Symmetric products of the Euclidean spaces and the spheres

Naotsugu Chinen

Abstract. By $F_n(X)$, $n \geq 1$, we denote the n-th symmetric product of a metric space (X, d) as the space of the non-empty finite subsets of X with at most n elements endowed with the Hausdorff metric d_H. In this paper we shall describe that every isometry from the n-th symmetric product $F_n(X)$ into itself is induced by some isometry from X into itself, where X is either the Euclidean space or the sphere with the usual metrics. Moreover, we study the n-th symmetric product of the Euclidean space up to bi-Lipschitz equivalence and present that the 2nd symmetric product of the plane is bi-Lipschitz equivalent to the 4-dimensional Euclidean space.

Keywords: isometry; symmetric product; bi-Lipschitz maps; Euclidean space; sphere

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1. Introduction

As an interesting construction in topology, Borsuk and Ulam [4] introduced the n-th *symmetric product* of a metric space (X, d), denoted by $F_n(X)$. Recall that $F_n(X)$ is the space of non-empty finite subsets of X with at most n elements endowed with the Hausdorff metric d_H, i.e., $F_n(X) = \{A \subset X : 1 \leq |A| \leq n\}$ and $d_H(A, A') = \inf\{\epsilon : A \subset B_d(A', \epsilon) \text{ and } A' \subset B_d(A, \epsilon)\} = \max\{d(a, A'), d(a', A) : a \in A, a' \in A'\}$ for any $A, A' \in F_n(X)$ (see [12, p. 6]). It was proved in [4] that $F_n(\mathbb{I})$ is homeomorphic to $\mathbb{I}^n$ (written $F_n(\mathbb{I}) \approx \mathbb{I}^n$) if and only if $1 \leq n \leq 3$ (cf. Remark 4.19 below), and that for $n \geq 4$, $F_n(\mathbb{I})$ cannot be embedded into $\mathbb{R}^n$, where $\mathbb{I} = [0, 1]$ has the usual metric. A considerable number of studies have been made on the topological structures of $F_n(X)$. For example, Molski [15] showed that $F_2(\mathbb{I}^2) \approx \mathbb{I}^4$(cf. Remark 4.19 below), and that for $n \geq 3$ neither $F_n(\mathbb{I}^2)$ nor $F_2(\mathbb{I}^n)$ can be embedded into $\mathbb{R}^{2n}$.

For the symmetric products of $\mathbb{R}$, it is easily seen that $F_2(\mathbb{R}) \approx \{(x, y) \in \mathbb{R}^2 : x \leq y\} \approx \mathbb{R} \times [0, \infty)$. Indeed, the map $h : \{(x, y) \in \mathbb{R}^2 : x \leq y\} \to F_2(\mathbb{R})$ defined by $h(x, y) = \{x, y\}$ is a homeomorphism. It was known that $F_3(\mathbb{R})$ and $\mathbb{R}^3$ are homeomorphic, in particular, there is a bi-Lipschitz equivalence (see [6] or Section 4 for details). Turning toward the symmetric product $F_n(\mathbb{S}^1)$ of the circle $\mathbb{S}^1$, in [10], it was proved that for $n \in \mathbb{N}$, both $F_{2n-1}(\mathbb{S}^1)$ and $F_{2n}(\mathbb{S}^1)$ have the

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same homotopy type of the $(2n-1)$-sphere $\mathbb{S}^{2n-1}$. In [8], Bott corrected Borsuk's statement [5] and showed that $F_3(\mathbb{S}^1) \approx \mathbb{S}^3$. In [10], another proof of it was given.

For a metric space (X, d), we denote by $\mathrm{Isom}_d(X)$ ($\mathrm{Isom}(X)$ for short) the group of all isometries from X into itself, i.e., $\phi : X \to X \in \mathrm{Isom}_d(X)$ if ϕ is a bijection satisfying that $d(x, x') = d(\phi(x), \phi(x'))$ for any $x, x' \in X$. Let $n \in \mathbb{N}$. Every isometry $\phi : X \to X$ induces an isometry $\chi_n(\phi) : (F_n(X), d_H) \to (F_n(X), d_H)$ defined by $\chi_n(\phi)(A) = \phi(A)$ for each $A \in F_n(X)$. Thus, there exists a natural monomorphism $\chi_n : \mathrm{Isom}_d(X) \to \mathrm{Isom}_{d_H}(F_n(X))$. It is clear that $\chi_n : \mathrm{Isom}_d(X) \to \mathrm{Isom}_{d_H}(F_n(X))$ is an isomorphism if and only if χ_n is an epimorphism, i.e., for every $\Phi \in \mathrm{Isom}_{d_H}(F_n(X))$ there exists $\phi \in \mathrm{Isom}_d(X)$ such that $\Phi = \chi_n(\phi)$.

Recently, Borovikova and Ibragimov [6] proved that $(F_3(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R}^3, d)$ and that $\chi_3 : \mathrm{Isom}_d(\mathbb{R}) \to \mathrm{Isom}_{d_H}(F_3(\mathbb{R}))$ is an isomorphism, where $\mathbb{R}$ has the usual metric d. It is of interest to know whether $\chi_n : \mathrm{Isom}_d(X) \to \mathrm{Isom}_{d_H}(F_n(X))$ is an isomorphism for a metric space (X, d). In the first part of this paper, we prove the following result which is a generalization of the result above and the affirmative answer to [7, p. 60, Conjecture 2.1].

Theorem 1.1. *Let $l \in \mathbb{N}$ and let X be either $\mathbb{R}^l$ or $\mathbb{S}^l$ with the usual metric d. Then $\chi_n : \mathrm{Isom}_d(X) \to \mathrm{Isom}_{d_H}(F_n(X))$ is an isomorphism for each $n \in \mathbb{N}$.*

We note that there exists a compact metric space (X, d) such that neither $\chi_n : \mathrm{Isom}_d(X) \to \mathrm{Isom}_{d_H}(F_n(X))$ is an isomorphism for $n > 1$ (see Section 3).

In the second part of this paper, we wish to find a metric space which is bi-Lipschitz equivalent to $(F_n(\mathbb{R}^l), d_H)$ for $l \in \mathbb{N}$ and $n \geq 2$. In [14], by use of the minimal element in $A \in F_n(\mathbb{R})$, it is proved that for every $n \geq 2$, $F_n(\mathbb{R})$ is bi-Lipschitz equivalent to the product of $\mathbb{R}$ with the open cone over some compact subset of $F_n(\mathbb{I})$. In Section 4, for every $l \in \mathbb{N}$, by use of the Chebyshev center of $A \in F_n(\mathbb{R}^l)$, we construct a homeomorphism h_{cheb} from $F_n(\mathbb{R}^l)$ to the product of $\mathbb{R}^l$ with the open cone $\mathrm{Cone}^o(F_n^{\mathrm{cheb},1}(\mathbb{B}^l))$ over some compact subset $F_n^{\mathrm{cheb},1}(\mathbb{B}^l)$ of $F_n(\mathbb{B}^l)$ and indicate that h_{cheb} is a bi-Lipschitz equivalence map if and only if either $l = 1$ or $n = 2$ holds. Moreover, we show that $(F_2(\mathbb{R}^2), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R}^4, d)$.

2. Preliminaries

Notation 2.1. Let us denote the set of all natural numbers and real numbers by $\mathbb{N}$ and $\mathbb{R}$, respectively. Let d be the usual metric on $\mathbb{R}^l$, i.e., $d(x, y) = \{\sum_{i=1}^{l}(x_i - y_i)^2\}^{1/2}$ for any $x = (x_1, \ldots, x_l), y = (y_1, \ldots, y_l) \in \mathbb{R}^l$. Write $\mathbb{S}^l = \{x = (x_1, \ldots, x_{l+1}) \in \mathbb{R}^{l+1} : \sum_{i=1}^{l+1} x_i^2 = 1\}$ with the length metric d. See [9] for length metrics. Denote the identity map from X into itself by id_X.

Definition 2.2. Let (X, d) be a metric space, let $x \in X$, let Y, Z be subsets of X and let $\epsilon > 0$. Set $\mathrm{diam} Y = \sup\{d(y, y') : y, y' \in Y\}$, $d(Y, Z) = \inf\{d(y, x) : y \in Y, z \in Z\}$, $B_d(Y, \epsilon) = \{x \in X : d(x, Y) \leq \epsilon\}$ and $S_d(Y, \epsilon) = \{x \in X :$

$d(x, Y) = \epsilon\}$. If $Y = \{y\}$, for simplicity of notation, we write $B_d(y, \epsilon) = B_d(Y, \epsilon)$ and $S_d(y, \epsilon) = S_d(Y, \epsilon)$.

For $n \in \mathbb{N}$, the n-th *symmetric product* of X is defined by

$$F_n(X) = \{A \subset X : 1 \leq |A| \leq n\}$$

endowed with the Hausdorff metric d_H, i.e., $d_H(A, B) = \inf\{\epsilon : A \subset B_d(B, \epsilon)$ and $B \subset B_d(A, \epsilon)\} = \max\{d(a, B), d(b, A) : a \in A, b \in B\}$ for any $A, B \in F_n(X)$ (see [12, p. 6]). Here $|A|$ is the cardinality of A. Write $F_{(m)}(X) = \{A \subset X : |A| = m\}$ for each $m \in \mathbb{N}$. Let $\mathrm{Isom}(X, Y) = \{\phi \in \mathrm{Isom}(X) : \phi(y) = y$ for each $y \in Y\}$ for $Y \subset X$. Set $r(A) = \min\{\{1\} \cup \{d(a, a') : a, a' \in A, a \neq a'\}\}$ for each $A \in F_n(X)$.

Lemma 2.3. *Let $n \in \mathbb{N}$ and let (X, d) be a metric space. Then, $\chi_n : \mathrm{Isom}(X) \to \mathrm{Isom}(F_n(X))$ is an isomorphism if and only if*

(1) $\Phi|_{F_1(X)} \in \mathrm{Isom}(F_1(X))$ *for each* $\Phi \in \mathrm{Isom}(F_n(X))$, *and*
(2) $\mathrm{Isom}(F_n(X), F_1(X)) = \{\mathrm{id}_{F_n(X)}\}$.

Proof: The part of "only if" is easy from the definition of χ_n.

Suppose that (1) and (2) hold. Let $\Phi \in \mathrm{Isom}(F_n(X))$ and let $\phi = \Phi|_{F_1(X)} \in \mathrm{Isom}(F_1(X))$. Set $\Phi' = \chi_n(\phi^{-1}) \circ \Phi \in \mathrm{Isom}(F_n(X))$. We claim that $\Phi'|_{F_1(X)} = \mathrm{id}|_{F_1(X)}$. Indeed, $\Phi|_{F_1(X)} = \chi_n(\phi)|_{F_1(X)}$ and $\chi_n(\phi^{-1}) = \chi_n(\phi)^{-1}$. By assumption, we have that $\Phi' = \mathrm{id}_{F_n(X)}$, therefore, $\Phi = \chi_n(\phi)$, which completes the proof. ☐

3. Isometries on symmetric products

Definition 3.1. Let (X, d) be a metric space, let $n \in \mathbb{N}$, let $\epsilon > 0$ and let $A \in F_n(X)$. Define

$$D_n(A, \epsilon) = \sup\{k \in \mathbb{N} : A_1, \dots, A_k \in S_{d_H}(A, \epsilon), d_H(A_i, A_j) = 2\epsilon \quad \text{for} \quad 1 \leq i < j \leq k\}. \tag{3.1}$$

Lemma 3.2. *Let $l, n \in \mathbb{N}$, let X be either $\mathbb{R}^l$ or $\mathbb{S}^l$ and let $\Phi \in \mathrm{Isom}(F_n(X))$. Then, $\Phi|_{F_1(X)} \in \mathrm{Isom}(F_1(X))$.*

Proof: Let $n \in \mathbb{N}$ with $n \geq 2$. Let $x \in X$, let $\epsilon > 0$ with $\epsilon < 1$ and let $y \in B_d(x, \epsilon)$. It is clear that

(i) if $y \in S_d(x, \epsilon)$, then there exists the unique $y' \in B_d(x, \epsilon)$ such that $d(y, y') = 2\epsilon$, and
(ii) if $y \notin S_d(x, \epsilon)$, then there exists no $y' \in B_d(x, \epsilon)$ such that $d(y, y') = 2\epsilon$.

Let $A \in F_1(X)$. We show that $D_n(A, \epsilon) = 3$. It follows from (i) and (ii) that for any $B, C \in F_n(B_d(A, \epsilon)) \setminus F_1(B_d(A, \epsilon))$ we have $d_H(B, C) < 2\epsilon$, and that for any $A_1, \dots, A_m \in S_{d_H}(A, \epsilon) \cap F_1(X)$ with $d_H(A_i, A_j) = 2\epsilon$ for $1 \leq i < j \leq m$ we see that $m \leq 2$. This shows that $D_n(A, \epsilon) \leq 3$.

Let $a, a' \in S_d(A, \epsilon)$ with $d(a, a') = 2\epsilon$. Set $B_1 = \{a\}$, $B_2 = \{a'\}$ and $B_3 = \{a, a'\}$. Then, $B_j \in S_{d_H}(A, \epsilon)$ for each $j = 1, 2, 3$ and $d_H(B_j, B_{j'}) = 2\epsilon$ whenever $j \neq j'$. Hence, $D_n(A, \epsilon) \geq 3$. Therefore, $D_n(A, \epsilon) = 3$.

Let $m \in \mathbb{N}$ with $m \geq 2$, let $A = \{a_1, \dots, a_m\} \in F_{(m)}(X)$ and let $\epsilon > 0$ with $\epsilon < r(A)/5$. We show that $D_n(A, \epsilon) > 3$. For every $j = 1, \dots, m$ and $k = 0, 1$, let $a_{j,k} \in S_d(a_j, \epsilon)$ such that $d(a_{j,0}, a_{j,1}) = 2\epsilon$. Set $A_\theta = \{a_{1,\theta_1}, \dots, a_{m,\theta_m}\}$ for each $\theta = (\theta_1, \dots, \theta_m) \in \{0,1\}^m$. We see that $A_\theta \in S_{d_H}(A, \epsilon)$ for each $\theta \in \{0,1\}^m$ and that $d_H(A_\theta, A_{\theta'}) = 2\epsilon$ whenever $\theta \neq \theta'$, therefore, $D_n(A, \epsilon) \geq 2^m \geq 2^2 > 3$.

Let $\Phi \in \mathrm{Isom}(F_n(X))$, let $A \in F_n(X)$ and let $\epsilon > 0$ be such that $\epsilon < \min\{r(A), r(\Phi(A))\}$. From the definition of $D_n(A, \epsilon)$, we obtain $D_n(A, \epsilon) = D_n(\Phi(A), \epsilon)$. By the above, we see that $A \in F_1(X)$ if and only if $\Phi(A) \in F_1(X)$. Therefore, $\Phi|_{F_1(X)} \in \mathrm{Isom}(F_1(X))$. □

Corollary 3.3. *Let $l, n \in \mathbb{N}$ and let d be a metric on $\mathbb{R}^{l+1}$ as in Notation 2.1. Suppose that $\mathbb{S}^l$ has a metric $\rho = d|_{\mathbb{S}^l}$. Let $\Phi \in \mathrm{Isom}_{\rho_H}(F_n(\mathbb{S}^l))$. Then, $\Phi|_{F_1(\mathbb{S}^l)} \in \mathrm{Isom}_{\rho_H}(F_1(\mathbb{S}^l))$.*

Proof: Let $A \in F_n(\mathbb{S}^l)$ and let $\epsilon > 0$ be such that $\epsilon < r(A)/5$. Define $r_\epsilon = \mathrm{diam} B_\rho((1, 0, \dots, 0), \epsilon)$ and

$$D'_n(A, \epsilon) = \sup\{k \in \mathbb{N} : A_1, \dots, A_k \in S_{\rho_H}(A, \epsilon), \\ \rho_H(A_i, A_j) = r_\epsilon \ \text{ for } \ 1 \leq i < j \leq k\} \in \mathbb{N} \cup \{\infty\}.$$

Analysis similar to that for $D_n(A, \epsilon)$ in the proof of Lemma 3.2 can show that $D'_n(A, \epsilon) = 3$ if and only if $A \in F_1(\mathbb{S}^l)$. Therefore, $\Phi|_{F_1(\mathbb{S}^l)} \in \mathrm{Isom}_{\rho_H}(F_1(\mathbb{S}^l))$. □

Notation 3.4. Let $l, n \in \mathbb{N}$ and let $A \in F_n(\mathbb{R}^l)$. Denote the minimal convex subset of $\mathbb{R}^l$ containing A by $\mathrm{conv}(A)$, and the set of all vertices of $\mathrm{conv}(A)$ by $\mathrm{conv}(A)^{(0)}$ (see [17] for details). We note that $\mathrm{conv}(A)^{(0)}$ is contained in A.

Lemma 3.5. *Let $l, n \in \mathbb{N}$, let $A \in F_n(\mathbb{R}^l)$ and let $\Phi \in \mathrm{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l))$. Then, $\mathrm{conv}(A)^{(0)} \subset \Phi(A) \subset \mathrm{conv}(A)$.*

Proof: Let $a \in \mathrm{conv}(A)^{(0)}$. We show that $a \in \Phi(A)$. Let H be a hyperplane in $\mathbb{R}^l$ with dimension $l-1$ such that $H \cap \mathrm{conv}(A) = \{a\}$, let C be the closed half-space bounded by H containing $\mathrm{conv}(A)$, and let L be the line containing a which is vertical to H. See [17] for hyperplanes and half-spaces. There exists $x \in C \cap L$ such that $\mathrm{conv}(A) \subset B_d(x, r)$ and $\mathrm{conv}(A) \cap S_d(x, r) = \{a\}$, where $r = d(x, a)$.

Since $d_H(\{x\}, \Phi(A)) = d_H(\Phi(\{x\}), \Phi(A)) = d_H(\{x\}, A) = r$, we have that $\Phi(A) \subset B_d(x, r)$ and $S_d(x, r) \cap \Phi(A) \neq \emptyset$. Let $x' \in C \cap L$ such that $r' = d(x', a) > r$. By a similar argument, we see that $S_d(x', r') \cap \Phi(A) \neq \emptyset$ and $S_d(x', r') \cap B_d(x, r) = \{a\}$. Thus, $a \in \Phi(A)$.

We show that $\Phi(A) \subset \mathrm{conv}(A)$. If similar arguments apply to $\Phi(A)$ and Φ^{-1}, we obtain

$$\mathrm{conv}(\Phi(A))^{(0)} \subset \Phi^{-1}(\Phi(A)) = A.$$

Therefore, $\Phi(A) \subset \mathrm{conv}(\mathrm{conv}(\Phi(A))^{(0)}) \subset \mathrm{conv}(A)$. □

Definition 3.6. Let $l, n \in \mathbb{N}$, let $\epsilon > 0$ and let $A \in F_n(\mathbb{R}^l)$. Define $S^c_{d_H}(A,\epsilon) = \{B \in S_{d_H}(A,\epsilon) : \mathrm{conv}(A) = \mathrm{conv}(B)\}$, and

$$(3.2)\quad D^c_n(A,\epsilon) = \sup\{k \in \mathbb{N} : A_1,\dots,A_k \in S^c_{d_H}(A,\epsilon), d_H(A_i,A_j) = 2\epsilon \text{ for } 1 \le i < j \le k\}.$$

Lemma 3.7. *Let $l, n \in \mathbb{N}$ and let $\Phi \in \mathrm{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l))$. Then, $\Phi|_{F_2(\mathbb{R}^l)} = \mathrm{id}_{F_2(\mathbb{R}^l)}$.*

Proof: Let $A \in F_{(2)}(\mathbb{R}^l)$. Since $A = \mathrm{conv}(A)^{(0)}$, by Lemma 3.5, $A \subset \Phi(A)$. Thus, if $\Phi(A) \in F_{(2)}(\mathbb{R}^l)$, then $A = \Phi(A)$. Therefore, it suffices to show that $\Phi(A) \in F_{(2)}(\mathbb{R}^l)$. We may assume that $n \ge 3$.

Suppose that $l = 1$. By [7], $\Phi(F_{(2)}(\mathbb{R})) = F_{(2)}(\mathbb{R})$, but we give another short proof of it. Let $A \in F_{(2)}(\mathbb{R})$ and let $\epsilon > 0$ with $\epsilon < r(A)/5$. We claim that $D^c_n(A,\epsilon) = 1$. Indeed, on the contrary, suppose that $D^c_n(A,\epsilon) \ge 2$, i.e., there exist $A_1, A_2 \in S^c_{d_H}(A,\epsilon)$ such that $d_H(A_1,A_2) = 2\epsilon$. Since $A \subset A_1 \cap A_2$, $A_1 \cup A_2 \subset B_d(A,\epsilon) \subset B_d(A_1,\epsilon) \cap B_d(A_2,\epsilon)$, thus $d_H(A_1,A_2) \le \epsilon$, a contradiction.

Let $B \in F_{(m)}(\mathbb{R})$ with $3 \le m \le n$ and let $\epsilon > 0$ with $\epsilon < r(B)/5$. We claim that $D^c_n(B,\epsilon) \ge 2$. Indeed, if we choose $b \in B \setminus \{\min B, \max B\}$, we define $B_1 = (B \setminus \{b\}) \cup \{b - \epsilon\}$ and $B_2 = (B \setminus \{b\}) \cup \{b + \epsilon\}$. Then, $B_1, B_2 \in S^c_{d_H}(B,\epsilon)$ and $d_H(B_1,B_2) = 2\epsilon$, thus, $D^c_n(B,\epsilon) \ge 2$.

Let $A \in F_n(\mathbb{R}) \setminus F_1(\mathbb{R})$ and let $\epsilon > 0$ with $\epsilon < \min\{r(A)/5, r(\Phi(A))/5\}$. By Lemma 3.5, $\Phi(S^c_{d_H}(A,\epsilon)) = S^c_{d_H}(\Phi(A),\epsilon)$. Thus, $D^c_n(A,\epsilon) = D^c_n(\Phi(A),\epsilon)$. By the above, $\Phi(A) \in F_{(2)}(\mathbb{R}^l)$.

Suppose that $l \ge 2$. Let $A \in F_{(2)}(\mathbb{R}^l)$ and let L be the line in $\mathbb{R}^l$ containing A. By Lemma 3.5, $\Phi(F_n(L)) = F_n(L)$, i.e., $\Phi|_{F_n(L)} \in \mathrm{Isom}(F_n(L))$. Applying to the case $l = 1$, $\Phi(A) = A$, which completes the proof. ☐

Lemma 3.8. *Let $l, n \in \mathbb{N}$. Then,* $\mathrm{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l)) = \{\mathrm{id}_{F_n(\mathbb{R}^l)}\}$.

Proof: Let $\Phi \in \mathrm{Isom}(F_n(\mathbb{R}^l), F_1(\mathbb{R}^l))$ and let $A \in F_{(m)}(\mathbb{R}^l)$. We show that $\Phi(A) \subset A$. On the contrary, suppose that there exists $z \in \Phi(A) \setminus A$. By Lemma 3.5, we note that $\mathrm{conv}(A)^{(0)} \subset \Phi(A) \subset \mathrm{conv}(A)$. There exist a hyperplane H in $\mathbb{R}^l$ with dimension $l-1$ containing z and a line L in $\mathbb{R}^l$ containing z such that H is vertical to L, $A \cap H = \emptyset$, and, $A \cap C_k \neq \emptyset$ for $k = 0, 1$, where C_0 and C_1 are the closed half-spaces bounded by H with $C_0 \cup C_1 = \mathbb{R}^l$. As in the proof of Lemma 3.5, there exist a sufficiently large $r > 0$ and $x_k \in L \cap \mathrm{Int}_{\mathbb{R}^l} C_k$ for $k = 0, 1$ such that $r = d(x_0, z) = d(x_1, z)$, $A \cap (S_d(x_0, r) \cup S_d(x_1, r)) = \emptyset$, and $A \subset B_d(x_0, r) \cup B_d(x_1, r)$. Set $A_1 = \{x_0, x_1\}$. Since $d(z, A_1) = r$, we see $d_H(\Phi(A), A_1) \ge r$. Since $A \cap S_d(A_1, r) = \emptyset$, $A \subset B_d(A_1, r)$ and $A_1 \subset B_d(A, r)$, we have $d_H(A, A_1) < r$. By Lemma 3.7, we have $r \le d_H(\Phi(A), A_1) = d_H(\Phi(A), \Phi(A_1)) = d_H(A, A_1) < r$, a contradiction.

If similar arguments apply to $\Phi(A)$ and Φ^{-1}, we obtain $A = \Phi^{-1}(\Phi(A)) \subset \Phi(A)$, therefore, $A = \Phi(A)$, which completes the proof. ☐

Lemma 3.9. *Let $l, n \in \mathbb{N}$. Then* $\mathrm{Isom}(F_n(\mathbb{S}^l), F_1(\mathbb{S}^l)) = \{\mathrm{id}_{F_n(\mathbb{S}^l)}\}$.

Proof: Let $\Phi \in \mathrm{Isom}(F_n(\mathbb{S}^l), F_1(\mathbb{S}^l))$, $m \in \mathbb{N}$ with $2 \leq m \leq n$ and let $A \in F_{(m)}(\mathbb{S}^l)$. We show that $A = \Phi(A)$. Let $a \in A$ and let $a' \in \mathbb{S}^l$ be the anti-point of a. Since $d_H(\{a'\}, \Phi(A)) = d_H(\Phi(\{a'\}), \Phi(A)) = d_H(\{a'\}, A) = \pi$, we have $a \in \Phi(A)$, therefore, $A \subset \Phi(A)$. If similar arguments apply to $\Phi(A)$ and Φ^{-1}, we obtain $\Phi(A) \subset \Phi^{-1}(\Phi(A)) = A$, therefore, $A = \Phi(A)$, which completes the proof. □

Proof of Theorem 1.1: By Lemmas 3.2, 3.8 and 3.9, the conditions in Lemma 2.3 hold for (X, d), which completes the proof. □

Corollary 3.10. *Let $l, n \in \mathbb{N}$ and let d be a metric on $\mathbb{R}^{l+1}$ as in Notation 2.1. Suppose $\mathbb{S}^l$ has a metric $\rho = d|_{\mathbb{S}^l}$. Then $\chi_n : \mathrm{Isom}_\rho(\mathbb{S}^l) \to \mathrm{Isom}_{\rho_H}(F_n(\mathbb{S}^l))$ is an isomorphism for each $n \in \mathbb{N}$.*

Proof: By similar arguments as in the proof of Lemma 3.9, we have $\mathrm{Isom}_{\rho_H}(F_n(\mathbb{S}^l), F_1(\mathbb{S}^l)) = \{\mathrm{id}_{F_n(\mathbb{S}^l)}\}$. By Corollary 3.3, the conditions in Lemma 2.3 hold for $(\mathbb{S}^l, \rho)$, which completes the proof. □

Question 3.11. *Let $l, n \in \mathbb{N}$ with $n \geq 2$. Is $\chi_n : \mathrm{Isom}_d(X) \to \mathrm{Isom}_{d_H}(F_n(X))$ an isomorphism when*

(1) *X is a convex subset of $\mathbb{R}^l$,*
(2) *X is an $\mathbb{R}$-tree (see [3] for $\mathbb{R}$-trees) or*
(3) *X is the hyperbolic l-space (see [9] for the hyperbolic l-space)?*

Remark 3.12. Let $n, m \in \mathbb{N}$ with $2 \leq n \leq m$ and let (X, d) be an m-points discrete metric space satisfying that $d(x, x') = 1$ whenever $x \neq x'$. Then, $F_n(X)$ is a discrete metric space such that $d_H(A, A') = 1$ for any $A, A' \in F_n(X)$ with $A \neq A'$. Thus, $|\mathrm{Isom}(X)| = |X|! < |F_n(X)|! = |\mathrm{Isom}(F_n(X))|$, therefore, $\chi_n : \mathrm{Isom}_d(X) \to \mathrm{Isom}_{d_H}(F_n(X))$ is not an isomorphism.

By [1, p. 182], there exists $\Phi \in \mathrm{Isom}_{\xi_H}(F_2(\mathbb{R}^2)) \setminus \{\mathrm{id}_{F_2(\mathbb{R}^2)}\}$ such that $\Phi|_{F_1(\mathbb{R}^2)} = \mathrm{id}_{F_1(\mathbb{R}^2)}$. Hence, by Lemma 2.3, $\chi_2 : \mathrm{Isom}_\xi(\mathbb{R}^2) \to \mathrm{Isom}_{\xi_H}(F_2(\mathbb{R}^2))$ is not an isomorphism.

Remark 3.13. Recall that $F(X)$ is the space of non-empty compact subsets of a metric space (X, d) endowed with the Hausdorff metric d_H. Similarly, we can define a natural monomorphism $\chi : \mathrm{Isom}_d(X) \to \mathrm{Isom}_{d_H}(F(X))$. There are quite general results for some underlying spaces X corresponding to Theorem 1.1 and Question 3.11 on an epimorphism $\chi : \mathrm{Isom}_d(X) \to \mathrm{Isom}_{d_H}(F(X))$ (see [1] and [11]).

4. Bi-Lipschitz equivalence

Definition 4.1. Let $K > 0$ and let $f : (X, d) \to (Y, \rho)$ be a map from a metric space (X, d) to a metric space (Y, ρ). The map f is said to *K-Lipschitz* if for any $x, x' \in X$, $\rho(f(x), f(x')) \leq K d(x, x')$. If f is a bijection and for any $x, x' \in X$,

$$K^{-1} d(x, x') \leq \rho(f(x), f(x')) \leq K d(x, x'),$$

then f is said to be *K-bi-Lipschitz equivalence* (*bi-Lipschitz equivalence* for short).

Remark 4.2. Let d be a metric on $\mathbb{R}^2$ as in Notation 2.1, let $\rho = d|_{\mathbb{S}^1}$ be a metric on $\mathbb{S}^1$, and let θ be the length metric on $\mathbb{S}^1$. We see that the identity map $\mathrm{id}_{\mathbb{S}^1} : (\mathbb{S}^1, \rho) \to (\mathbb{S}^1, \theta)$ is a π-bi-Lipschitz equivalence map. Indeed, $\rho < \theta$ and, for every $x_t = e^{2\pi i t} \in \mathbb{S}^1$, we have that $\pi^2 \rho(x_0, x_t)^2 - \theta(x_0, x_t)^2 = 2\pi^2(1 - \cos t) - t^2 \geq 0$ for $0 \leq t \leq \pi/3$, and that $\pi\rho(x_0, x_t) \geq \pi \geq t = \theta(x_0, x_t)$ for $\pi/3 \leq t \leq \pi$, therefore $\theta \leq \pi\, \rho$.

Notation 4.3. Let $l, n \in \mathbb{N}$, let $t \in [0, \infty)$, let $a = (a_1, \ldots, a_l), x = (x_1, \ldots, x_l) \in \mathbb{R}^l$ and let $A \in F_n(\mathbb{R}^l)$. Write $a \pm x = (a_1 \pm x_1, \ldots, a_l \pm x_l)$, $ta = (ta_1, \ldots, ta_l)$, $A \pm x = \{a \pm x : a \in A\}$ and $tA = \{ta : a \in A\}$.

Definition 4.4. Let $l, n \in \mathbb{N}$ with $n > 1$, let $z_0 = (0, \ldots, 0) \in \mathbb{R}^l$, let $c : (F_n(\mathbb{R}^l), d_H) \to (\mathbb{R}^l, d)$ be a map, and let $F_n^c(\mathbb{R}^l) = \{A \in F_n(\mathbb{R}^l) : c(A) = z_0\}$. Let us define two maps $\overline{c}_0 : \mathbb{R}^l \times F_n^c(\mathbb{R}^l) \to F_n(\mathbb{R}^l)$ and $\overline{c}_1 : F_n(\mathbb{R}^l) \to \mathbb{R}^l \times F_n(\mathbb{R}^l)$ by $\overline{c}_0(x, A) = A + x$ and $\overline{c}_1(A') = (c(A'), A' - c(A'))$ for each $A \in F_n^c(\mathbb{R}^l)$, each $A' \in F_n(\mathbb{R}^l)$ and each $x \in \mathbb{R}^l$.

The proof of the following lemma is based on the proof of [14, Lemma 2.4].

Lemma 4.5. *Let $l, n \in \mathbb{N}$ with $n > 1$, let $c : (F_n(\mathbb{R}^l), d_H) \to (\mathbb{R}^l, d)$ be a map and let $\overline{c}_0 : (\mathbb{R}^l \times F_n^c(\mathbb{R}^l), \rho) \to (F_n(\mathbb{R}^l), d_H)$ and $\overline{c}_1 : (F_n(\mathbb{R}^l), d_H) \to (\mathbb{R}^l \times F_n(\mathbb{R}^l), \rho)$ be two maps as in Definition 4.4, where $\rho = \sqrt{d^2 + d_H^2}$ is the metric compatible with the topology on $\mathbb{R}^l \times F_n(\mathbb{R}^l)$. Then, the following statements hold.*

(1) *The map $\overline{c}_0$ is a $\sqrt{2}$-Lipschitz map.*
(2) *If the map c is a K-Lipschitz map for some $K > 0$, then the map $\overline{c}_1$ is a $\sqrt{2K^2 + 2K + 1}$-Lipschitz map.*
(3) *If $c(A + x) = c(A) + x$ for each $A \in F_n(\mathbb{R}^l)$ and each $x \in \mathbb{R}^l$, then $\overline{c}_1(F_n(\mathbb{R}^l)) = \mathbb{R}^l \times F_n^c(\mathbb{R}^l)$ and ${\overline{c}_1}^{-1} = \overline{c}_0$.*
(4) *If c satisfies (2) and (3), then the map $\overline{c}_1 : (F_n(\mathbb{R}^l), d_H) \to (\mathbb{R}^l \times F_n^c(\mathbb{R}^l), \rho)$ is a K'-bi-Lipschitz equivalence map, where $K' = \max\{\sqrt{2}, \sqrt{2K^2 + 2K + 1}\}$.*

Proof: (1) Let $(x, A), (x', A') \in \mathbb{R}^l \times F_n^c(\mathbb{R}^l)$, let $\epsilon > 0$ such that $A \subset B_d(A', \epsilon)$ and $A' \subset B_d(A, \epsilon)$ and let $a \in A$. Then, there exists $a' \in A'$ such that $d(a, a') < \epsilon$. Thus,

$$d(a + x, a' + x') = d(a, a' + x' - x) \leq d(a, a') + d(a', a' + x' - x) \leq \epsilon + d(x, x').$$

Hence, $a + x \in B_d(A' + x', \epsilon + d(x, x'))$, therefore, $A + x \subset B_d(A' + x', \epsilon + d(x, x'))$. Similarly, $A' + x' \subset B_d(A + x, \epsilon + d(x, x'))$. We conclude that $d_H(A + x, A' + x')^2 \leq \{d_H(A, A') + d(x, x')\}^2 \leq 2\{d(x, x')^2 + d_H(A, A')^2\} = 2\rho((x, A), (x', A'))^2$, hence, the map $\overline{c}_0$ is a $\sqrt{2}$-Lipschitz map.

(2) Let $A, A' \in F_n(\mathbb{R}^l)$ and let $\epsilon > 0$ such that $A \subset B_d(A', \epsilon)$ and $A' \subset B_d(A, \epsilon)$. Let $a \in A$. Then, there exists $a' \in A'$ such that $d(a, a') < \epsilon$. We have

$$\begin{aligned} d(a - c(A), a' - c(A')) &= d(a, a' - (c(A') - c(A))) \\ &\leq d(a, a') + d(a', a' - (c(A') - c(A))) \end{aligned}$$

$$= d(a, a') + d(c(A'), c(A))$$
$$< \epsilon + d(c(A'), c(A)).$$

Thus, $a - c(A) \in B_d(A', \epsilon + d(c(A'), c(A)))$, therefore, $A - c(A) \subset B_d(A', \epsilon + d(c(A'), c(A)))$. Similarly, we obtain $A' - c(A') \subset B_d(A, \epsilon + d(c(A'), c(A)))$. We conclude

$$\begin{aligned} d_H(A - c(A), A' - c(A')) &\le d_H(A, A') + d(c(A), c(A')) \\ &\le d_H(A, A') + K d_H(A, A') = (K+1) d_H(A, A'), \end{aligned}$$

therefore, the map $\overline{c}_1$ is a $(\sqrt{2K^2 + 2K + 1})$-Lipschitz map.

(3) By assumption, it is clear that $\overline{c}_1(F_n(\mathbb{R}^l)) = \mathbb{R}^l \times F_n^c(\mathbb{R}^l)$. Let $A \in F_n^c(\mathbb{R}^l)$ and let $x \in \mathbb{R}^l$. Then $\overline{c}_1 \circ \overline{c}_0(x, A) = \overline{c}_1(A + x) = (c(A+x), A + x - c(A+x)) = (c(A) + x, A + x - (c(A) + x)) = (x, A)$. Therefore, $\overline{c}_1 \circ \overline{c}_0 = \mathrm{id}_{\mathbb{R}^l \times F_n^c(\mathbb{R}^l)}$. It is clear that $\overline{c}_0 \circ \overline{c}_1 = \mathrm{id}_{F_n(\mathbb{R}^l)}$.

(4) It is clear from (1),(2) and (3). □

Definition 4.6 ([14])**.** Let (X, d) be a metric space with $\mathrm{diam} X \le 2$. The quotient space $\mathrm{Cone}^o(X) = X \times [0, \infty)/X \times \{0\}$ is called an *open cone over* X. Let $p : X \times [0, \infty) \to \mathrm{Cone}^o(X)$ be the natural projection. Denote $p(x, t)$ by $[x, t] \in \mathrm{Cone}^o(X)$. Let us define a metric d_c on $\mathrm{Cone}^o(X)$ compatible with the topology on $\mathrm{Cone}^o(X)$ by

$$d_c([x, t], [x', t']) = \min\{t, t'\} d(x, x') + |t - t'|$$

for any $[x, t], [x', t'] \in \mathrm{Cone}^o(X)$.

Remark 4.7. Let (X, d) and (Y, ρ) be metric spaces and let $f : (X, d) \to (Y, \rho)$ be a K-Lipschitz map for some $K > 0$. Then, $\chi_n(f) : (F_n(X), d_H) \to (F_n(Y), \rho_H)$ defined by $\chi_n(f)(A) = f(A)$ for each $A \in F_n(X)$ is a K-Lipschitz map. If $\max\{\mathrm{diam} X, \mathrm{diam} Y\} \le 2$ and $K \ge 1$, then $\overline{f} : (\mathrm{Cone}^o(X), d_c) \to (\mathrm{Cone}^o(Y), \rho_c)$ defined by $\overline{f}([x, t]) = [f(x), t]$ for each $[x, t] \in \mathrm{Cone}^o(X)$ is a K-Lipschitz map.

The following lemma is obtained from the proof of [14, Lemma 2.2].

Lemma 4.8. *Let (X, d) be a metric space with $\mathrm{diam} X \le 2$, let $K > 0$, and let ρ be a metric on $\mathrm{Cone}^o(X)$ compatible with the topology on $\mathrm{Cone}^o(X)$ such that*

(1) $\rho([x, t], [x', t]) = t d(x, x')$,
(2) $\rho([x, t], [x', t']) \ge |t - t'|$, *and,*
(3) $\rho([x, t], [x, t']) \le K|t - t'|$

for any $t, t' \in [0, \infty)$ and any $x, x' \in X$. Then, $\mathrm{id}_{\mathrm{Cone}^o(X)} : (\mathrm{Cone}^o(X), \rho) \to (\mathrm{Cone}^o(X), d_c)$ is a K-Lipschitz map and $\mathrm{id}_{\mathrm{Cone}^o(X)} : (\mathrm{Cone}^o(X), d_c) \to (\mathrm{Cone}^o(X), \rho)$ is a $(K+2)$-Lipschitz map and, thus, $\mathrm{id}_{\mathrm{Cone}^o(X)}$ is a $(K+2)$-bi-Lipschitz equivalence map.

Definition 4.9. Let $l, n \in \mathbb{N}$ with $n > 1$, let $\mathbb{B}^l = \{x \in \mathbb{R}^l : d(x, z_0) \le 1\}$, and let $c : F_n(\mathbb{R}^l) \to \mathbb{R}^l$ be a map. Set $F_n^{c,1}(\mathbb{B}^l) = \{A \in F_n(\mathbb{B}^l) : c(A) = z_0$ and

$d_H(\{z_0\}, A) = 1\}$. Let us define $\widetilde{c} : \mathrm{Cone}^o(F_n^{c,1}(\mathbb{B}^l)) \to F_n(\mathbb{R}^l)$ by $\widetilde{c}([A,t]) = tA$ for each $A \in F_n^{c,1}(\mathbb{B}^l)$ and each $t \in [0,\infty)$.

The proof of the following lemma is based on the proof of [14, Lemma 2.4].

Lemma 4.10. *If $c(tA) = z_0$ for each $A \in F_n^c(\mathbb{R}^l)$ and each $t \in [0,\infty)$, then $\widetilde{c}(\mathrm{Cone}^o(F_n^{c,1}(\mathbb{B}^l))) = F_n^c(\mathbb{R}^l)$ and $\widetilde{c} : (\mathrm{Cone}^o(F_n^{c,1}(\mathbb{B}^l)), (d_H)_c) \to (F_n^c(\mathbb{R}^l), d_H)$ is a 3-bi-Lipschitz equivalence map, where $\widetilde{c}$ is the map as in Definition 4.9. In particular, $\widetilde{c}$ is a 3-Lipschitz map and $\widetilde{c}^{-1}$ is a 1-Lipschitz map.*

Proof: It is clear that $\widetilde{c}(\mathrm{Cone}^o(F_n^{c,1}(\mathbb{B}^l))) = F_n^c(\mathbb{R}^l)$. It suffices to show three conditions with $K = 1$ from Lemma 4.8 for $d = \rho = d_H$.

Since $d(tx, tx') = td(x, x')$ for any $x, x' \in \mathbb{B}^l$ and each $t \in [0,\infty)$, $d_H(tA, tA') = td_H(A, A')$ for each $A \in F_n^{c,1}(\mathbb{B}^l)$ and each $t \in [0,\infty)$.

Let $t, t' \in [0,\infty)$ with $t \le t'$ and let $A, A' \in F_n^{c,1}(\mathbb{B}^l)$. Since $S_d(z_0, t) \cap (tA) \neq \emptyset$ and $S_d(z_0, t') \cap (t'A') \neq \emptyset$, we have $d_H(tA, t'A') \ge d_H(S_d(z_0,t), S_d(z_0,t')) = t' - t$.

Let $t, t' \in [0,\infty)$ and let $A \in F_n^{c,1}(\mathbb{B}^l)$. Let $x \in A$. Since

$$d(tx, t'x) = |t - t'| d(z_0, x) \le |t - t'|,$$

$t'x \in B_d(tA, |t - t'|)$. Hence, $t'A \subset B_d(tA, |t - t'|)$. Similarly, we see that $tA \subset B_d(t'A, |t - t'|)$, therefore, $d_H(tA, t'A) \le |t - t'|$. □

Proposition 4.11. *Let $l, n \in \mathbb{N}$ with $n > 1$ and let $c : (F_n(\mathbb{R}^l), d_H) \to (\mathbb{R}^l, d)$ be a map such that $c(A + x) = c(A) + x$ for each $A \in F_n(\mathbb{R}^l)$ and each $x \in \mathbb{R}^l$, and that $c(tA') = z_0$ for each $A' \in F_n^c(\mathbb{R}^l)$ and each $t \in [0,\infty)$. Let $\sigma = \sqrt{d^2 + (d_H)_c^2}$ be the metric compatible with the topology on $\mathbb{R}^l \times \mathrm{Cone}^o(F_n^{c,1}(\mathbb{B}^l))$ and let $h_c = (\mathrm{id}_{\mathbb{R}^l} \times \widetilde{c}^{-1}) \circ \overline{c}_1 : (F_n(\mathbb{R}^l), d_H) \to (\mathbb{R}^l \times \mathrm{Cone}^o(F_n^{c,1}(\mathbb{B}^l)), \sigma)$ be a map, where $\overline{c}_1$ and $\widetilde{c}$ are the maps as in Definitions 4.4 and 4.9, respectively.*

1. *If c is a K-Lipschitz map for some $K > 0$, then h_c is a K'-bi-Lipschitz equivalence map, where $K' = \max\{3\sqrt{2}, \sqrt{2K^2 + 2K + 1}\}$. In particular, h_c is a $\sqrt{2K^2 + 2K + 1}$-Lipschitz map and h_c^{-1} is a $3\sqrt{2}$-Lipschitz map.*
2. *Conversely, if h_c is K''-Lipschitz map for some $K'' > 0$, then c is a K''-Lipschitz map.*

Proof: (1) By Lemma 4.10,

$$\mathrm{id}_{\mathbb{R}^l} \times \widetilde{c}^{-1} : (\mathbb{R}^l \times F_n^c(\mathbb{R}^l), \rho) \to (\mathbb{R}^l \times \mathrm{Cone}^o(F_n^{c,1}(\mathbb{B}^l)), \sigma)$$

is a 3-bi-Lipschitz equivalence map. Thus, by Lemma 4.5, h_c is a K'-bi-Lipschitz equivalence map.

(2) Let $p : (\mathbb{R}^l \times \mathrm{Cone}^o(F_n^{c,1}(\mathbb{B}^l)), \sigma) \to (\mathbb{R}^l, d)$ be the projection map which is an 1-Lipschitz map. Since $c = p \circ h_c$, c is a K''-Lipschitz map. □

If c satisfies the assumptions in Proposition 4.11, then c is a Lipschitz map if and only if h_c is a bi-Lipschitz equivalence map.

Definition 4.12. Let $l, n \in \mathbb{N}$ with $n > 1$ and let $A \in F_n(\mathbb{R}^l)$. A point $\mathrm{cheb}(A)$ of $\mathbb{R}^l$ is said to be the *Chebyshev center* of A if

$$(*)\qquad \begin{aligned} &\max_{a\in A} d(\mathrm{cheb}(A), a) = \min_{x\in\mathbb{R}^l} \max_{a\in A} d(x, a) \\ &(d_H(\{\mathrm{cheb}(A)\}, A) = \min_{x\in\mathbb{R}^l} d_H(\{x\}, A) = d_H(F_1(\mathbb{R}^l), A)). \end{aligned}$$

Set $R(A) = \max_{a\in A} d(\mathrm{cheb}(A), a) = d_H(\{\mathrm{cheb}(A)\}, A)$, called a *Chebyshev radius* of A. It is known that such a point satisfying $(*)$ is unique and the map $\mathrm{cheb} : F_n(\mathbb{R}^l) \to \mathbb{R}^l : A \mapsto \mathrm{cheb}(A)$ is well-defined and continuous (see [2] or [13]). It is clear that $R : F_n(\mathbb{R}^l) \to \mathbb{R} : A \mapsto R(A)$ is continuous by $(*)$ and that cheb satisfies the assumptions for $c = \mathrm{cheb}$ in Proposition 4.11.

Let $F_n^{\mathrm{cheb},1}(\mathbb{B}^l) = \{A \in F_n(\mathbb{B}^l) : \mathrm{cheb}(A) = z_0 \text{ and } R(A) = 1\}$, and let $\mathrm{Cone}^o(F_n^{\mathrm{cheb},1}(\mathbb{B}^l))$ be the open cone over $F_n^{\mathrm{cheb},1}(\mathbb{B}^l)$ with the metric $(d_H)_c$. Fix $A_0 \in F_n^{\mathrm{cheb},1}(\mathbb{B}^l)$. Let us define a map $h_{\mathrm{cheb}} : F_n(\mathbb{R}^l) \to \mathbb{R}^l \times \mathrm{Cone}^o(F_n^{\mathrm{cheb},1}(\mathbb{B}^l))$ by

$$h_{\mathrm{cheb}}(A) = \begin{cases} (\mathrm{cheb}(A), [(A - \mathrm{cheb}(A))/R(A), R(A)]) & \text{if } A \in F_n(\mathbb{R}^l) \setminus F_1(\mathbb{R}^l) \\ (\mathrm{cheb}(A), [A_0, 0]) & \text{if } A \in F_1(\mathbb{R}^l). \end{cases}$$

It is clear that $h_{\mathrm{cheb}} = (\mathrm{id}_{\mathbb{R}^l} \times \widetilde{\mathrm{cheb}}^{-1}) \circ \overline{\mathrm{cheb}_1}$, where $\overline{\mathrm{cheb}_1}$ and $\widetilde{\mathrm{cheb}}$ are the maps as in Definitions 4.4 and 4.9 for $c = \mathrm{cheb}$, respectively.

By definition, it is easy to check the following result.

Proposition 4.13. *Let $l, n \in \mathbb{N}$ with $n > 1$. The map $h_{\mathrm{cheb}} : F_n(\mathbb{R}^l) \to \mathbb{R}^l \times \mathrm{Cone}^o(F_n^{\mathrm{cheb},1}(\mathbb{B}^l))$ defined in Definition* 4.12 *is a homeomorphism.*

We note that $F_2^{\mathrm{cheb},1}(\mathbb{B})$ is one point, $F_3^{\mathrm{cheb},1}(\mathbb{B}) = \{\{-1, t, 1\} : -1 \le t \le 1\}$ is a circle, and, $F_2^{\mathrm{cheb},1}(\mathbb{B}^l) = \{\{-x, x\} \subset \mathbb{B}^l : d(x, z_0) = 1\}$ is the real projective $(l-1)$-space $\mathbb{RP}^{l-1}$ for each $l \ge 2$. Hence, it is obtained that $F_2(\mathbb{R}) \approx \mathbb{R} \times [0, \infty)$, $F_3(\mathbb{R}) \approx \mathbb{R} \times \mathbb{R}^2 \approx \mathbb{R}^3$, $F_2(\mathbb{R}^l) \approx \mathbb{R}^l \times \mathrm{Cone}^o(\mathbb{RP}^{l-1})$ for each $l \ge 2$, in particular, $F_2(\mathbb{R}^2) \approx \mathbb{R}^2 \times \mathbb{R}^2 \approx \mathbb{R}^4$.

We obtain the following result from Proposition 4.11 and [13, Lemmas 1,2 and 3].

Corollary 4.14. *Let $l, n \in \mathbb{N}$ with $n > 1$ and let $h_{\mathrm{cheb}} : (F_n(\mathbb{R}^l), d_H) \to (\mathbb{R}^l \times \mathrm{Cone}^o(F_n^{\mathrm{cheb},1}(\mathbb{B}^l)), \sigma)$ be the map defined in Definition* 4.12. *Then, the following conditions are equivalent:*

(1) *h_{cheb} is a bi-Lipschitz equivalence map;*
(2) *h_{cheb} is a $3\sqrt{2}$-bi-Lipschitz equivalence map;*
(3) *either $l = 1$ or $n = 2$ holds.*

In particular, if either $l = 1$ or $n = 2$ holds, then h_{cheb} is a $\sqrt{5}$-Lipschitz map and h_{cheb}^{-1} is a $3\sqrt{2}$-Lipschitz map.

Remark 4.15. Let $n \in \mathbb{N}$ with $n > 1$. Let us define $\min : (F_n(\mathbb{R}), d_H) \to (\mathbb{R}, d)$ by $\min(A) = \min\{a : a \in A\}$ for each $A \in F_n(\mathbb{R})$. It is clear that min is a 1-Lipschitz map satisfying the assumptions for $c = \min$ in Proposition 4.11. By Proposition 4.11(1), $h_{\min} : (F_n(\mathbb{R}), d_H) \to (\mathbb{R} \times \mathrm{Cone}^o(F_n^{\min,1}(\mathbb{B})), \sigma)$ is a $3\sqrt{2}$-bi-Lipschitz equivalence map. We note that $F_n^{\min,1}(\mathbb{B}) = \mathbb{I}_*^{(n)}$ which is bi-Lipschitz equivalent to $F_n^{\mathrm{cheb},1}(\mathbb{B})$. Here $\mathbb{I}_*^{(n)} = \{A \in F_n(\mathbb{I}) : \{0,1\} \subset A\}$ is induced in [14].

Question 4.16. *Let $l > 1$ and let $n > 2$. Are spaces $(F_n(\mathbb{R}^l), d_H)$ and $(\mathbb{R}^l \times \mathrm{Cone}^o(F_n^{\mathrm{cheb},1}(\mathbb{B}^l)), \sigma)$ bi-Lipschitz non-equivalent?*

Since $\mathrm{Cone}^o(F_2^{\mathrm{cheb},1}(\mathbb{B}))$ is one point, by Corollary 4.14, $F_2(\mathbb{R})$ is $3\sqrt{2}$-bi-Lipschitz equivalent to $\mathbb{R} \times [0, \infty)$. The following result was first proved in [6].

Corollary 4.17. *$(F_3(\mathbb{R}), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R}^3, d)$.*

Proof: We note that $F_3^{\mathrm{cheb},1}(\mathbb{B}) = \{A_t = \{-1, t, 1\} : -1 \le t \le 1\}$ has the metric d_H and $\mathbb{S}^1 = \{e^{(t+1)\pi i} \in \mathbb{S}^1 : -1 \le t \le 1\}$ has the length metric θ, where $M(t,t') = \max\{d(t, A_1), d(t', A_1)\}$, $d_H(A_t, A_{t'}) = \min\{|t - t'|, M(t,t')\}$ and $\theta(t,t') = \pi \min\{|t-t'|, 2 - |t-t'|\}$ for each $-1 \le t \le 1$. Let us define $\alpha : F_3^{\mathrm{cheb},1}(\mathbb{B}) \to \mathbb{S}^1$ by $\alpha(A_t) = e^{(t+1)\pi i}$ for each $-1 \le t \le 1$. We note that

$$(*) \qquad M(t,t') \le d(t, A_1) + d(t', A_1) = 2 - |t - t'| \le 2M(t,t')$$

for any $t, t' \in [-1, 1]$. Hence, $d_H(A_t, A_{t'}) \le \theta(t,t')$ for any $t, t' \in [-1,1]$ and $\alpha^{-1} : (\mathbb{S}^1, \theta) \to (F_3^{\mathrm{cheb},1}(\mathbb{B}), d_H)$ is a 1-Lipschitz map. We show that $\alpha : (F_3^{\mathrm{cheb},1}(\mathbb{B}), d_H) \to (\mathbb{S}^1, \theta)$ is a (2π)-Lipschitz map. If $d_H(A_t, A_{t'}) = |t - t'|$, then $\theta(t,t') = \pi|t-t'|$ by $(*)$. If $d_H(A_t, A_{t'}) = M(t,t')$, by $(*)$, then

$$\frac{1}{\pi}\theta(t,t') \le 2 - |t-t'| \le 2M(t,t') \le 2d_H(A_t, A_{t'}),$$

thus, $\alpha : (F_3^{\mathrm{cheb},1}(\mathbb{B}), d_H) \to (\mathbb{S}^1, \theta)$ is a (2π)-bi-Lipschitz equivalence map. By Remark 4.2, $\mathrm{id}_{\mathbb{S}^1} \circ \alpha : (F_3^{\mathrm{cheb},1}(\mathbb{B}), d_H) \to (\mathbb{S}^1, \theta) \to (\mathbb{S}^1, \rho)$ is a (2π)-Lipschitz map and its inverse is a π-Lipschitz map. Therefore, by Remark 4.7, the natural extension map $\overline{\alpha} : (\mathrm{Cone}^o(F_3^{\mathrm{cheb},1}(\mathbb{B})), (d_H)_c) \to (\mathrm{Cone}^o(\mathbb{S}^1), \rho_c)$ of $\mathrm{id}_{\mathbb{S}^1} \circ \alpha$ is a (2π)-Lipschitz map and its inverse is a π-Lipschitz map.

Let us define $\beta : (\mathbb{R}^2, d) \to (\mathrm{Cone}^o(\mathbb{S}^1), \rho_c)$ by $\beta(x) = [x/d(x, z_0), d(x, z_0)]$ for each $x \in \mathbb{R}^2 \setminus \{z_0\}$ and $\beta(z_0) = [e^{\pi i}, 0]$. We show that β is a 1-Lipschitz map and its inverse is a 3-Lipschitz map. It suffices to show three conditions with $K = 1$ from Lemma 4.8 for d. It is clear that $d(tx, tx') = td(x, x') = t\rho(x, x')$ for each $t \in [0, \infty)$ and any $x, x' \in \mathbb{S}^1$. Let $t, t' \in [0, \infty)$ with $t \le t'$ and let $x, x' \in \mathbb{S}^1$. Since $tx \in S_d(z_0, t)$ and $t'x' \in S_d(z_0, t')$, we have $d_H(tx, t'x') \ge d_H(S_d(z_0, t), S_d(z_0, t')) = t' - t$. Let $t, t' \in [0, \infty)$ and let $x \in \mathbb{S}^1$. Then $d(tx, t'x) = |t - t'|d(z_0, x) = |t - t'|$.

By Corollary 4.14, $(\mathrm{id}_{\mathbb{R}} \times \beta^{-1}) \circ (\mathrm{id}_{\mathbb{R}} \times \overline{\alpha}) \circ h_{\mathrm{cheb}} : (F_3(\mathbb{R}), d_H) \to (\mathbb{R} \times \mathrm{Cone}^o(F_3^{\mathrm{cheb},1}(\mathbb{B}^1)), \sigma) \to (\mathbb{R} \times \mathrm{Cone}^o(\mathbb{S}^1), \sqrt{d^2 + \rho_c^2}) \to (\mathbb{R}^3, d)$ is a $6\sqrt{5}\,\pi$-bi-Lipschitz equivalence map. $\square$

Corollary 4.18. *$(F_2(\mathbb{R}^2), d_H)$ is bi-Lipschitz equivalent to $(\mathbb{R}^4, d)$.*

Proof: We note that $\mathbb{S}^1 = \{e^{2\pi it} \in \mathbb{B}^2 : 0 \le t \le 1\}$ has the length metric θ, $F_2^{\mathrm{cheb},1}(\mathbb{B}^2) = \{A_t = \{-e^{\pi it}, e^{\pi it}\} : 0 \le t \le 1\}$. Let θ_H be the metric on $F_2^{\mathrm{cheb},1}(\mathbb{B}^2)$ induced by θ. It is clear that the map $\alpha : (F_2^{\mathrm{cheb},1}(\mathbb{B}^2), \theta_H) \to (\mathbb{S}^1, \theta)$ defined by $\alpha(A_t) = e^{2\pi it}$ for each $t \in [0,1]$ is a 2-Lipschitz map and its inverse is a 1/2-Lipschitz map. By Remarks 4.2 and 4.7, the identity maps $\mathrm{id}_{\mathbb{S}^1} : (\mathbb{S}^1, \theta) \to (\mathbb{S}^1, \rho)$ and $\mathrm{id}_{F_2^{\mathrm{cheb},1}(\mathbb{B}^2)} : (F_2^{\mathrm{cheb},1}(\mathbb{B}^2), \theta_H) \to (F_2^{\mathrm{cheb},1}(\mathbb{B}^2), d_H)$ are 1-Lipschitz and its inverses are π-Lipschitz. Therefore, by Remark 4.7, the natural extension map $\overline{\alpha} : (\mathrm{Cone}^o(F_2^{\mathrm{cheb},1}(\mathbb{B}^2)), (d_H)_c) \to (\mathrm{Cone}^o(\mathbb{S}^1), (\rho_H)_c)$ of $\mathrm{id}_{\mathbb{S}^1} \circ \alpha \circ (\mathrm{id}_{F_2^{\mathrm{cheb},1}(\mathbb{B}^2)})^{-1}$ is a (2π)-Lipschitz map and its inverse is a $(\pi/2)$-Lipschitz map.

Let $\beta : (\mathbb{R}^2, d) \to (\mathrm{Cone}^o(\mathbb{S}^1), \rho_c)$ be a 1-Lipschitz map such that its inverse is a 3-Lipschitz map as in the proof of Corollary 4.17. By Corollary 4.14, $(\mathrm{id}_{\mathbb{R}^2} \times \beta^{-1}) \circ (\mathrm{id}_{\mathbb{R}^2} \times \overline{\alpha}) \circ h_{\mathrm{cheb}} : (F_2(\mathbb{R}^2), d_H) \to (\mathbb{R}^2 \times \mathrm{Cone}^o(F_2^{\mathrm{cheb},1}(\mathbb{B}^2)), \sigma) \to (\mathbb{R}^2 \times \mathrm{Cone}^o(\mathbb{S}^1), \sqrt{d^2 + \rho_c^2}) \to (\mathbb{R}^4, d)$ is a $6\sqrt{5}\,\pi$-bi-Lipschitz equivalence map. □

Remark 4.19. Let (X, d) be a metric space with $\mathrm{diam} X \le 2$. Set $\mathrm{Cone}(X) = X \times [0,1]/X \times \{0\}$ which is called a *cone over* X. Let us consider $F_n(\mathbb{B}^l)$ and the restriction map $h'_{\mathrm{cheb}} = h_{\mathrm{cheb}}|_{F_n(\mathbb{B}^l)} : (F_n(\mathbb{B}^l), d_H) \to (\mathbb{B}^l \times \mathrm{Cone}(F_n^{\mathrm{cheb},1}(\mathbb{B}^l)), \sigma)$ of h_{cheb} defined in Definition 4.12. It is clear that h'_{cheb} is a homeomorphism. If similar arguments above apply to the case $(\mathbb{B}^l, d)$, we obtain that the following conditions are equivalent:

(1) h'_{cheb} is a bi-Lipschitz equivalence map;
(2) h'_{cheb} is a $3\sqrt{2}$-bi-Lipschitz equivalence map;
(3) either $l = 1$ or $n = 2$ holds.

Moreover, $(F_2(\mathbb{B}), d_H)$, $(F_3(\mathbb{B}), d_H)$ and $(F_2(\mathbb{B}^2), d_H)$ are bi-Lipschitz equivalent to $(\mathbb{B}^2, d)$, $(\mathbb{B}^3, d)$ and $(\mathbb{B}^4, d)$, respectively.

Question 4.20. *Since $F_3(\mathbb{S}^1) \approx \mathbb{S}^3$, it is natural to ask a question whether $F_3(\mathbb{S}^1)$ is bi-Lipschitz equivalent to $\mathbb{S}^3$.*

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Department of Mathematics, National Defense Academy of Japan, Yokosuka 239-8686, Japan

E-mail: naochin@nda.ac.jp

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