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Todorcevic orderings as examples of ccc forcings without adding random reals

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Abstract. In [*Two examples of Borel partially ordered sets with the countable chain condition*, Proc. Amer. Math. Soc. **112** (1991), no. 4, 1125–1128], Todorcevic introduced a ccc forcing which is Borel definable in a separable metric space. In [*On Todorcevic orderings*, Fund. Math., to appear], Balcar, Pazák and Thümmel applied it to more general topological spaces and called such forcings *Todorcevic orderings*. There they analyze Todorcevic orderings quite deeply. A significant remark is that Thümmel solved the problem of Horn and Tarski by use of Todorcevic ordering [*The problem of Horn and Tarski*, Proc. Amer. Math. Soc. **142** (2014), no. 6, 1997–2000].

This paper supplements the analysis of Todorcevic orderings due to Balcar, Pazák and Thümmel in [*On Todorcevic orderings*, Fund. Math., to appear]. More precisely, it is proved that Todorcevic orderings add no random reals whenever they have the countable chain condition.

Keywords: Todorcevic orderings; random reals

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1. Introduction

In [13], Todorcevic introduced a Borel definable ccc forcing which consistently does not have property K. He defined it on a separable metric space. By generalizing it and applying it to other topological spaces, Thümmel discovered a forcing notion which has the σ-finite chain condition but does not have the σ-bounded chain condition, and so he solved the problem of Horn and Tarski [11]. (For Horn-Tarski's problem, see [5], [14].) Right after Thümmel's result, Todorcevic introduced a Borel definable solution of the problem of Horn and Tarski [15].

In [2], Balcar, Pazák and Thümmel applied Todorcevic's Borel definable ccc forcing in [13] to more general topological spaces. They called such forcings *Todorcevic orderings*. In [2], they analyzed Todorcevic orderings from several view points. One of them is about the countable chain condition. They introduced a sufficient condition for topological spaces (which is called *condition* 1, see below) for which Todorcevic orderings have the countable chain condition.

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Moreover, they introduced a necessary and sufficient condition for topological spaces for which Todorcevic orderings have the countable chain condition under the Proper Forcing Axiom (in fact, Todorcevic's dichotomy for ω_1-generated ideals suffices). They also found the space for which Todorcevic ordering does not have the countable chain condition.

In [9], Solovay introduced notion of random reals in terms of forcing theory. The phrase "adding a random real" is equivalent to the phrase "having a regular subalgebra which supports a measure in terms of Boolean algebras". In this paper, we will show that Todorcevic orderings add no random reals whenever they have the countable chain condition (ccc). Todorcevic [13] and Balcar, Pazák and Thümmel [2] provided two sufficient conditions for topological spaces with the cccness of Todorcevic orderings. These conditions cover a wide class of topological spaces. So many Todorcevic orderings add no random reals.

In fact, it is proved that ccc Todorcevic orderings satisfy the ${}^{\omega}2$-hyper-properness. The hyper-properness was introduced by Dow-Steprāns [4]. In [4], the hyper-properness is formulated for the Baire space ${}^{\omega}\omega$. The ${}^{\omega}2$-hyper-properness is Dow-Steprāns' hyper-properness for the Cantor space ${}^{\omega}2$ instead of ${}^{\omega}\omega$. The ${}^{\omega}2$-hyper-properness is stronger than the property "adding no random reals".

In §2, we state the definition of Todorcevic orderings, some basic observations, two sufficient conditions for topological spaces with the cccness of Todorcevic orderings, Dow-Steprāns' hyper-properness, and the main theorem in this paper. In §3, it is proved that Todorcevic orderings with the countable chain condition are ${}^{\omega}2$-hyper-proper.

2. Preliminaries

2.1 Todorcevic orderings. As said in [2], when Todorcevic ordering is applied to a topological space, it is natural to require it to be sequential and have the unique limit property[1]. A topological space X is called sequential if for any $Z \subseteq X$, Z is closed in X iff for any $A \subseteq Z$ and $x \in X$ to which A converges, x belongs to Z. A topological space X has the unique limit property if any converging subset of X converges to the unique point. For example, Hausdorff spaces have the unique limit property. For a subset F of a topological space, let F^d denote the first Cantor-Bendixson derivative of F, that is, the set of all accumulation points of F.

Definition 2.1 (Todorcevic [13], see also [2], [11])**.** For a topological space X, $\mathbb{T}(X)$ is the set of all subsets of X which are unions of finitely many converging sequences including their limit points, and for each p and q in $\mathbb{T}(X)$, $q \leq_{\mathbb{T}(X)} p$ iff $q \supseteq p$ and $q^d \cap p = p^d$.[2]

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[1]Todorcevic ordering for a space X has the ccc whenever X satisfies condition 1. For condition 1, see below.

[2]This definition is slightly different from the original one, in [13], which introduced the set of all finite sets σ of convergent sequences in X including their limit points such that for any

For $p, q \in \mathbb{T}(X)$, the statement $q \leq_{\mathbb{T}(X)} p$ means that q is an extension of p (as the subset relation) and the isolated points in p are still isolated in q. $\mathbb{T}(X)$ is called *Todorcevic ordering for the space* X in [2], [11].

In [15], Todorcevic introduced the Borel definable version of Todorcevic orderings which consists of all countable compact subsets whose first Cantor-Bendixson derivative is finite. In [2], Balcar, Pazák and Thümmel introduced a separable version of Todorcevic orderings which consists of all functions f from elements p of $\mathbb{T}(X)$ into $\{0,1\}$ such that $f^{-1}(1)$ is a finite set including p^d as a subset, ordered by the function-extension. In this paper, we adopt the definition of Todorcevic orderings as in Definition 2.1. However, all of the proofs in this paper can be applied for the other definitions without any change.

We note that $\mathbb{T}(X)$ is closed under finite unions, i.e. for every $p, q \in \mathbb{T}(X)$, $p \cup q$ is a condition of $\mathbb{T}(X)$ too. But then, $p \cup q$ may not be an extension of p or q. We also note that each condition of $\mathbb{T}(X)$ is a countable closed subset of X.

In [13], Todorcevic proved that Todorcevic ordering for the reals has the countable chain condition [13, Claim 3]. His proof can be extended to spaces which satisfy that any of its finite powers is hereditarily separable. Namely, for a space X, if each of finite powers of X is hereditarily separable, then Todorcevic ordering for X has the countable chain condition.

In [2], Balcar, Pazák and Thümmel analyzed for what spaces Todorcevic orderings have the countable chain conditions. They gave the property, called *condition* 1, for topological spaces X which is sufficient for Todorcevic orderings to have the countable chain condition:

> Condition 1: For any $x \in X$ and $Y \in [X]^{\aleph_1}$, there exists $Y' \in [Y]^{\aleph_1}$ such that any countable subset of Y' does not converge to the point x.

They pointed out that condition 1 is a weak property for topological spaces. Topological spaces with condition 1 include first countable spaces (hence metric spaces), linear ordered spaces, and hereditarily Lindelöf T_1-spaces. It seems that condition 1 is not related to the hereditary separability. A nonseparable metric space has condition 1 but is not hereditarily separable. If a hereditarily separable space does not have condition 1, it has to be an S-space. (For S-spaces, see e.g. [12].)

In [11], Thümmel discovered a counterexample to the problem of Horn and Tarski in [5] whether every poset with the σ-finite chain condition also has the σ-bounded chain condition. More precisely, he proved that Todorcevic ordering for the ordered topological space $(\bigcup_{\alpha\in\omega_1} {}^{\alpha+1}(\omega^*), <_{\text{lex}})$ [3] has the σ-finite chain

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$A, B \in \sigma$,

$$\lim(A) \notin (B \setminus \{\lim(B)\}),$$

ordered by the reverse inclusion. But essentially, both are same. In fact, both are forcing-equivalent.

[3] ω^* stands for the set of natural numbers with the reverse order, and $<_{\text{lex}}$ stands for the lexicographic order. We note that the space $(\bigcup_{\alpha\in\omega_1} {}^{\alpha+1}(\omega^*), <_{\text{lex}})$ satisfies the condition 1.

condition but does not have the σ-bounded chain condition. At first, the author proved that his Todorcevic ordering adds no random reals.

Since a Boolean algebra which supports a measure has the σ-bounded chain condition (see e.g. [1, §4], [14, §1]), it follows from Thümmel's result that the Boolean completion of his Todorcevic ordering does not support a measure. We recall that the phrase "adding a random real" means the phrase "having a regular subalgebra which supports a measure in terms of Boolean algebras". So it follows from the author's result that a Boolean completion of Thümmel's Todorcevic ordering has no regular subalgebras which support a measure.

2.2 The ${}^\omega 2$-hyper-properness. In [4, Definition 3.8], Dow and Steprāns introduced the property for forcing notions like the following definition, called the hyper-properness. This is a useful property to show some preservation theorems of the iterated forcings. In fact, the hyper-properness is preserved by countable support iterations. The original definition is formulated for the Baire space ${}^\omega\omega$. But in this paper, we consider it on the Cantor space ${}^\omega 2$. Two of them are slightly different, so in this paper, we call the following *the ${}^\omega 2$-hyper properness*.

Definition 2.2 (Dow-Steprāns)**.** A forcing notion $\mathbb{P}$ is called ${}^\omega 2$-hyper-proper if for any regular cardinal $\kappa > 2^{|\mathbb{P}|}$, countable elementary submodel N of $H(\kappa)$ which contains $\mathbb{P}$ as a member, $p \in \mathbb{P} \cap N$, and countable family $\mathcal{A}$ of open subsets of ${}^\omega 2$, if ${}^\omega 2 \cap N \subseteq \bigcap \mathcal{A}$, then there is an extension q of p in $\mathbb{P}$ such that q is $(N, \mathbb{P})$-generic and

$$q \Vdash_{\mathbb{P}} \text{“ } {}^\omega 2 \cap N[\dot{G}] \subseteq \bigcap \mathcal{A}\text{”}.$$

Since the countable set can be covered by a G_δ Lebesgue measure zero set, the ${}^\omega 2$-hyper proper forcings add no random reals.

In §3, the following theorem is proved.

Theorem 2.3. *For a topological space X, if $\mathbb{T}(X)$ has the countable chain condition, then $\mathbb{T}(X)$ is ${}^\omega 2$-hyper-proper, and hence $\mathbb{T}(X)$ adds no random reals.*

2.3 Remarks on the theorem. In the first draft of the paper, the author proved that, for a topological space X satisfying one of the following properties. $\mathbb{T}(X)$ adds no random reals.

- Any of finite powers of X is hereditarily separable.
- X satisfies condition 1.

Proofs in both cases are similar, see [19, Theorem 5.4] and [22]. The only difference in two cases is a proof of Claim in §3. The referee let the author know about Dow-Steprāns' hyper-properness and gave a proof of Claim in the case that Todorcevic orderings have the ccc.

There are several non-ccc forcings which add no random reals, e.g. forcings with the Laver property. But one does not know so many such ccc-examples. A σ-centered forcing was the only well known ccc forcing without adding random reals which is proved in ZFC (due to Judah and Repický [6, Lemma 6], see also [3, Theorem 6.5.30])[4]. As consistent examples, Suslin tree is such a typical example (because this is a ccc forcing without adding new reals), and it is consistent that there exists a ccc perfect poset (due to Velickovic [16, §4]). Talagrand found a weakly distributive ccc σ-complete Boolean algebra which does not carry a measure [10] (which answers Maharam's problem); however, it is not known whether the completion of Talagrand's algebra has a regular subalgebra which carries a measure, that is, which adds a random real.

In [19, Theorem 5.4], the author found a subclass of ccc forcings (which was implicitly introduced by Larson and Todorcevic [7]) whose members add no random reals ([19, Theorem 5.3], [21]). This subclass is somewhat wide, for example, it includes a specializing Aronszajn tree and a freezing gap [17], [18], [19], [20], [21]. The theorem in this paper gives a new class of ccc forcings without adding random reals. This is the main motivation of this research.

3. Proof of Theorem 2.3

Suppose that X is a topological space such that $\mathbb{T}(X)$ has the countable chain condition. We will show that $\mathbb{T}(X)$ is ${}^{\omega}2$-hyper-proper.

Let κ and λ be large enough regular cardinals such that

$$\left(2^{|X|}\right)^{+} < \lambda < \left(2^{\lambda}\right)^{+} < \kappa,$$

N a countable elementary submodel of $H(\kappa)$ which contains all of finitely many objects we need in the proof (in the current case, N contains X, $\mathcal{P}(X)$ and $H(\lambda)$ as members), $p \in \mathbb{T}(X) \cap N$, $\langle U_n; n \in \omega \rangle$ a sequence of open subsets of ${}^{\omega}2$ such that

$${}^{\omega}2 \cap N \subseteq \bigcap_{n \in \omega} U_n,$$

and $\dot{x}$ a $\mathbb{T}(X)$-name for a real in ${}^{\omega}2$. We will show that

$$p \Vdash_{\mathbb{T}(X)} \text{“} \dot{x} \in \bigcap_{n \in \omega} U_n \text{”}.$$

This is what we want.

Suppose that

$$p \nVdash_{\mathbb{T}(X)} \text{“} \dot{x} \in \bigcap_{n \in \omega} U_n \text{”},$$

and take an extension q of p in $\mathbb{T}(X)$ and $m \in \omega$ such that

$$q \Vdash_{\mathbb{T}(X)} \text{“} \dot{x} \notin U_m \text{”}.$$

For each $k \in \omega$, define

[4] Osuga and Kamo develop Judah-Repický's result to σ-linked forcings of a strong form in some sense [8].

$$S_k := \Big\{ v \in {}^k 2\,;\ \text{there exists a countable elementary submodel } M \text{ of } H(\lambda) \text{ which contains the set } \{X, \mathcal{P}(X), \dot{x}, q^d \cap N\} \text{ as a member such that for every } r \in \mathbb{T}(X), \text{ if } r \text{ satisfies the statement}$$

(*) r^d includes $q^d \cap N$ as a subset, the size of r^d is equal to the size of q^d, and $(r^d \setminus (q^d \cap N)) \cap M = \emptyset$,

then $r \nVdash_{\mathbb{T}(X)}$ "$\dot{x}{\restriction}k \neq v$" $\Big\}$.

We note that the sequence $\langle S_k; k \in \omega\rangle$ belongs to the model N and for each $k \in \omega$, it holds that

$$\{v{\restriction}k\,; v \in S_{k+1}\} \subseteq S_k,$$

that is, $\bigcup_{k\in\omega} S_k$ forms a subtree of $2^{<\omega}$ (with respect to the subset relation).

The following is the key point of the proof.

Claim. *For every $k \in \omega$, S_k is not empty.*

We will show this later, and now we finish the proof assuming Claim.

By our assumption and the elementarity of the model N, we can find $u \in {}^\omega 2 \cap N$ such that for every $k \in \omega$, $u{\restriction}k \in S_k$. Since ${}^\omega 2 \cap N$ is covered by the intersection of the open sets U_n, we can take $l \in \omega$ such that

$$[u{\restriction}l] := \{y \in {}^\omega 2; u{\restriction}l \subseteq y\} \subseteq U_m.$$

Since $u{\restriction}l \in S_l$ in the model N, there exists a countable elementary submodel $M \in N$ of $H(\lambda)$ which witnesses the statement $u{\restriction}l \in S_l$. Then since q satisfies the statement **(*)** above for this M in the definition of S_l, it follows that

$$q \nVdash_{\mathbb{T}(X)} \text{“}\dot{x}{\restriction}l \neq u{\restriction}l\text{”}.$$

Therefore there exists $q' \leq_{\mathbb{T}(X)} q$ such that

$$q' \Vdash_{\mathbb{T}(X)} \text{“}\dot{x}{\restriction}l = u{\restriction}l\text{”}.$$

But then

$$q' \Vdash_{\mathbb{T}(X)} \text{“}\dot{x} \in [\dot{x}{\restriction}l] = [u{\restriction}l] \subseteq U_m\text{”},$$

which is a contradiction.

Proof of Claim (due to the referee): Let $k \in \omega$. We show that S_k is not empty. Assume the converse, and let $\{v_j; j < 2^k\}$ be an enumeration of the set ${}^k 2$. Then by our assumption, there exists a sequence $\langle r^j_\xi; j < 2^k,\ \xi \in \omega_1\rangle$ of conditions of $\mathbb{T}(X)$ such that

- for each $j < 2^k$ and $\xi \in \omega_1$,

$$r^j_\xi \Vdash_{\mathbb{T}(X)} \text{“}\dot{x}{\restriction}k \neq v_j\text{”},$$

- the set $\{(r^j_\xi)^d; j < 2^k,\ \xi \in \omega_1\}$ forms a Δ-system with root $q^d \cap N$.

Then the set $I := \{\bigcup_{j<2^k} r^j_\xi; \xi \in \omega_1\}$ is an uncountable set of conditions of $\mathbb{T}(X)$. (Here we do not say that $\bigcup_{j<2^k} r^j_\xi$ is a common extension of the set $\{r^j_\xi; j < 2^k\}$ in $\mathbb{T}(X)$. But each $\bigcup_{j<2^k} r^j_\xi$ is a condition of $\mathbb{T}(X)$.) Since $\mathbb{T}(X)$ is ccc, there exists $s \in \mathbb{T}(X)$ which forces that $I \cap \dot{G}$ is uncountable (here $\dot{G}$ is the canonical $\mathbb{T}(X)$-name for a generic filter). Then we can find $s' \leq_{\mathbb{T}(X)} s$ and $\{\xi_i; i < 2^k\} \in [\omega_1]^{2^k}$ such that s' forces that $\{\bigcup_{j<2^k} r^j_{\xi_i}; i < 2^k\} \subseteq \dot{G}$. Then the set $\{\bigcup_{j<2^k} r^j_{\xi_i}; i < 2^k\}$ has a common extension in $\mathbb{T}(X)$.

Therefore the set $\{r^i_{\xi_i}; i < 2^k\}$ has a common extension in $\mathbb{T}(X)$, actually, the set $\bigcup_{i<2^k} r^i_{\xi_i}$ is its common extension. Then it follows that

$$\bigcup_{i<2^k} r^i_{\xi_i} \Vdash_{\mathbb{T}(X)} \text{“}\dot{x}{\restriction}k \notin {}^k2\text{”},$$

which is a contradiction. □

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References

[1] Balcar B., Jech T., *Weak distributivity, a problem of von Neumann and the mystery of measurability*, Bull. Symbolic Logic **12** (2006), no. 2, 241–266.

[2] Balcar B., Pazák T., Thümmel E., *On Todorcevic orderings*, Fund. Math., to appear.

[3] Bartoszyński T., Judah H., *Set Theory. On the Structure of the Real Line*, A K Peters, Ltd., Wellesley, MA, 1995.

[4] Dow A., Steprāns J., *Countable Fréchet α_1-spaces may be first countable*, Arch. Math. Logic **32** (1992), no. 1, 33–50.

[5] Horn A., Tarski A., *Measures in Boolean algebras*, Trans. Amer. Math. Soc. **64** (1948), 467–497.

[6] Judah H., Repický M., *No random reals in countable support iterations*, Israel J. Math. **92** (1995), no. 1–3, 349–359.

[7] Larson P., Todorcevic S., *Katětov's problem*, Trans. Amer. Math. Soc. **354** (2002), no. 5, 1783–1791.

[8] Osuga N., Kamo S., *Many different covering numbers of Yorioka's ideals*, Arch. Math. Logic **53** (2014), no. 1–2, 43–56.

[9] Solovay R., *A model of set-theory in which every set of reals is Lebesgue measurable*, Ann. of Math. (2) **92** (1970), 1–56.

[10] Talagrand M., *Maharam's problem*, Ann. of Math. (2) **168** (2008), no. 3, 981–1009.

[11] Thümmel E., *The problem of Horn and Tarski*, Proc. Amer. Math. Soc. **142** (2014), no. 6, 1997–2000.

[12] Todorcevic S., *Partition Problems in Topology*, Contemporary Mathematics, 84, American Mathematical Society, Providence, Rhode Island, 1989.

[13] Todorcevic S., *Two examples of Borel partially ordered sets with the countable chain condition*, Proc. Amer. Math. Soc. **112** (1991), no. 4, 1125–1128.

[14] Todorcevic S., *A problem of von Neumann and Maharam about algebras supporting continuous submeasures*, Fund. Math. **183** (2004), no. 2, 169–183.

[15] Todorcevic S., *A Borel solution to the Horn-Tarski problem*, Acta Math. Hungar. **142** (2014), no. 2, 526–533.

[16] Velickovic B., *CCC posets of perfect trees*, Compos. Math. **79** (1991), no. 3, 279–294.

[17] Yorioka T., *Some weak fragments of Martin's axiom related to the rectangle refining property*, Arch. Math. Logic **47** (2008), no. 1, 79–90.

[18] Yorioka T., *The inequality $\mathfrak{b} > \aleph_1$ can be considered as an analogue of Suslin's Hypothesis*, Axiomatic Set Theory and Set-theoretic Topology (Kyoto 2007), Sūrikaisekikenkyūsho Kōkyūroku No. 1595 (2008), 84–88.

[19] Yorioka T., *A non-implication between fragments of Martin's Axiom related to a property which comes from Aronszajn trees*, Ann. Pure Appl. Logic **161** (2010), no. 4, 469–487.

[20] Yorioka T., *Uniformizing ladder system colorings and the rectangle refining property*, Proc. Amer. Math. Soc. **138** (2010), no. 8, 2961–2971.

[21] Yorioka T., *A correction to "A non-implication between fragments of Martin's Axiom related to a property which comes from Aronszajn trees"*, Ann. Pure Appl. Logic **162** (2011), 752–754.

[22] Yorioka T., *Keeping the covering number of the null ideal small*, preprint, 2013.

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