)| + 1$. Then $G + H$ is also s-maximal.*

PROOF: Put $n = |V(G)|$ and $k = |V(H)|$. We have $k \leq n + 1$. Also, let $G' = G + H$.

For $U \subset V(G')$ put $a = |U \cap V(G)|$ and $b = |U \cap V(H)|$.

If $b = 0$, then $2l(U) \geq a(n - a) + ak = a(n + k - a) = |U|(|V(G')| - |U|)$.

Now let $b \geq 1$. Since $l(U) = l(V(G') - U)$, without loss of generality we can assume that $k \geq 2b$. Therefore

$$\begin{array}{rcl} 2l(U) & = & 2(e_G(U \cap V(G), V(G) - U) + e_G(U \cap V(G), V(H) - U) \\ & + & e_G(U \cap V(H), V(G) - U) + e_G(U \cap V(H), V(H) - U)) \\ & \geq & a(n-a) + 2a(k-b) + 2b(n-a) \\ & = & (a+b)(n+k-a-b) + a(k-2b) + b(b+n-k) \\ & \geq & (a+b)(n+k-a-b) + b(b-1) \\ & \geq & (a+b)(n+k-a-b) = |U|(|V(G')| - |U|) \end{array}$$

and the desired is proved. □

Theorem 3.5. *Suppose that we have $m \geq 2$ and graphs $G_1, \ldots, G_m$ with $\||V(G_i)| - |V(G_j)|\| \leq 1$ for all $1 \leq i, j \leq m$. Then $\sum_{i=1}^{m} G_i$ is s-maximal.*

Proof: We will prove this theorem using induction argument.

Firstly, let $m = 2$. Consider two graphs G_1 and G_2 with $n_i = |V(G_i)|$, $i = 1, 2$ and suppose that $|n_1 - n_2| \leq 1$. Also, let $G = G_1 + G_2$.

For every nonempty $U \subset V(G)$ put $a_i = |U \cap V(G_i)|$, $i = 1, 2$.

If $a_1 = 0$, then

$$\begin{array}{rcl} 2l_G(U) - |U|(|V(G)| - |U|) & = & 2a_2n_1 - a_2(n_1 + n_2 - a_2) \\ & = & 2a_2n_1 - a_2n_1 - a_2n_2 + a^2 \\ & = & a_2(n_1 - n_2) + a_2^2 \\ & \geq & a_2(a_2 - 1) \geq 0. \end{array}$$

Now let, without loss of generality, $a_1 \geq a_2 \geq 1$. We have

$$\begin{array}{rcl} 2l_G(U) - |U|(|V(G)| - |U|) & \geq & 2(a_1(n_2 - a_2) + a_2(n_1 - a_1)) \\ & - & (a_1 + a_2)(n_1 + n_2 - a_1 - a_2) \\ & = & (n_2 - n_1)(a_1 - a_2) + (a_1 - a_2)^2 \\ & \geq & (a_2 - a_1) + (a_1 - a_2)^2 \\ & = & (a_1 - a_2)(a_1 - a_2 - 1) \geq 0. \end{array}$$

Now consider $m+1$ graphs $G_1, \ldots, G_{m+1}$ with $\||V(G_i)| - |V(G_j)|\| \leq 1$ for $1 \leq i, j \leq m+1$ and put $G = \sum_{i=1}^{m} G_i$. From induction hypothesis it follows that G is s-maximal. Furthermore, $|V(G_{m+1})| \leq |V(G)| + 1$. Thus Lemma 3.4 implies that $G + G_{m+1} = \sum_{i=1}^{m+1} G_i$ is also s-maximal. □

Example 3.6. There exist s-maximal graphs which cannot be expressed as a join of two graphs. Consider the complement of the path with n vertices $G = \overline{P}_n$ for each $n \geq 8$. Then $\delta(G) = n - 1 - \Delta(\overline{G}) = n - 3 \geq \frac{3n}{4} - 1$. Thus from Proposition 3.2 it follows that G is s-maximal, but clearly G is not the join of two graphs as $\overline{G}$ is connected.

We say that the edge $e = uv \in E(G)$ is *dominating* if the set $\{u, v\}$ is dominating.

Lemma 3.7. *Each edge in an s-maximal graph is either dominating or lies in a triangle.*

PROOF: Let G be an s-maximal graph and $e = uv \in E(G)$. Put $U = \{u, v\}$. Then $d(u) + d(v) - 2 = l(U) \geq n - 2$. Thus $d(u) + d(v) \geq n$. Now if $N(u) \cap N(v)$ is empty, then e is dominating. Otherwise, for every $x \in N(u) \cap N(v)$ the triple (u, v, x) forms a triangle. □

Theorem 3.8. *Let G be triangle-free s-maximal graph. Then $G \simeq K_{n,n}$ or $G \simeq K_{n,n+1}$, where $n \geq 0$.*

PROOF: If $|V(G)| = 1$, then $G \simeq K_1 = K_{1,0}$. Similarly, if $|V(G)| = 2$, then $G \simeq K_2 = K_{1,1}$. Now let $|V(G)| \geq 3$. From Theorem 3.1(3) it follows that $|E(G)| \geq 1$.

Consider some edge $e = uv \in E(G)$. Since G is triangle-free, from Lemma 3.7 it follows that e is dominating. Thus $N[u] \cup N[v] = V(G)$.

For every $x \in N(u) - \{v\}$ the edge $e' = ux$ is also dominating. But $(N(v) - \{u\}) \cap N(u)$ is empty, otherwise there would be a triangle. It means that $N(v) - \{u\} \subset N(x)$. Similarly, $N(x) - \{u\} \subset N(v)$.

Thus for all $x \in N(u)$ we have $N(x) = N(v)$. Therefore $(N(v) \cup \{u\}, N(u) \cup \{v\})$ is a bipartition of complete bipartite graph G.

Further, let $G \simeq K_{a,b}$ with bipartition (A, B) and $a = |A|$, $b = |B|$. Assume that $a \geq b + 2$. Then for all $x \in A$ we obtain

$$\frac{|V(G)| - 1}{2} = \frac{a + b - 1}{2} \geq \frac{b + 2 + b - 1}{2} = b + \frac{1}{2} > b = d(x),$$

a contradiction with s-maximality of G. Therefore $a \leq b + 1$. Analogously, $b \leq a + 1$. Thus $|a - b| \leq 1$ and the desired is proved. □

Remark 3.9. Note that from Theorem 3.5 it follows that $K_{n,n}$ and $K_{n,n+1}$ are s-maximal graphs for all $n \geq 0$. Thus Theorem 3.8 gives a complete characterization of triangle-free s-maximal graphs, as well as bipartite s-maximal graphs.

Now we turn to the characterization of non-hamiltonian s-maximal graphs.

Theorem 3.10. *Let G be a non-hamiltonian s-maximal graph. Then $G \simeq K_2$ or $G \simeq \overline{K}_{k+1} + H$ for some graph H with $k \geq 0$ vertices.*

PROOF: Put $n = |V(G)|$. If $n = 1$, then $G \simeq \overline{K}_{k+1} + H$, where $k = 0$ and $H \simeq K_0$.

Now let $n \geq 2$ and suppose that G is acyclic. Using Theorem 3.1, part 3 we obtain

$$\frac{n(n-1) + 2}{4} \leq |E(G)| \leq n - 1.$$

This yields $2 \leq n \leq 3$. If $n = 2$, then $G \simeq K_2$. If $n = 3$, then $G \simeq \overline{K}_{k+1} + H$, where $k = 1$ and $H \simeq K_1$.

Now suppose that G has a cycle. Fix the longest cycle C in G and put $c = |V(C)|$. Also, let $V(C) = \{u_1, \dots, u_c\}$ with $\{u_i u_{i+1} : 1 \le i \le c-1\} \cup \{u_c u_1\} \subset E(G)$.

Note that since G is non-hamiltonian, the set $U = V(G) - V(C)$ is nonempty.

Claim 1. For all $v \in U$ we have $|N(v) \cap V(C)| = \frac{c}{2}$.

At first, suppose that there exists a vertex $v_0 \in U$ with $|N(v) \cap V(C)| > \frac{c}{2}$. Then one can find two distinct vertices $x, y \in N(v_0) \cap V(C)$ with $xy \in E(C)$. This means that v_0 can be inserted into C to obtain a longer cycle which is a contradiction. Thus $|N(v) \cap V(C)| \le \frac{c}{2}$ for all $v \in U$.

On the other hand, if there exists a vertex $v_0 \in U$ with $|N(v) \cap V(C)| < \frac{c}{2}$, then

$$2l(U) = 2\sum_{v \in U} |N(v) \cap V(C)| < |U|c = |U|(n - |U|)$$

which contradicts the s-maximality of G.

Claim 2. The set U is independent.

Suppose that there exist two vertices $v_1, v_2 \in U$ with $v_1 v_2 \in E(G)$.

Since for all $v \in U$ the set $N(v) \cap V(C)$ is independent (otherwise v can be inserted into C) of cardinality $\frac{c}{2}$, without loss of generality we can assume that $N(v_1) \cap V(C) = \{u_1, u_3, \dots, u_{c-1}\}$.

If $N(v_2) \cap V(C) = N(v_1) \cap V(C)$, then

$$v_1 - u_1 - u_2 - \cdots - u_{c-1} - v_2 - v_1$$

is a cycle longer than C which is a contradiction.

Similarly, if $N(v_2) \cap V(C) \neq N(v_1) \cap V(C)$, then it is easy to see that $N(v_2) \cap V(C) = \{u_2, \dots, u_c\}$. In this case

$$v_1 - u_1 - u_2 - \cdots - u_c - v_2 - v_1$$

is a cycle longer than C which again is a contradiction.

Claim 3. $|U| = 1$.

From Claim 2 it follows that for all $v \in U$ we have $d(v) = |N(v)| = |N(v) \cap V(C)| = \frac{c}{2}$. Since G is s-maximal and U is nonempty, for all $v \in U$ we have

$$\frac{n-1}{2} \le d(v) = \frac{c}{2} = \frac{n - |U|}{2}.$$

Thus $|U| = 1$ and therefore there exists a vertex $v_0 \in V(G)$ with $U = \{v_0\}$. Note that $d(v_0) = \frac{n-1}{2}$.

Claim 4. The set $M := V(C) - N(v_0)$ is independent.

Without loss of generality we can assume that $N(v_0) = N(v_0) \cap V(C) = \{u_2, \dots, u_c\}$. To the contrary, let there exists an edge $u_{2k+1}u_{2l+1} \in E(G)$, where $k < l$. Then

$$v_0 - u_{2k+2} - \cdots - u_{2l} - u_{2l+1} - u_{2k+1} - u_{2k} - \cdots - u_{2l+2} - v_0$$

is a cycle longer than C which is a contradiction.

Claim 5. For all $u \in M$ we have $N(u) = N(v_0)$.

From Claim 4 it follows that $N(u) \subset N(v_0)$. But from the s-maximality of G we have $d(u) \geq \frac{n-1}{2} = d(v_0)$. Therefore $N(u) = N(v_0)$. This leads to

$$G = G[M \cup \{v_0\}] + G[N(v_0)] \simeq \overline{K}_{k+1} + H,$$

where $k = d(v_0) = \frac{c}{2} = \frac{n-1}{2}$. □

Remark 3.11. It is obvious that K_2 is an s-maximal graph. Furthermore, from Theorem 3.5 it follows that for every graph H with $k \geq 0$ vertices the graph $\overline{K}_{k+1}+H$ is also s-maximal. Thus Theorem 3.10 gives a complete characterization of non-hamiltonian s-maximal graphs.

Corollary 3.12. *Every s-maximal graph with even number $n \geq 4$ of vertices is hamiltonian.*

Corollary 3.13. *Let G be a non-hamiltonian s-maximal graph with $n \geq 1$ vertices. Then there exists a vertex $v \in V(G)$ such that $G - v$ is also s-maximal.*

PROOF: From Theorem 3.10 it follows that $G \simeq K_2$ or $G \simeq \overline{K}_{k+1} + H$ for some graph H with $k \geq 0$ vertices. If $G \simeq K_2$, then for all $v \in V(G)$ we have $G - v \simeq K_1$, and thus $G - v$ is s-maximal. If $G \simeq \overline{K}_{k+1} + H$, then there exists a vertex $v \in V(G)$ such that $G - v \simeq \overline{K}_k + H$. But since $|V(H)| = k$ the graph $G - v$ appears to be s-maximal as it follows from Theorem 3.5. □

We do not know if every nontrivial s-maximal graph G contains a vertex $v \in V(G)$ such that $G - v$ is also s-maximal. However, we can prove the following result.

Theorem 3.14. *Let G be an s-maximal graph with n vertices. If $\delta(G) = \frac{n-1}{2}$, then there exists $v \in V(G)$ such that $G - v$ is s-maximal.*

PROOF: Consider $v \in V(G)$ with $d(v) = \frac{n-1}{2}$ and assume that $G - v$ is not s-maximal. Then there exists $U \subset V(G)$ such that $l_{G-v}(U) < \frac{m(n-1-m)}{2}$, where $m = |U|$.

Since G is s-maximal, we have

$$l_G(U) \geq \frac{m(n-m)}{2}$$

and

$$l_G(U') \geq \frac{(m+1)(n-m-1)}{2}$$

for $U' = U \cup \{v\}$.

Consider the next equalities

$$l_G(U) = l_{G-v}(U) + |N_G(v) \cap U|,$$
$$l_G(U') = l_{G-v}(U) + |N_G(v) \cap (V(G) - U)|.$$

Adding these we obtain

$$l_G(U) + l_G(U') = 2l_{G-v}(U) + d_G(v).$$

Hence

$$\begin{aligned} d_G(v) &= l_G(U) + l_G(U') - 2l_{G-v}(U) \\ &> \frac{m(n-m)}{2} + \frac{(m+1)(n-m-1)}{2} - m(n-1-m) \\ &= \frac{n-1}{2} \end{aligned}$$

which is a contradiction. □

Finally, we should say a few words about unique s-maximal graphs in their s-classes. In [4] Hage proved the following result.

Theorem 3.15 ([4])**.** *Let G be a graph with $n \geq 3$ vertices. Then G is s-equivalent to an s-maximal pancyclic graph if and only if G is not s-equivalent to $\overline{K}_n$.*

Therefore, if G is unique s-maximal graph in its s-class, then $G \simeq K_{n,n}$ or $G \simeq K_{n,n+1}$ or G is pancyclic.

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