Universally divergent Fourier series via Landau's extremal functions

Gerd Herzog, Peer Chr. Kunstmann

Abstract. We prove the existence of functions $f \in A(\mathbb{D})$, the Fourier series of which being universally divergent on countable subsets of $\mathbb{T} = \partial\mathbb{D}$. The proof is based on a uniform estimate of the Taylor polynomials of Landau's extremal functions on $\mathbb{T} \setminus \{1\}$.

Keywords: Fourier series; universal functions; Landau's extremal functions

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1. Introduction

Let $\mathbb{D}$ denote the unit disc in $\mathbb{C}$ and $\mathbb{T} = \partial\mathbb{D}$. For $f \in L^1(\mathbb{T})$ let

$$\sum_{k=-\infty}^{\infty} \widehat{f}(k) e_k$$

denote the corresponding Fourier series, i.e.

$$e_k(t) = \exp(ikt),\ \ \widehat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} f(\exp(it)) \exp(-ikt)\, dt\ \ (k \in \mathbb{Z}).$$

There is a tremendous amount of classical convergence and divergence results on Fourier series [14]. Moreover, several results on universally divergent Fourier (and trigonometric) series are known, see for example [3], [5], [7], [10], [11] and the references given there. Roughly speaking, f has a universally divergent Fourier series if the set of restrictions

$$\left\{ \Big(\sum_{k=-n}^{n} \widehat{f}(k) e_k \Big)\Big|_E : n \in \mathbb{N} \right\}$$

in a function space Y, of functions over a subset E of $\mathbb{T}$, is dense in Y. In [10] Müller proved that given a *countable* set $E \subseteq \mathbb{T}$, then the set of functions $f \in C(\mathbb{T})$

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having the property that

$$\left\{\Big(\sum_{k=-n}^{n}\widehat{f}(k)e_k\Big)\Big|_E : n\in\mathbb{N}\right\} \quad \text{is dense in} \quad \mathbb{C}^E$$

is a dense G_δ-subset of $C(\mathbb{T})$. Here $\mathbb{C}^E$ is the Fréchet space of all functions $w: E\to\mathbb{C}$ endowed with the topology of pointwise convergence.

In this paper, we prove that there are even functions $f\in C(\mathbb{T})$ with universal divergent Fourier series in the sense of Müller such that in addition $\widehat{f}(-k)=0$ $(k\in\mathbb{N})$. While Müller's proof is based on the Localization Principle of Fourier series we will use Landau's extremal functions [9, §2].

To formulate our results let $H(\mathbb{D})$ denote the Fréchet space of all analytic functions on $\mathbb{D}$ endowed with the compact open topology, let $A(\mathbb{D})=C(\overline{\mathbb{D}})\cap H(\mathbb{D})$ denote the disc-algebra endowed with the maximum norm $\|\cdot\|_\infty$, and for $f\in H(\mathbb{D})$ let

$$S_n(f,z)=\sum_{k=0}^{n}\frac{f^{(k)}(0)}{k!}z^k \quad (n\in\mathbb{N}_0).$$

For a given countable set $E\subseteq\mathbb{T}$ consider the continuous linear operators

$$L_n: A(\mathbb{D})\to\mathbb{C}^E, \quad L_n(f)=S_n(f,\cdot)|_E.$$

We will prove

Theorem 1. *The set of all $f\in A(\mathbb{D})$ with the property*

$$\{L_n(f): n\in\mathbb{N}_0\} \text{ is dense in } \mathbb{C}^E$$

is a dense G_δ-subset of $A(\mathbb{D})$.

Remark. According to a result of Fejér [4], no q-to-one function $f\in A(\mathbb{D})$ $(q\in\mathbb{N})$ can share the property in Theorem 1, since for those functions

$$|S_n(f,1)|\le\left(1+\sqrt{\frac{q}{2}}\right)\|f\|_\infty \quad (n\in\mathbb{N}_0).$$

Just as Müller's Theorem extends to L^p-spaces, we can extend Theorem 1 to certain Banach spaces of analytic functions, such as the Hardy spaces $H^p(\mathbb{D})$ $(1\le p<\infty)$, or the little Bloch space

$$\mathcal{B}_0=\{f\in H(\mathbb{D}): \lim_{|z|\to 1-}(1-|z|^2)|f'(z)|=0\},$$
$$\|f\|=|f(0)|+\max_{|z|<1}(1-|z|^2)|f'(z)|$$

(compare [1]), for example.

Theorem 2. *Let $(X, \|\cdot\|)$ be a Banach space with $A(\mathbb{D}) \subseteq X \subseteq H(\mathbb{D})$ such that*

(1) *$A(\mathbb{D})$ is dense in X,*
(2) *$\|\cdot\|$ is weaker than $\|\cdot\|_\infty$ on $A(\mathbb{D})$,*
(3) *convergence in X implies convergence in $H(\mathbb{D})$.*

Let $E \subseteq \mathbb{T}$ be countable, and let

$$\widetilde{L}_n : X \to \mathbb{C}^E, \quad \widetilde{L}_n(f) = S_n(f, \cdot)|_E.$$

Then, the set of all $f \in X$ with the property

$$\{\widetilde{L}_n(f) : n \in \mathbb{N}_0\} \text{ is dense in } \mathbb{C}^E$$

is a dense G_δ-subset of X.

In [10] Müller also discusses the problem of replacing pointwise convergence by uniform convergence for compact subsets E of $\mathbb{T}$. We note here that the general topological category argument [10, Lemma 2] is also applicable in our setting. Combined with Theorem 1 it proves that there are many and even uncountable compact subsets $E \subseteq \mathbb{T}$ such that the set of all $f \in A(\mathbb{D})$ with the property

$$\{S_n(f, \cdot)|_E : n \in \mathbb{N}_0\} \text{ is dense in } C(E)$$

is a dense G_δ-subset of $A(\mathbb{D})$. Here $C(E)$ denotes the Banach space of all continuous functions $w : E \to \mathbb{C}$ endowed with the maximum norm.

It follows from [8, Cor. 3.3] that there are countable compact subsets $E \subseteq \mathbb{T}$ with a single accumulation point such that, for no function $f \in A(\mathbb{D})$, the set

$$\{S_n(f, \cdot)|_E : n \in \mathbb{N}_0\}$$

is dense in $C(E)$. On other hand, recent results in [2] show that, for any compact subset $E \subseteq \mathbb{T}$ of arc length measure 0, the set of all $f \in H^p(\mathbb{D})$, $p \in [1, \infty)$, with the property

$$\{S_n(f, \cdot)|_E : n \in \mathbb{N}_0\} \text{ is dense in } C(E)$$

is a dense G_δ-subset of $H^p(\mathbb{D})$.

Remark. After submitting this paper Jürgen Müller informed us that he and George Costakis have independently found Theorem 1. Their proof is based on Fejér polynomials

$$F_n(z) = \sum_{k=0, k\neq n}^{2n} \frac{z^k}{n-k},$$

instead of Landau's extremal functions and has not been published up to now.

2. Landau's extremal functions

In the sequel let

$$a_k := (-1)^k \binom{-1/2}{k} \quad (k \in \mathbb{N}_0),$$

that is

$$a_0 = 1,\ a_k = \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2k} \quad (k \in \mathbb{N}),$$

and

$$K_n(z) := a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n \quad (n \in \mathbb{N}_0).$$

In [9] Landau proved that $0 \notin K_n(\overline{\mathbb{D}})$, that

$$\gamma_n := \sup\{|S_n(f,1)| : f \in A(\mathbb{D}),\ \|f\|_\infty \le 1\} \quad (n \in \mathbb{N}_0)$$

is a maximum which is attained at

$$f(z) = R_n(z) := \frac{z^n K_n(1/z)}{K_n(z)} = \frac{a_n + a_{n-1} z + a_{n-2} z^2 + \dots + a_0 z^n}{a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n},$$

and that

$$S_n(R_n, 1) = \gamma_n = \sum_{k=0}^n a_k^2 \sim \frac{\log(n)}{\pi} \quad (n \to \infty).$$

In particular we have

$$S_n(R_n, 1) = \gamma_n \to \infty \quad (n \to \infty).$$

Some immediate facts on the rational functions R_n are

$$|R_n(z)| = 1 \ (z \in \mathbb{T}), \quad \|R_n\|_\infty = 1.$$

To utilize the functions R_n in the proof of Theorem 1 we prove the following theorem, which seems to us of some interest on its own. It asserts that the functions $S_n(R_n, \cdot)$, $n \in \mathbb{N}_0$, have a common majorant on $\overline{\mathbb{D}} \setminus \{1\}$. We shall use this on $\mathbb{T} \setminus \{1\}$.

Theorem 3. *For each $z \in \overline{\mathbb{D}} \setminus \{1\}$ we have*

$$|S_n(R_n, z)| \le \frac{4}{|1-z|} \quad (n \in \mathbb{N}_0).$$

Proof: We fix $n \in \mathbb{N}_0$, and consider

$$S_n(R_n, z) = \sum_{k=0}^n b_k z^k.$$

Observe that here $b_k = b_{k,n}$ depend on n, but we shall ignore this in notation. Since the a_k are real, it is clear that $b_k \in \mathbb{R}$, and from $\|R_n\|_\infty = 1$ we get by

Cauchy's formula $|b_k| \leq 1$ $(k = 0, \ldots, n)$. In [13], Wintner proved that in fact $b_k \in [0,1]$ $(k = 0, \ldots, n)$. From

$$(a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n) R_n(z) = a_n + a_{n-1} z + a_{n-2} z^2 + \cdots + a_0 z^n$$

we obtain (cf. [13])

$$\sum_{k=0}^{m} a_{m-k} b_k = a_{n-m} \quad (m = 0, \ldots, n).$$

By setting $P_k := \operatorname{diag}([1, \ldots, 1], -k) \in \mathbb{R}^{(n+1)\times(n+1)}$ (the band matrix with 1's in the k-th subdiagonal and 0's else), $b := (b_0, b_1, \ldots, b_n)^\top \in \mathbb{R}^{n+1}$ and $a := (a_n, a_{n-1}, \ldots, a_0)^\top \in \mathbb{R}^{n+1}$, the linear system above can be written as

$$\left(\sum_{k=0}^{n} a_k P_k \right) b = a.$$

We claim that

$$\left(\sum_{k=0}^{n} a_k P_k \right)^{-1} = a_0 P_0 + (a_1 - a_0) P_1 + (a_2 - a_1) P_2 + \cdots + (a_n - a_{n-1}) P_n.$$

In order to see this we start with the Taylor expansion

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \quad (|x| < 1)$$

for $\alpha = -\frac{1}{2}$ and $\alpha = \frac{1}{2}$ and take the Cauchy product

$$1 = (1+x)^{-1/2}(1+x)^{1/2} = \sum_{k=0}^{\infty} \sum_{\nu=0}^{k} \binom{-1/2}{\nu} \binom{1/2}{k-\nu} x^k.$$

We conclude that

$$\sum_{\nu=0}^{k} \binom{-1/2}{\nu} \binom{1/2}{k-\nu} = \delta_{0k} \quad (k \in \mathbb{N}_0).$$

For $k \in \mathbb{N}$ we get by a well-known formula

$$a_k - a_{k-1} = (-1)^k \left[\binom{-1/2}{k} + \binom{-1/2}{k-1} \right] = (-1)^k \binom{1/2}{k}.$$

Then

$$I + \sum_{k=1}^{n} (a_k - a_{k-1}) P_k = I + \sum_{k=1}^{n} (-1)^k \binom{1/2}{k} P_k = \sum_{k=0}^{n} (-1)^k \binom{1/2}{k} P_k$$

and

$$\sum_{k=0}^{n} a_k P_k = I + \sum_{k=1}^{n} (-1)^k \binom{-1/2}{k} P_k = \sum_{k=0}^{n} (-1)^k \binom{-1/2}{k} P_k.$$

We take the product, respect $P_k P_l = P_{l+k}$ (setting $P_j = 0$ for $j > n$), and arrive at

$$\begin{aligned}
&\Big(\sum_{k=0}^{n} a_k P_k\Big)\Big(I + \sum_{k=1}^{n} (a_k - a_{k-1}) P_k\Big) \\
&= \Big(\sum_{k=0}^{n} (-1)^k \binom{-1/2}{k} P_k\Big)\Big(\sum_{k=0}^{n} (-1)^k \binom{1/2}{k} P_k\Big) \\
&= \sum_{k,l=0}^{n} (-1)^{k+l} \binom{-1/2}{k} \binom{1/2}{l} P_{k+l} \\
&= \sum_{k=0}^{n} (-1)^k \Big(\sum_{\nu=0}^{k} \binom{-1/2}{\nu} \binom{1/2}{k-\nu}\Big) P_k = I,
\end{aligned}$$

and our claim is proved. We now rewrite

$$\begin{aligned}
\Big(\sum_{k=0}^{n} a_k P_k\Big)^{-1} &= a_0 P_0 + (a_1 - a_0) P_1 + (a_2 - a_1) P_2 + \cdots + (a_n - a_{n-1}) P_n \\
&= I - \Big(\frac{a_0}{2} P_1 + \frac{a_1}{4} P_2 + \cdots + \frac{a_{n-1}}{2n} P_n\Big) =: I - Q.
\end{aligned}$$

Now a is an increasing vector with positive entries, therefore each vector

$$\frac{a_k}{2k+2} P_{k+1} a \quad (k = 0, \ldots, n-1)$$

is an increasing vector. Thus, with $c := Qa$ we have $b = a - c$ where a and c are increasing, and from $a_k, b_k \in [0,1]$, $c_k \geq 0$ $(k = 0, \ldots, n)$ we get that all entries of c are in $[0,1]$. This proves that b has variation

$$\sum_{k=0}^{n-1} |b_{k+1} - b_k| \leq 2.$$

Next, we consider

$$\begin{aligned}
(1-z) S_n(R_n, z) = (1-z) \sum_{k=0}^{n} b_k z^k &= \sum_{k=0}^{n} b_k z^k - \sum_{k=1}^{n+1} b_{k-1} z^k \\
&= b_0 - b_n z^{n+1} + \sum_{k=1}^{n} (b_k - b_{k-1}) z^k,
\end{aligned}$$

thus, for each $z \in \overline{\mathbb{D}}$

$$|(1-z)S_n(R_n, z)| \le b_0 + b_n + \sum_{k=1}^{n} |b_k - b_{k-1}| \le 1 + 1 + 2 = 4.$$

$\square$

Remark. We note that the constant 4 in Theorem 3 is most certainly not optimal. Numerical experiments suggest that the vector b is always convex, and that

$$|S_n(R_n, z)| \le \frac{2}{|1-z|} \quad (n \in \mathbb{N}_0,\ z \in \overline{\mathbb{D}} \setminus \{1\}).$$

3. Universality criterion

To prove Theorem 1 we use the universality criterion of Grosse-Erdmann [6, Theorem 1.57].

Theorem 4. *Let X be a complete metric space, Y a separable metric space, and $T_n : X \to Y$, $n \in \mathbb{N}_0$, continuous maps. Denoting $\mathcal{U} = \mathcal{U}((T_n)_{n\in\mathbb{N}_0})$ as the set of all $x \in X$ such that*

$$\{T_n x : n \in \mathbb{N}_0\} \text{ is dense in } Y,$$

the following assertions are equivalent.

(1) *The family $(T_n)_{n\in\mathbb{N}_0}$ is topologically transitive, i.e. for any pair $U \subseteq X$, $V \subseteq Y$ of nonempty open sets, there is some $n \in \mathbb{N}_0$ such that*

$$T_n(U) \cap V \neq \emptyset.$$

(2) *The set $\mathcal{U}$ is a dense G_δ-subset of X.*
(3) *The set $\mathcal{U}$ is dense in X.*

In the proof of Theorem 1 we shall apply Theorem 4 to the situation

$$X = A(\mathbb{D}), \quad Y = \mathbb{C}^E, \quad T_n = L_n \ (n \in \mathbb{N}_0)$$

and check that (1) holds. Here we already observe that the equivalence of (2) and (3) in Theorem 4 shows that Theorem 1 implies Theorem 2.

Proof of Theorem 2: Assumption (3) of Theorem 2 implies that $\widetilde{L}_n : X \to \mathbb{C}^E$ is continuous ($n \in \mathbb{N}_0$). By Theorem 1, the set $\mathcal{U}((L_n)_{n\in\mathbb{N}_0})$ is dense in $A(\mathbb{D})$. Since $A(\mathbb{D})$ is densely and continuously embedded in X, we obtain that $\mathcal{U}((L_n)_{n\in\mathbb{N}_0})$ is dense in X. Hence also the superset $\mathcal{U}((\widetilde{L}_n)_{n\in\mathbb{N}_0})$ is dense in X, and is a dense G_δ-subset of X by Theorem 4. $\square$

Open Problem. It would be interesting to know whether $\mathcal{U}((\widetilde{L}_n)_{n\in\mathbb{N}_0})$ is a dense G_δ-set also in case $X = H^\infty(\mathbb{D})$. Clearly $\mathcal{U}((L_n)_{n\in\mathbb{N}_0}) \subseteq H^\infty(\mathbb{D})$, but assumption (1) of Theorem 2 is not satisfied.

We prepare the proof of Theorem 1 and note that if, in the situation of Theorem 4, D is a dense subset of X, and if d and ρ are the metrics on X and Y, respectively, then topological transitivity of $(T_n)_{n\in\mathbb{N}_0}$ is equivalent to the following condition:

$$\forall \varepsilon > 0 \ \forall (x,y) \in D \times Y \ \exists (n,z) \in \mathbb{N}_0 \times X : \ d(x,z) < \varepsilon \ \wedge \ \rho(T_n z, y) < \varepsilon.$$

In our situation, we let D denote the set of all polynomials, which is known to be a dense subset of $X = A(\mathbb{D})$, [12, p. 366]. Moreover, let $E = \{z_k : k \in \mathbb{N}\}$ with $z_k \neq z_j$ $(k \neq j)$ and let $Y = \mathbb{C}^E$ be endowed with the usual Fréchet metric

$$\rho(v,w) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|v(z_k) - w(z_k)|}{1 + |v(z_k) - w(z_k)|}.$$

4. Proof of Theorem 1

Let $\varepsilon > 0$, $p \in D$, $n_0 :=$ grad p and $w \in \mathbb{C}^E$. Let $m \in \mathbb{N}$ be such that

$$\sum_{k=m+1}^{\infty} \frac{1}{2^k} < \varepsilon.$$

It is sufficient to find a function $f \in A(\mathbb{D})$ and some $n \geq n_0$ such that

$$\|f\|_\infty < \varepsilon \ \wedge \ L_n(f)(z_k) = S_n(f, z_k) = w(z_k) - p(z_k) =: \zeta_k \ (k = 1, \ldots, m).$$

Once such n and f are known we set $g := f + p$ and obtain $\|p - g\|_\infty < \varepsilon$ and

$$\begin{aligned} \rho(L_n(g), w) &= \rho((S_n(f,\cdot) + p)|_E, w) \\ &= \rho(S_n(f,\cdot)|_E, w - p|_E) \leq \sum_{k=m+1}^{\infty} \frac{1}{2^k} < \varepsilon. \end{aligned}$$

To construct f and n we set $\zeta = (\zeta_1, \ldots, \zeta_m)^\top$ and we make the ansatz

$$f(z) = \lambda_1 R_n(\overline{z_1} z) + \cdots + \lambda_m R_n(\overline{z_m} z)$$

where $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ have to be chosen. Recalling that $\|R_n(\overline{z_k}\ \cdot)\|_\infty = 1$ $(k = 1, \ldots, m)$ we already find

$$\|f\|_\infty \leq \|\lambda\|_1$$

where $\|\cdot\|_1$ denotes the l^1-norm on $\mathbb{C}^m$.

Now we consider the $m \times m$-matrix Q_n with entries

$$q_{kj}^{(n)} = S_n(R_n(\overline{z_j}\ \cdot), z_k) \quad (k, j \in \{1, \ldots, m\}).$$

Note that if $\lambda = (\lambda_1, \ldots, \lambda_m)^\top$ solves $Q_n \lambda = \zeta$, then

$$S_n(f, z_k) = \zeta_k \quad (k = 1, \ldots, m).$$

Moreover

$$q_{kk}^{(n)} = S_n(R_n(\overline{z_k}\ \cdot), z_k) = S_n(R_n, 1) = \gamma_n, \quad (k = 1, \dots, m),$$

and according to Theorem 3 we have, for $k \neq j$,

$$|q_{kj}^{(n)}| = |S_n(R_n(\overline{z_j}\ \cdot), z_k)| \leq \frac{4}{|1 - \overline{z_j} z_k|} = \frac{4}{|z_k - z_j|} \leq c,$$

where

$$c := \max\left\{ \frac{4}{|z_k - z_j|} : j, k = 1, \dots, m,\ k \neq j \right\}$$

does not depend on n.

Since $\gamma_n \to \infty$ $(n \to \infty)$ we thus find

$$I - \frac{Q_n}{\gamma_n} \to 0 \quad (n \to \infty).$$

So for large n we have by Neumann's series

$$Q_n^{-1} = \frac{1}{\gamma_n}\left(I - \left(I - \frac{Q_n}{\gamma_n}\right)\right)^{-1} = \frac{1}{\gamma_n}\sum_{r=0}^{\infty}\left(I - \frac{Q_n}{\gamma_n}\right)^r,$$

and we conclude that $Q_n^{-1} \to 0$ as $n \to \infty$. In particular, we can choose $n \in \mathbb{N}$ such that

$$\|\lambda\|_1 = \|Q_n^{-1}\zeta\|_1 < \varepsilon.$$

This ends the proof. $\square$

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Institut für Analysis, Karlsruher Institut für Technologie (KIT), D-76128 Karlsruhe, Germany

E-mail: Gerd.Herzog2@kit.edu,
Peer.Kunstmann@kit.edu

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