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Results on generalized models and singular products of distributions in the Colombeau algebra $\mathcal{G}(\mathbb{R})$

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Abstract. Models of singularities given by discontinuous functions or distributions by means of generalized functions of Colombeau have proved useful in many problems posed by physical phenomena. In this paper, we introduce in a systematic way generalized functions that model singularities given by distributions with singular point support. Furthermore, we evaluate various products of such generalized models when the results admit associated distributions. The obtained results follow the idea of a well-known result of Jan Mikusiński on balancing of singular distributional products.

Keywords: Colombeau algebra; singular products of distributions

Classification: 46F30, 46F10

1. Introduction

The Colombeau algebra of generalized functions $\mathcal{G}$ [1] has become a useful tool for treating differential equations with singular coefficients and data as well as singular products of Schwartz distributions. The flexibility of Colombeau theory allows us to model such singularities by means of appropriately chosen generalized functions, treat them in this framework and obtain results on distributional level, using the association process in $\mathcal{G}$. A detailed presentation of results on that topic and list of references can be found in [9] and [4]; see also the recent paper [8] and the references included.

We recall next the known result published by Jan Mikusiński in [7]:

$$x^{-1} \cdot x^{-1} - \pi^2 \delta(x) \cdot \delta(x) = x^{-2}, \quad x \in \mathbb{R}. \tag{1}$$

Though, neither of the products on the left-hand side here exists, their difference still has a correct meaning in the distribution space $\mathcal{D}'(\mathbb{R})$. Formulas including such balanced singular products of distributions can be found in mathematics and physics literature. For balanced products of this kind, we used the name 'products of Mikusiński type' in a previous paper [2], where we derived a generalization of equation (1) in the Colombeau algebra of equation (1) such that the distributions x^{-p} and $\delta^{(q)}$ for arbitrary natural p and q were involved. Furthermore, we have

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introduced in a *unified way* generalized functions of Colombeau that model singularities of certain type and have additional properties [3]. The singularities we considered in that paper were given by distributions with singular support (the complement to the maximal open set where the distribution is a C^∞-function) in a point x on the real line $\mathbb{R}$. For $x = 0$, such are Dirac δ-function and its derivatives, Heaviside step function, the non-differentiable functions $x_\pm^p$, and the distributions $x_\pm^a$, $a \in \mathbb{R}\backslash\mathbb{Z}$.

In the present paper, we study generalized models in $\mathcal{G}(\mathbb{R})$ of the distributions $x_\pm^{-p}$, $p \in \mathbb{N}$ and evaluate various products of such models when the result admits an associated distribution. We note that when computed for the canonical embedding of the distributions in $\mathcal{G}$, none of the singular products computed in the paper admits an associated distribution.

2. Notation and definitions

2.1 We recall first the basic definitions of Colombeau algebra $\mathcal{G}(\mathbb{R})$ [1].

Notation 1. Let $\mathbb{N}$ denote the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and for $i, j \in \mathbb{N}_0$. Then we put for arbitrary $q \in \mathbb{N}_0$:

$$A_q(\mathbb{R}) = \{\varphi(x) \in \mathcal{D}(\mathbb{R}) : \int_{\mathbb{R}} x^j\, \varphi(x)\, dx = \delta_{0j},\ j = 0, 1, \ldots, q\},$$

where $\mathcal{D}(\mathbb{R})$ is the space of infinitely differentiable functions with compact support. For $\varphi \in A_0(\mathbb{R})$ and $\varepsilon > 0$, we will use the following notation throughout the paper: $\varphi_\varepsilon = \varepsilon^{-1}\varphi(\varepsilon^{-1}x)$ and $s \equiv s(\varphi) := \sup\,\{|x| : \varphi(x) \neq 0\}$. Then clearly $s(\varphi_\varepsilon) = \varepsilon s(\varphi)$, and denoting $\sigma \equiv \sigma(\varphi, \varepsilon) := s(\varphi_\varepsilon) > 0$, we have $\sigma := \varepsilon s = O(\varepsilon)$, as $\varepsilon \to 0$, for each $\varphi \in A_0(\mathbb{R})$. Finally, the shorthand notation $\partial_x = d/dx$ will be used in the one-dimensional case too.

Definition 1. Let $\mathcal{E}\,[\mathbb{R}]$ be the algebra of functions $F(\varphi, x) : A_0(\mathbb{R}) \times \mathbb{R} \to \mathbb{C}$ that are infinitely differentiable for fixed 'parameter' φ. Then the generalized functions of Colombeau are elements of the quotient algebra $\mathcal{G} \equiv \mathcal{G}(\mathbb{R}) = \mathcal{E}_{\mathrm{M}}[\mathbb{R}]\,/\,\mathcal{I}\,[\mathbb{R}]$. Here $\mathcal{E}_{\mathrm{M}}[\mathbb{R}]$ is the subalgebra of 'moderate' functions such that for each compact subset K of $\mathbb{R}$ and $p \in \mathbb{N}_0$ there is a $q \in \mathbb{N}$ such that, for each $\varphi \in A_q(\mathbb{R})$, $\sup_{x\in K} |\partial^p F(\varphi_\varepsilon, x)\,| = O(\varepsilon^{-q})$, as $\varepsilon \to 0_+$, where ∂^p denotes the derivative of order p. The ideal $\mathcal{I}\,[\mathbb{R}]$ of $\mathcal{E}_{\mathrm{M}}[\mathbb{R}]$ consists of all functions such that for each compact $K \subset \mathbb{R}$ and any $p \in \mathbb{N}_0$ there is a $q \in \mathbb{N}$ such that, for every $r \geq q$ and $\varphi \in A_r(\mathbb{R})$, $\sup_{x\in K} |\partial^p F(\varphi_\varepsilon, x)\,| = O(\varepsilon^{r-q})$, as $\varepsilon \to 0_+$.

The differential algebra $\mathcal{G}(\mathbb{R})$ contains the distributions on $\mathbb{R}$, canonically embedded as a $\mathbb{C}$-vector subspace by the map

$$i : \mathcal{D}'(\mathbb{R}) \to \mathcal{G} : u \mapsto \widetilde{u} = \{\widetilde{u}(\varphi, x) := (u * \check{\varphi})(x) | \varphi \in A_q(\mathbb{R})\},\ \text{where } \check{\varphi}(x) = \varphi(-x).$$

The equality of generalized functions in $\mathcal{G}$ is very strict and so it is introduced a weaker form of equality in the sense of *association* that plays a fundamental role in Colombeau theory.

Definition 2. (a) Two generalized functions $F, G \in \mathcal{G}(\mathbb{R})$ are said to be 'associated', denoted $F \approx G$, if for some representatives $F(\varphi_\varepsilon, x)$, $G(\varphi_\varepsilon, x)$ and arbitrary $\psi(x) \in \mathcal{D}(\mathbb{R})$ there is a $q \in \mathbb{N}_0$, such that for any $\varphi(x) \in A_q(\mathbb{R})$, $\lim_{\varepsilon \to 0_+} \int_{\mathbb{R}} [F(\varphi_\varepsilon, x) - G(\varphi_\varepsilon, x)]\psi(x)\, dx = 0$.

(b) A generalized function $F \in \mathcal{G}(\mathbb{R})$ is said to be 'associated' with a distribution $u \in \mathcal{D}'(\mathbb{R})$, denoted $F \approx u$, if for some representative $F(\varphi_\varepsilon, x)$, and arbitrary $\psi(x) \in \mathcal{D}(\mathbb{R})$ there is a $q \in \mathbb{N}_0$, such that for any $\varphi(x) \in A_q(\mathbb{R})$, $\lim_{\varepsilon \to 0_+} \int_{\mathbb{R}} f(\varphi_\varepsilon, x)\psi(x)\, dx = \langle u, \psi \rangle$.

These definitions are independent of the representatives chosen, and the association is a faithful generalization of the equality of distributions. The following relations hold in $\mathcal{G}$:

$$F \approx u \quad \& \quad F_1 \approx u_1 \implies F + F_1 \approx u + u_1, \quad \partial F \approx \partial u. \tag{2}$$

2.2 We next recall the definition of some distributions to be used in the sequel.

Notation 2. If $a \in \mathbb{C}$ and Re $a > -1$, denote as usual the locally-integrable functions:

$$x_+^a = \begin{cases} x^a & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases} \qquad x_-^a = \begin{cases} (-x)^a & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$

$$\ln x_+ = \begin{cases} \ln x & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases} \qquad \ln x_- = \begin{cases} \ln(-x) & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$

$$\ln|x| = \ln x_+ + \ln x_-, \qquad \ln|x| \operatorname{sgn} x = \ln x_+ - \ln x_-.$$

The distributions $x_\pm^a$ are defined for any $a \in \Omega := \{a \in \mathbb{R} : a \neq -1, -2, \ldots\}$, by setting

$$x_+^a = \partial^r x_+^{a+r}(x), \qquad x_-^a = (-1)^r \partial^r x_-^{a+r}(x),$$

where $r \in \mathbb{N}_0$ is such that $a + r > -1$ and the derivatives are in distributional sense.

This definition can be extended also for negative integer values of a by a procedure due to M. Riesz (see [5, § 3.2]). For each $\psi(x) \in \mathcal{D}(\mathbb{R})$, $a \mapsto \langle x_+^a, \psi \rangle$ is an analytic function of a on the set Ω. The excluded points are simple poles of this function. For any $p \in \mathbb{N}_0$, the residue at $a = -p-1$ is $\lim_{a \to -p-1}(a+p+1)\, \langle x_+^a, \psi \rangle = \psi^{(p)}(0)/p!$. Subtracting the singular part, one gets for any $p \in \mathbb{N}_0$:

$$\lim_{a \to -p-1} \langle x_+^a,\ \psi \rangle - \frac{1}{p!}\, \frac{\psi^{(p)}(0)}{a+p+1} = -\frac{1}{p!} \int_0^\infty \ln x\, \psi^{(p+1)}\, dx + \frac{\psi^{(p)}(0)}{p!} \sum_{k=1}^{p} \frac{1}{k}.$$

The right-hand side of this equation, which is the principal part of the Laurent expansion, was proposed by Hörmander in [5] to define the distribution x_+^{-p-1}, acting here on the test-function $\psi(x)$. In view of the notation in 2.2, this is equivalent to the following definition of x_+^{-p-1} for arbitrary $p \in \mathbb{N}_0$ $(x \in \mathbb{R})$:

$$x_+^{-p-1} = \frac{(-1)^p}{p!}\, \partial_x^{p+1} \ln x_+ \;+\; \frac{(-1)^p\, \kappa_p}{p!}\, \delta^{(p)}(x). \tag{3}$$

It is introduced here the shorthand notation $\kappa_p := \sum_{k=1}^{p} 1/k$ $(p \in \mathbb{N}_0)$; note that $\kappa_0 = 0$. Similar consideration leads to the defining equation

$$x_-^{-p-1} = \frac{-1}{p!}\, \partial_x^{p+1} \ln x_- \;+\; \frac{\kappa_p}{p!}\, \delta^{(p)}(x). \tag{4}$$

One checks that the distributions $x_\pm^{-p}$ satisfy:

$$\partial_x\, x_+^{-p} \;=\; -p\, x_+^{-p-1} \;+\; \frac{(-1)^p}{p!}\, \delta^{(p)}(x), \quad \partial_x\, x_-^{-p} \;=\; p\, x_-^{-p-1} \;-\; \frac{1}{p!}\, \delta^{(p)}(x).$$

Moreover, it follows immediately that

$$x_+^{-p}\,|_{x \mapsto -x} = x_-^{-p} \qquad \text{and also} \qquad x_+^{-p} \;+\; (-1)^p\, x_-^{-p} \;=\; x^{-p} \quad (p \in \mathbb{N}), \tag{5}$$

where x^{-p} is defined, as usual, as a distributional derivative of order p of $\ln |x|$.

Similarly, we define the distribution

$$x^{-p}\, \mathrm{sgn} x \;:=\; x_+^{-p} \;-\; (-1)^p\, x_-^{-p} \qquad (p \in \mathbb{N}_0). \tag{6}$$

Note that $x^{-p}\, \mathrm{sgn} x \;\neq x^{-p}$ for arbitrary $p \in \mathbb{N}_0$; it also differs from the 'odd' and 'even' compositions $|x|^{-p}\, \mathrm{sgn} x := x_+^{-p} + x_-^{-p} = x^{-p}$ for odd natural p and $|x|^{-p} := x_+^{-p} - x_-^{-p} = x^{-p}$ for even p.

Recall finally the definition of the distributions $(x \pm i0)^{-p-1}$ for $p \in \mathbb{N}_0$:

$$(x \pm i0)^{-p-1} := \lim_{y \to 0_+} (x \pm iy)^{-p-1} = x^{-p-1} \;\mp\; \frac{(-1)^p\; i\,\pi}{p!}\, \delta^{(p)}(x), \qquad x \in \mathbb{R}. \tag{7}$$

3. Modelling of singularities in the Colombeau algebra

Consider first generalized functions that model the δ-type singularity in the sense of association, i.e. being associated with the δ-function. Since there is an abundant variety of such functions (together with the canonical imbedding $\widetilde{\delta}$ in $\mathcal{G}$ of the distribution δ), we can put on the generalized functions in question an additional requirement. So define, following [9, §10], a generalized function $D \in \mathcal{G}$ with the properties:

$$D \approx \delta, \qquad D^{\,2} \approx \delta. \tag{8}$$

To this aim, we let $\varphi \in A_0(\mathbb{R})$, $s \equiv s(\varphi)$, and $\sigma = s(\varphi_\varepsilon) = \varepsilon s$ be as in Notation 1, and $D \in \mathcal{G}$ be the class $[\varphi \mapsto D(s(\varphi), x)]$. We specify further that

$D(s,x) = f(x) + \lambda_s\ g(x)$, where $f, g \in \mathcal{D}(\mathbb{R})$ are real-valued, symmetric, with disjoint support, and satisfying:

$$\int_{\mathbb{R}} f(x)\,dx = 1, \quad \int_{\mathbb{R}} g(x)\,dx = 0, \quad \text{and} \quad \lambda_s^2 = \frac{s - \int f^{\,2}(x)\,dx}{\int g^{\,2}(x)\,dx}.$$

It is not difficult to check that, for each $\varphi \in A_0(\mathbb{R})$, the representative $D(s,x)$ of the generalized function D satisfies the conditions:

$$\text{(9)} \quad D(\cdot,x) \in \mathcal{D}(\mathbb{R}),\ D(\cdot,-x) = D(\cdot,x),\ \frac{1}{s}\int_{\mathbb{R}} D^{\,2}(s,x)\,dx = \int_{\mathbb{R}} D(s,x)\,dx = 1$$

for each real positive value of the parameter s. Moreover, the generalized function D so defined satisfies the association relations (8). To show this, denote by

$$\text{(10)} \qquad D_\sigma(x) \ := \ \frac{1}{\sigma} D\Big(\sigma, \frac{x}{\sigma}\Big)\,, \quad \text{where} \ \ \sigma = s(\varphi_\varepsilon).$$

Now, for an arbitrary test-function $\psi \in \mathcal{D}(\mathbb{R})$, evaluate the functional values

$$I_1(\sigma) = \langle D_\sigma(x), \psi(x)\rangle, \qquad I_2(\sigma) = \langle D_\sigma^2(x), \psi(x)\rangle,$$

as $\varepsilon \to 0_+$, or equivalently, as $\sigma \to 0_+$. But in view of (9), it is immediate to see that $\lim_{\sigma\to 0_+} I_1(\sigma) = \lim_{\sigma\to 0_+} I_2(\sigma) = \langle \delta, \psi\rangle$; which according to Definition 2 (b) gives (8).

The first equation in (8) is in consistency with the observation that $D_\sigma(x)$ is a strict δ-net as defined in distribution theory [9, §7]. But notice that D is not the canonical embedding $\widetilde{\delta}$ of the δ-function since $\widetilde{\delta}^{\,2}$ does not admit associated distribution.

The flexible approach to modelling singularities allowed by generalized functions in $\mathcal{G}$ so that the models satisfy auxiliary conditions, can be systematically applied to defining generalized models of particular singularities. We will consider models of singularities given by distributions with singular point support. For their definition, we intend to take advantage of the properties of δ-modelling function D. Observe that

$$(\delta * D(s,\cdot))(x) = \langle \delta_y,\, D(s,x-y)\rangle = D(s,x).$$

$$\begin{aligned}(\delta' * D(s,\cdot))(x) &= \langle \delta'_y,\, D(s,x-y)\rangle = -\langle \delta_y\ \partial_y D(s,x-y)\rangle \\ &= \langle \delta_y,\, D'(s,x-y)\rangle = D'(s,x).\end{aligned}$$

This can be continued by induction for any derivative to define a generalized function $D^{\,(p)}(x)$ that models $\delta^{\,(p)}(x)$ and has representative $D^{\,(p)}(s,x) = (\delta^{\,(p)} * D(s,\cdot))(x)$.

Clearly, this definition is in consistency with the differentiation: $\partial_x D^{(p)}(x) = D^{(p+1)}(x)$, $p \in \mathbb{N}_0$. Moreover,

$$D^{(p)}(-x) \;=\; (-1)^p D^{(p)}(x). \tag{11}$$

In [3] we have employed such procedure for unified modelling of singularities given by distributions with singular point support, i.e. (besides $\delta^{(p)}$) the distributions $x_\pm^a$, $a \in \Omega$. Namely, choosing an arbitrary generalized function D with representative $D(s,x)$ that satisfies (9) for each $\varphi \in A_0(\mathbb{R})$, we have introduced generalized functions $X_\pm^a(x)$, modelling the above singularities, with representatives

$$X_\pm^a(s,x) := (y_\pm^a * D(s,y))(x), \quad a \in \Omega. \tag{12}$$

This is consistent with the differentiation: $\partial_x X_\pm^a(x) = a X_\pm^{a-1}(x)$; in particular, $H' = D$, where $H \in \mathcal{G}$ is model of the step-function θ, with representative $H(s,x) = \theta * D(s,\cdot)(x)$.

Extending now definition (12) to the distributions $x_\pm^{-p-1}$, $p \in \mathbb{N}_0$, we obtain

$$X_\pm^{-p-1}(s,x) := (y_\pm^{-p-1} * D(s,y))(x). \tag{13}$$

Similarly, we put $\operatorname{Ln} x_\pm := (\ln y_\pm * D(s,y))(x)$.

Note that generalized functions so introduced are indeed models of the corresponding singularities: it is straightforward to show that for each $a \in \Omega$

$$X_\pm^a(x) \;\approx\; x_\pm^a(x); \;\; \text{in particular, } H \approx \theta, \text{ and } H^p \approx \theta \text{ for each } p \in \mathbb{N}.$$

It was also proved in [3] that — as it can be expected — the functions H and D that model correspondingly the θ- and δ-type singularities satisfy the relation $H \,.\, D \;\approx\; \frac{1}{2}\,\delta$. Moreover, these generalized models were proved to satisfy

$$H \,\cdot\, D' \;\approx\; -\,\delta \,+\, \frac{1}{2}\,\delta'. \tag{14}$$

Concerning the singularities given by the distributions $x_\pm^{-p}$, $p \in \mathbb{N}$, it can be easily checked that $\operatorname{Ln}_\pm x \approx \ln_\pm x$ for the latter locally-integrable function. Then the modelling property for the generalized functions $X_\pm^{-p}(x)$ follows in view of relation (2) for consistency between the differentiation and association in $\mathcal{G}$.

Finally, we shall need below the representatives of the generalized models when they depend on φ_ε, or rather on the value $s(\varphi_\varepsilon) = \varepsilon s(\varphi) = \sigma$. In view of equations (3), (4), (10), (12), and (13), we obtain for the corresponding representatives ($p \in \mathbb{N}_0$):

$$X_{+\,\sigma}^p(x) = \frac{1}{\sigma}\int_0^\infty y^p D\left(\sigma, \frac{x-y}{\sigma}\right)\, dy, \tag{15}$$

$$X^{p}_{-\,\sigma}(x) = \frac{1}{\sigma}\int_{-\infty}^{0} (-y)^p D\Big(\sigma, \frac{x-y}{\sigma}\Big)\, dy,$$

$$X^{-p-1}_{+\,\sigma}(x) = \frac{(-1)^p}{\sigma^{p+2}\; p!}\int_0^{\infty} \ln y\, D^{(p+1)}\left(\sigma, \frac{x-y}{\sigma}\right) dy$$
$$+\; \frac{(-1)^p\,\kappa_p}{\sigma^{p+1}\,p!}\, D^{(p)}\left(\sigma, \frac{x}{\sigma}\right),$$

$$X^{-p-1}_{-\,\sigma}(x) = \frac{-1}{\sigma^{p+2}\; p!}\int_{-\infty}^{0} \ln(-y)\, D^{(p+1)}\left(\sigma, \frac{x-y}{\sigma}\right) dy$$
$$+\; \frac{\kappa_p}{\sigma^{p+1}\,p!}\, D^{(p)}\left(\sigma, \frac{x}{\sigma}\right). \tag{16}$$

4. Products of some singularities modelled in $\mathcal{G}(\mathbb{R})$

The models of singularities we consider have products in the Colombeau algebra as generalized functions, but we are seeking results that can be evaluated back in terms of distributions, i.e. such that admit associated distributions. We will establish first certain balanced products of generalized models in the algebra $\mathcal{G}(\mathbb{R})$ that exist on distributional level, proving the following.

Theorem 1. *The generalized models of the distributions $x_{\pm}^{-2}, \theta, \check{\theta}$, and $\delta'(x)$ satisfy:*

$$X_{-}^{-2}\;\cdot\; H - \operatorname{Ln} x_{+}\;\cdot\; D' \approx\; -\,\delta, \tag{17}$$

$$X_{+}^{-2}\;\cdot\; \check{H} + \operatorname{Ln} x_{-}\;\cdot\; D' \approx\; -\,\delta. \tag{18}$$

Proof: (i) For an arbitrary test-function $\psi(x) \in \mathcal{D}(\mathbb{R})$, denote $I(\sigma) := \langle\, X^{-2}_{-\,\sigma}\cdot H_\sigma,\, \psi(x)\,\rangle$. Suppose (without loss of generality) that $\operatorname{supp} D(\sigma,x) \subseteq [-l,l]$ for some $l \in \mathbb{R}_+$; then $-l \le x/\sigma \le l$ implies $-l\sigma \le x \le l\sigma$. Now from equations (16) for $p=1$ and (15) for $p=0$, we get on transforming the variables $y = \sigma u + x$, $z = \sigma v + x$, and $x = -\sigma w$:

$$\begin{aligned} I(\sigma) = &-\frac{1}{\sigma^4}\int_{-\sigma l}^{\sigma l} dx\,\psi(x)\int_0^{\sigma l + x} dy\, D\Big(\sigma, \frac{x-y}{\sigma}\Big) \\ &\times \int_{-\sigma l + x}^{0} \ln(-z)\, D''\left(\sigma, \frac{x-z}{\sigma}\right) dz \\ &+ \frac{1}{\sigma^3}\int_{-\sigma l}^{\sigma l} dx\,\psi(x) D'\left(\sigma, \frac{x}{\sigma}\right)\int_0^{\sigma l + x} D\Big(\sigma, \frac{x-y}{\sigma}\Big)\, dy \\ = &-\frac{1}{\sigma}\int_{-l}^{l} dw\,\psi(-\sigma w)\int_w^{l} du\, D(\sigma,u)\int_{-l}^{w} \ln(\sigma w - \sigma v)\, D''(\sigma,v)\, dv \\ &+ \frac{1}{\sigma}\int_{-l}^{l} dw\,\psi(-\sigma w) D'(\sigma,w)\int_w^{l} D(\sigma,u)\, du \;=:\; I_1 + I_2. \end{aligned} \tag{19}$$

Applying Taylor theorem to the function ψ and changing the order of integration, we get

$$
\begin{aligned}
I_1 \quad = \quad & -\frac{\psi(0)}{\sigma}\int_{-l}^{l} du\, D(\sigma,u)\int_{-l}^{u} dv\, D''(\sigma,v)\int_{v}^{u}\ln(\sigma w-\sigma v)\, dw \\
& +\psi'(0)\int_{-l}^{l} du\, D(\sigma,u)\int_{-l}^{u} dv\, D''(\sigma,v)\int_{v}^{u}\ln(\sigma w-\sigma v)\, w\, dw \;+\; o\,(1).
\end{aligned}
$$

Here the Landau symbol $o(1)$ stands for an arbitrary function of asymptotic order less than any constant, and the asymptotic evaluation is obtained taking into account that the third term in the Taylor expansion is multiplied by definite integrals majorizable by constants. Now the substitution $w \rightarrow t=(w-v)/(u-v)$, together with $w-v=(u-v)t$, yields

$$
\begin{aligned}
I_1 \quad = \quad & -\frac{\psi(0)}{\sigma}\int_{-l}^{l} du\, D(\sigma,u)\int_{-l}^{u} dv\, D''(\sigma,v)(u-v)\left[\ln(\sigma u-\sigma v)+\int_0^1 \ln t\, dt\right] \\
& +\psi'(0)\int_{-l}^{l} du\, D(\sigma,u)\int_{-l}^{u} dv\, D''(\sigma,v)(u-v)^2\left[\frac{\ln(\sigma u-\sigma v)}{2}+\int_0^1 t\ln t\, dt\right] \\
& -\psi'(0)\int_{-l}^{l} du\, D(\sigma,u)\int_{-l}^{u} dv\, D''(\sigma,v)(u-v)^2\left[\ln(\sigma u-\sigma v)+\int_0^1 \ln t\, dt\right] \\
& +\psi'(0)\int_{-l}^{l} du\, u\, D(\sigma,u)\int_{-l}^{u} dv\, D''(\sigma,v)(u-v)\left[\ln(\sigma u-\sigma v)+\int_0^1 \ln t\, dt\right] \\
& + o\,(1).
\end{aligned}
$$

Calculating the integrals $\int_0^1 \ln t\, dt=-1$, $\int_0^1 \ln t\, dt=-1/4$, replacing $v=u-(u-v)$, and integrating by parts in the variable v (the integrated part being 0) we get

$$
\begin{aligned}
I_1 \quad = \quad & -\frac{\psi(0)}{\sigma}\int_{-l}^{l} du\, D(\sigma,u)\int_{-l}^{u}\ln(\sigma u-\sigma v)\; D'(\sigma,v)\, dv \;-\; 2\,\psi(0) \\
& +\psi'(0)\int_{-l}^{l} du\, u\, D(\sigma,u)\int_{-l}^{u}\ln(\sigma u-\sigma v)\, D'(\sigma,v)\, dv \\
& -\psi'(0)\int_{-l}^{l} du\, D(\sigma,u)\int_{-l}^{u}\ln(\sigma u-\sigma v)\, D(\sigma,v)\, dv \;+\; o\,(1).
\end{aligned}
\tag{20}
$$

To obtain the latter result, we have used equation (9) and also that

$$
\int_{-l}^{l} du\, D(\sigma,u)\int_{-l}^{u} D(\sigma,v)\, dv=\frac{1}{2}. \tag{21}
$$

Applying again Taylor theorem to the function ψ, changing the order of integration, and integrating by parts in the variable w, we obtain for the second term in (19):

$$I_2 = \psi(0) \; - \; \frac{1}{2}\, \psi\,'(0) \; + \; o\,(1),$$

where equation (21) is used again. Thus

$$\begin{aligned} I(\sigma) = &-\frac{\psi(0)}{\sigma}\int_{-l}^{l} du\, D(\sigma,u)\int_{-l}^{u} \ln(\sigma u - \sigma v) D\,'(\sigma,v)\, dv \\ &+ \psi\,'(0)\int_{-l}^{l} du\, u\, D(\sigma,u) \int_{-l}^{u} \ln(\sigma u - \sigma v)\, D\,'(\sigma,v)\, dv \\ &- \psi\,'(0)\int_{-l}^{l} du\, D(\sigma,u)\int_{-l}^{u} \ln(\sigma u - \sigma v)\, D(\sigma,v)\, dv \\ &- \psi(0) - \frac{1}{2}\psi\,'(0) + o\,(1). \end{aligned} \tag{22}$$

(ii) On the other hand, denoting $J(\sigma) := \langle\, \mathrm{Ln}\, x_{+\sigma} \,.\, D\,'_{\sigma},\; \psi(x)\,\rangle$, we obtain on transforming the variables $y = \sigma u + x$ and $x = -\sigma v$, applying Taylor theorem to ψ, and changing the order of integration:

$$\begin{aligned} J(\sigma) \;&=\; \frac{1}{\sigma^3}\int_{-\sigma l}^{\sigma l} dx\, \psi(x)\, D\,'\Big(\sigma, \frac{x}{\sigma}\Big) \int_{0}^{\sigma l + x} \ln y \; D\Big(\sigma, \frac{x-y}{\sigma}\Big)\; dy \\ &=\; -\,\frac{\psi(0)}{\sigma}\int_{-l}^{l} du\, D(\sigma,u)\int_{-l}^{u} \ln(\sigma v - \sigma u)\, D\,'(\sigma,v)\, dv \\ &\quad + \psi\,'(0)\int_{-l}^{l} du\, D(\sigma,u)\int_{-l}^{u} \ln(\sigma v - \sigma u)\, v\, D\,'(\sigma,v)\, dv \; + \; o\,(1). \end{aligned}$$

Replacing $v = u + (v - u)$ in the last term and integrating by parts the third term, we get

$$\begin{aligned} J(\sigma) = &-\frac{\psi(0)}{\sigma}\int_{-l}^{l} du\, D(\sigma,u)\int_{-l}^{u} \ln(\sigma v - \sigma u)\, D\,'(\sigma,v)\, dv \\ &+ \; \psi\,'(0)\int_{-l}^{l} du\, u\, D(\sigma,u)\int_{-l}^{u} \ln(\sigma v - \sigma u)\, D\,'(\sigma,v)\, dv \\ &- \; \psi\,'(0)\int_{-l}^{l} du\, D(\sigma,u)\int_{-l}^{u} \ln(\sigma v - \sigma u)\, D(\sigma,v)\, dv \\ &- \frac{1}{2}\,\psi\,'(0) \; + \; o\,(1). \end{aligned} \tag{23}$$

Combining now equations (22) and (23), we obtain by linearity that

$$\lim_{\sigma\to 0_+}\int_{\mathbb{R}} \psi(x)\,\big[\, X_{-\,\sigma}^{-2}(x)\,\cdot\, H_\sigma(x) - \,\mathrm{Ln}\, x_{+\,\sigma}(x)\,\cdot\, D\,'_{\sigma}(x)\big]\; dx = -\,\psi(0) = -\,\langle\delta,\psi\rangle.$$

According to Definition 2(b), this proves the first equation in (17). The second equation follows on replacing $x \to -x$ in the first one and taking into account equations (5) and (11). This completes the proof. $\square$

The above balanced products of the functions $X_{\pm}^{-2}$ supported in the corresponding real half-lines can be employed further to get results on singular products of the generalized modelling functions $X^{-2} \operatorname{sgn} x$ and X^{-2} (obtained from equations (6), (5), and (13)).

Corollary 1. *The following balanced product holds for the generalized models of the distribution $x^{-2} \operatorname{sgn} x$, θ, and δ':*

$$X^{-2} \operatorname{sgn} x \cdot H + \operatorname{Ln}|x| \operatorname{sgn} x \cdot D' \approx x_{+}^{-2} + 2\,\delta. \tag{24}$$

Proof: Consider the following chain of identities and associations in $\mathcal{G}(\mathbb{R})$, taking into account equation (18) and the relation $H + \check{H} \approx 1$:

$$X_{+}^{-2} \cdot H = X_{+}^{-2} \cdot (1 - \check{H}) = X_{+}^{-2} - X_{+}^{-2} \cdot \check{H} \approx X_{+}^{-2} + \operatorname{Ln} x_{-} \cdot D' + \delta.$$

Thus

$$X_{+}^{-2} \cdot H - \operatorname{Ln} x_{-} \cdot D' \approx X_{+}^{-2} + \delta,$$

which, in view of the association $X_{+}^{-2} \approx x_{+}^{-2}$ and the linearity by (2) of the association in $\mathcal{G}$, leads to the balanced product

$$X_{+}^{-2} \cdot H - \operatorname{Ln} x_{-} \cdot D' \approx x_{+}^{-2} + \delta. \tag{25}$$

Further, equations (6) for $p = 2$, (17) and (25), will all yield

$$X^{-2} \operatorname{sgn} x \cdot H = \left(X_{+}^{-2} - X_{-}^{-2}\right) \cdot H \approx \operatorname{Ln} x_{-} \cdot D' + x_{+}^{-2} + \delta - \operatorname{Ln} x_{+} \cdot D' + \delta.$$

Due to relation (2) for linearity of the association, this proves equation (24).

Other consequences from the above results are given by this.

Corollary 2. *The generalized models in $\mathcal{G}$ of the distributions $(x \pm i0)^{-2}$, θ, and δ' satisfy*

$$(X \pm i0)^{-2} \cdot H - \operatorname{Ln}|x| \cdot D' \approx x_{+}^{-2} \mp i\pi\,\delta(x) \pm \frac{i\pi}{2}\,\delta'. \tag{26}$$

Proof: The second equation in (5), as well as equations (17) and (25), now give

$$X^{-2} \cdot H = \left(X_{+}^{-2} + X_{-}^{-2}\right) \cdot H \approx \operatorname{Ln} x_{-} \cdot D' + x_{+}^{-2} + \delta + \operatorname{Ln} x_{+} \cdot D' - \delta.$$

In view of (2), this yields

$$X^{-2} \cdot H - \operatorname{Ln}|x| \cdot D' \approx x_{+}^{-2}. \tag{27}$$

Employing further equations (7), (27) and (14), we get

$$(X \pm i0)^{-1} \cdot H \,=\, X^{-2} \cdot H \,\pm\, i\pi\, D'(x) \cdot H \;\approx \mathrm{Ln}\,|x| \cdot D' + x_+^{-2} \mp i\pi\,\delta \pm \frac{i\pi}{2}\,\delta',$$

which in view of linearity of association in $\mathcal{G}$ proves (26). □

Finally, we will evaluate some products of singularities given by the non-differentiable functions $x_\pm$ modelled by the generalized functions $X_\pm$ with derivatives of D. They only exist as balanced products, as demonstrated by this.

Theorem 2. *The following balanced products hold for the modelling generalized function $X_\pm, H$ and D:*

$$X_+ \cdot D^{(4)} \,+\, H \cdot D^{(3)} \;\approx\; \frac{5}{2}\,\delta'' \,-\, \frac{3}{2}\,\delta''', \tag{28}$$

$$X_- \cdot D^{(4)} \,+\, \check{H} \cdot D^{(3)} \approx\; \frac{5}{2}\,\delta'' \,+\, \frac{3}{2}\,\delta'''. \tag{29}$$

Proof: For an arbitrary $\psi(x) \in \mathcal{D}(\mathbb{R})$, we denote

$$I(\sigma) := \langle\, X_{+\,\sigma}(x) \,\cdot\, D_\sigma^{(4)}(x),\, \psi(x)\,\rangle.$$

From equations (10) and (15), we get on transforming the variables $y = \sigma v + x$, $x = -\sigma u$, changing the order of integration, and applying Taylor theorem

$$\begin{aligned}
I(\sigma) \;&=\; \frac{1}{\sigma^3}\int_{-l}^{l} du\,\psi(-\sigma u) D^{(4)}(\sigma,u) \int_u^l (v-u) D(\sigma,v)\,dv \\
&=\; \frac{\psi(0)}{\sigma^3}\int_{-l}^{l} dv\, D(\sigma,v)\int_{-l}^{v}(v-u)D^{(4)}(\sigma,u)\,du \\
&\quad -\frac{\psi'(0)}{\sigma^2}\int_{-l}^{l} dv\, D(\sigma,v)\int_{-l}^{v} u\,(v-u)D^{(4)}(\sigma,u)\,du \\
&\quad +\frac{\psi''(0)}{2\,\sigma}\int_{-l}^{l} dv\, D(\sigma,v)\int_{-l}^{v} u^2\,(v-u)D^{(4)}(\sigma,u)\,du \\
&\quad -\frac{\psi'''(0)}{6}\int_{-l}^{l} dv\, D(\sigma,v)\int_{-l}^{v} u^3\,(v-u)D^{(4)}(\sigma,u)\,du + O(\sigma) \\
&=: \; \psi(0)\; I_0 \,+\, \psi'(0)\; I_1 \,+\, \psi''(0)\; I_2 \,+\, \psi'''(0)\; I_3 \,+\, O(\sigma).
\end{aligned}$$

Denote further $J(\sigma) := \langle\, H_{\,\sigma}(x) \,\cdot\, D_\sigma^{(3)}(x),\, \psi(x)\,\rangle$. Proceeding as above, we get

$$J(\sigma) = \psi(0)\; J_0 \,+\, \psi'(0)\; J_1 \,+\, \psi''(0)\; J_2 \,+\, \psi'''(0)\; J_3 \,+\, O(\sigma).$$

Compute next the terms I_k, $k=(0,1,2,3)$. We shall use equations (9), (21), as well as that

$$\frac{1}{\sigma}\int_{-l}^{l} v\, D(\sigma,v)D'(\sigma,v)\,dv \;=\; -\frac{1}{2\,\sigma}\int_{-l}^{l} D^{\,2}(\sigma,v)\,dv \;=\; -\frac{1}{2}.$$

Also, due to the equality $D\,'(\cdot,-x) = -D\,'(\cdot,x)$, the following equations hold

$$\int_{-l}^{l} D(\sigma,v)D'(\sigma,v)\,dv \;=\; \int_{-l}^{l} v\; D^{\,2}(\sigma,v)\,dv \;=\; \int_{-l}^{l} v^{\,2}\; D(\sigma,v)D'(\sigma,v)\,dv \;=\; 0.$$

Integrating now by parts in the variable u, the integrated part being 0 each time, we obtain:

$$\begin{aligned}
I_0 \;&=\; \frac{1}{\sigma^{\,3}}\int_{-l}^{l} dv\, D(\sigma,v)\int_{-l}^{v} D^{\,(3)}(\sigma,u)\,du \;=\; -\,J_0,\\
I_1 \;&=\; -\,\frac{1}{\sigma^2}\int_{-l}^{l} dv\, D(\sigma,v)\int_{-l}^{v} u\, D^{(3)}(\sigma,u)\,du\\
&\quad +\,\frac{1}{\sigma^2}\int_{-l}^{l} dv\, D(\sigma,v)\int_{-l}^{v} (v-u)D^{(3)}(\sigma,u)\,du\\
&=\; -\,J_1 \;+\; \frac{1}{\sigma^{\,2}}\int_{-l}^{l} D(\sigma,v)D^{'}(\sigma,v)\,dv = \;-J_1,\\
I_2 \;&=\; \frac{1}{2\sigma}\int_{-l}^{l} dv\, D(\sigma,v)\int_{-l}^{v} u^{\,2}\, D^{\,(3)}(\sigma,u)\,du\\
&\quad -\,\frac{1}{\sigma}\int_{-l}^{l} dv\, D(\sigma,v)\int_{-l}^{v} u(v-u)\, D^{\,(3)}(\sigma,u)\,du\\
&=\; -\,J_2 \;+\; I\,'_2,
\end{aligned}$$

where

$$\begin{aligned}
I\,'_2 \;&=\; \frac{1}{\sigma}\int_{-l}^{l} dv\, D(\sigma,v)\int_{-l}^{v} (v-u)^2 D^{(3)}(\sigma,u)\,du\\
&\quad -\,\frac{1}{\sigma}\int_{-l}^{l} dv\, v\, D(\sigma,v)\int_{-l}^{v} (v-u)D^{(3)}(\sigma,u)\,du\\
&=\; \frac{2}{\sigma}\int_{-l}^{l} D^{\,2}(\sigma,v)\,dv - \frac{1}{\sigma}\int_{-l}^{l} v\; D(\sigma,v)\, D\,'(\sigma,v)\,dv \;=\; \frac{5}{2},\\
I_3 \;&=\; -\,\frac{1}{6}\int_{-l}^{l} dv\, D(\sigma,v)\int_{-l}^{v} u^{\,3}\, D^{\,(3)}(\sigma,u)\,du\\
&\quad +\,\frac{1}{2}\int_{-l}^{l} dv\, D(\sigma,v)\int_{-l}^{v} u^{\,2}\,(v-u)D^{\,(3)}(\sigma,u)\,du
\end{aligned}$$

$$= \; - \; J_3 \; + \frac{3}{2}.$$

Summing up, we get

$$\text{(30)} \quad I(\sigma) = -\psi(0)J_0 - \psi'(0)J_1 - \psi''(0)J_2 - \psi'''(0)J_3 + \frac{5}{2}\,\psi''(0) + \frac{3}{2}\,\psi'''(0) + O(\sigma).$$

Now from equation (30), we obtain by linearity that

$$\lim_{\sigma\to 0_+} \int_{\mathbb{R}} \psi(x) \left[X_{+\ \sigma}(x) \,\cdot\, D_\sigma^{(4)}(x) + H_{\ \sigma}(x) \,\cdot\, D_\sigma^{(3)}(x) \right] dx = \langle \frac{5}{2}\,\delta'' - \frac{3}{2}\,\delta''', \psi \rangle.$$

According to Definition 2(b), this proves equation (28), whereas equation (29) follows on replacing $x \to -x$ in the former. The proof is complete. $\square$

Remark. Note that when computed for the canonical embedding of distributions in $\mathcal{G}$, none of the above singular products can be balanced so as to admit associated distribution.

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