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Positive solutions for a system of third-order differential equation with multi-point and integral conditions

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Abstract. This paper concerns the following system of nonlinear third-order boundary value problem:

$$u_i'''(t) + f_i(t, u_1(t), \dots, u_n(t), u_1'(t), \dots, u_n'(t)) = 0,\ 0 < t < 1,\ i \in \{1, \dots, n\}$$

with the following multi-point and integral boundary conditions:

$$\begin{cases} u_i(0) = 0 \\ u_i'(0) = 0 \\ u_i'(1) = \sum_{j=1}^{p} \beta_{j,i} u_i'(\eta_{j,i}) + \int_0^1 h_i(u_1(s), \dots, u_n(s))\, ds \end{cases}$$

where $\beta_{j,i} > 0$, $0 < \eta_{1,i} < \dots < \eta_{p,i} < \frac{1}{2}$, $f_i : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $h_i : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions for all $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, p\}$. Using Guo-Krasnosel'skii fixed point theorem in cone, we discuss the existence of positive solutions of this problem. We also prove nonexistence of positive solutions and we give some examples to illustrate our results.

Keywords: third-order differential equation; multi-point and integral boundary conditions; Guo-Krasnosel'skii fixed point theorem in cone; positive solutions

Classification: 34B15, 34B18, 34B27

1. Introduction

The third-order ordinary differential equations arise in different areas of applied mathematics and physics among others the deflection of a curved beam having a constant or a varying cross section, the three-layer beam, the electromagnetic waves or the gravity driven flows and so on [1]. The aim of this paper is to investigate sufficient conditions for the existence of positive solutions for the following problem:

$$u_i'''(t) + f_i(t, u_1(t), \dots, u_n(t), u_1'(t), \dots, u_n'(t)) = 0, \quad 0 < t < 1,\ i \in \{1, \dots, n\} \tag{1.1}$$

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with the following multi-point and integral boundary conditions:

$$(1.2) \qquad \begin{cases} u_i(0) = 0 \\ u_i'(0) = 0 \\ u_i'(1) = \sum_{j=1}^{p} \beta_{j,i} u_i'(\eta_{j,i}) + \int_0^1 h_i(u_1(s), \ldots, u_n(s))\, ds \end{cases}$$

where $\beta_{j,i} > 0$, $0 < \eta_{1,i} < \cdots < \eta_{p,i} < \frac{1}{2}$, $f_i : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $h_i : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ are continuous functions for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, p\}$. Various types of boundary value problems were studied by many authors using fixed point theorems on cones, fixed point index theory, upper and lower solutions method, differential inequality, topological transversality and Leggett-Williams fixed point theorem [2], [3], [4], [5], [6], [7], [8], [11].

In [2], Yao and Feng used the upper and lower solutions method to prove some existence results for the following third-order two-point boundary value problem:

$$(1.3) \qquad u'''(t) + f(t, u(t)) = 0, \ 0 < t < 1$$

$$(1.4) \qquad u(0) = u'(0) = u'(1) = 0.$$

In [3], Sanyang Liu and Yuqiang Feng used the upper and lower solutions method and a new maximum principle to prove the existence of some solutions to the more general third-order two-point boundary value problem:

$$(1.5) \qquad u'''(t) + f(t, u(t), u'(t)) = 0, \ 0 < t < 1$$

$$(1.6) \qquad u(0) = u'(0) = u'(1) = 0.$$

In [8], Guo, Sun and Zhao studied the third-order three-point boundary value problem:

$$(1.7) \qquad u'''(t) + a(t)g(u(t)) = 0, \ 0 < t < 1$$

$$(1.8) \qquad u(0) = u'(0) = 0, \ u'(1) = \alpha u'(\eta),$$

where $0 < \eta < 1$, $1 < \alpha < \frac{1}{\eta}$ and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given function. The existence of at least one positive solution for (1.7)–(1.8) was proved when f is superlinear or sublinear using fixed point theorems in cones.

Zhang et al. [11] investigated the existence of positive solutions for the following third-order eigenvalue problem:

$$(1.9) \qquad u'''(t) + \lambda f(t, u(t), u'(t)) = 0, \ 0 < t < 1$$

$$(1.10) \qquad u(0) = u'(\eta) = u''(0) = 0.$$

In [7], Sun studied the following third-order nonhomogeneous boundary value problem:

$$(1.11) \qquad u''' + a(t)g(t, u(t)) = 0, \ 0 < t < 1$$

$$(1.12) \qquad u(0) = u'(0) = 0, u'(1) - \alpha u'(\eta) = \lambda.$$

Using the Guo-Krasnosel'skii fixed point theorem and Schauder's fixed point theorem, Sun investigated the existence and nonexistence of positive solutions for (1.11)–(1.12). For more knowledge about boundary value problem, we refer the reader to [12]–[23].

Our aim in this paper is to use the Guo-Krasnosel'skii fixed point theorem to prove the existence of at least one positive solution of our problem. To this end, we formulate the boundary value problem as a fixed point problem. The particularity of our method is in establishing the equation (1.1)–(1.2) so that the boundary conditions involve multipoint integral boundary conditions. Our work is new and more general than [7], [8]. For example, (1.7)–(1.8) is established for the following case $n = 1$, $p = 1$, $f_1(t, u(t), u'(t)) = a(t)g(u(t))$, $h_1 \equiv 0$, $\eta_{1,1} = \eta$, $\beta_{1,1} = \alpha$ and (1.11)–(1.12) in the case $n = 1$, $p = 1$, $h_1 \equiv \lambda$, $\eta_{1,1} = \eta$, $\beta_{1,1} = \alpha$, $f_1(t, u(t), u'(t)) = a(t)g(t, u(t))$.

This paper is organized as follows. In Section 2 we present some useful preliminaries to prove our results. Section 3 studies the positivity of solutions using Guo-Krasnosel'skii fixed point theorem. Section 4 studies the nonexistence of positive solutions. Finally, we give some examples to illustrate our main results.

2. Preliminaries and lemmas

Let $E = (C^1([0, 1]; \mathbb{R}))^n$ equipped with the norm $\|u\|_E = \sum_{i=1}^n \|u_i\|$ where $\|u_i\| = \max(\|u_i\|_\infty, \|u_i'\|_\infty)$, $u = (u_1, \dots, u_n) \in E$. The space E is then a Banach space. We denote $\|x\|_1 = \sum_{i=1}^n |x_i|$ for $x \in \mathbb{R}^n$, $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n; x_1 \in \mathbb{R}_+, \dots, x_n \in \mathbb{R}_+\}$. We assume that for all $i \in \{1, \dots, n\}$, $0 < \sum_{j=1}^p \beta_{j,i}\eta_{j,i} < 1$.

Definition 2.1. The function $u = (u_1, \dots, u_n)$ is called a nonnegative (resp. positive) solution of the system (1.1)–(1.2) if and only if u satisfies (1.1)–(1.2) and for all $i \in \{1, \dots, n\}$, $u_i(t) \geq 0$ for all $t \in [0, 1]$ (resp. $u_i(t) > 0$ for all $t \in]0, 1[$).

Lemma 2.2. *Let $i \in \{1, \dots, n\}$, g_i and $h_i \in C([0, 1])$, then the problem*

$$\begin{cases} u_i'''(t) + h_i(t) = 0, \ 0 < t < 1 \\ u_i(0) = 0 \\ u_i'(0) = 0 \\ u_i'(1) = \sum_{j=1}^p \beta_{j,i} u_i'(\eta_{j,i}) + \int_0^1 g_i(s)\, ds \end{cases} \tag{2.1}$$

has a unique solution $u = (u_1, \dots, u_n)$ in E such that

$$u_i(t) = \int_0^1 H_i(t, s) h_i(s)\, ds + \varphi_i(t),$$

where

$$\varphi_i(t) = \left[K_i \int_0^1 g_i(s)\, ds\right] t^2, \tag{2.2}$$

$$H_i(t,s) = G(t,s) + K_i t^2 \sum_{j=1}^{p} \beta_{j,i} \frac{\partial G(\eta_{j,i}, s)}{\partial t} \tag{2.3}$$

and

$$G(t,s) = \frac{1}{2} \begin{cases} (1-s)t^2 & \text{if } 0 \le t \le s, \\ (-s+2t-t^2)s & \text{if } s \le t \le 1, \end{cases} \tag{2.4}$$

$$K_i = \frac{1}{2\left(1 - \sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}\right)} \ge \frac{1}{2}\,. \tag{2.5}$$

Proof: Integrating the equation (2.1), it yields for all $i \in \{1, \dots, n\}$ that

$$u_i(t) = -\frac{1}{2}\int_0^t (t-s)^2 h_i(s)\,ds + C_{1,i}t^2 + C_{2,i}t + C_{3,i}.$$

From the boundary condition $u_i(0) = 0$, we deduce that $C_{3,i} = 0$ and from the boundary condition $u_i'(0) = 0$, we deduce that $C_{2,i} = 0$. From the condition $u_i'(1) = \sum_{j=1}^{p} \beta_{j,i} u_i'(\eta_{j,i}) + \int_0^1 g_i(s)\,ds$, we have

$$\begin{aligned} C_{1,i} = {} & \frac{1}{2\left(1 - \sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}\right)} \int_0^1 (1-s)h_i(s)\,ds - \frac{\sum_{j=1}^{p} \beta_{j,i}}{2\left(1 - \sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}\right)} \\ & \times \int_0^{\eta_{j,i}} (\eta_{j,i} - s)h_i(s)\,ds + \frac{1}{2\left(1 - \sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}\right)} \int_0^1 g_i(s)\,ds. \end{aligned}$$

Therefore

$$\begin{aligned} u_i(t) = {} & -\frac{1}{2}\int_0^t (t-s)^2 h_i(s)\,ds + \frac{t^2}{2\left(1 - \sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}\right)} \int_0^1 (1-s)h_i(s)\,ds \\ & - \frac{t^2 \sum_{j=1}^{p} \beta_{j,i}}{2\left(1 - \sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}\right)} \int_0^{\eta_{j,i}} (\eta_{j,i} - s)h_i(s)\,ds + \frac{\int_0^1 g_i(s)\,ds}{2\left(1 - \sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}\right)}\, t^2 \\ = {} & -\frac{1}{2}\int_0^t (t-s)^2 h_i(s)\,ds + \frac{t^2}{2}\int_0^1 (1-s)h_i(s)\,ds \\ & + \frac{t^2 \sum_{j=1}^{p} \beta_{j,i}}{2\left(1 - \sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}\right)} \int_0^1 \eta_{j,i}(1-s)h_i(s)\,ds - \frac{t^2 \sum_{j=1}^{p} \beta_{j,i}}{2\left(1 - \sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}\right)} \\ & \times \int_0^{\eta_{j,i}} (\eta_{j,i} - s)h_i(s)\,ds + \frac{\int_0^1 g_i(s)\,ds}{2\left(1 - \sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}\right)}\, t^2, \end{aligned}$$

and we conclude that

$$u_i(t) = \int_0^1 G(t,s)h_i(s)\,ds + \frac{t^2}{2\left(1-\sum_{j=1}^{p}\beta_{j,i}\eta_{j,i}\right)}\sum_{j=1}^{p}\beta_{j,i}\int_0^1 \frac{\partial G(\eta_{j,i},s)}{\partial t}h_i(s)\,ds$$
$$+\varphi_i(t) = \int_0^1 H_i(t,s)h_i(s)\,ds + \varphi_i(t)$$

where $H_i(t,s)$ and $\varphi_i(t)$ are given by (2.2) and (2.3), which achieve the proof of Lemma 2.2. $\square$

We denote by T the operator defined by

$$\begin{array}{rcccl} T & : & E & \longrightarrow & E \\ & & u & \longmapsto & (T_1(u),\ldots,T_n(u)), \end{array}$$

where for all $i \in \{1,\ldots,n\}$ and for all $t \in [0,1]$

$$T_i(u)(t) = P_i(t) + \int_0^1 H_i(t,s)f_i(s,u(s),u'(s))\,ds$$

and

$$P_i(t) = \left[K_i\int_0^1 h_i(s,u_1(s),\ldots,u_n(s))\,ds\right]t^2.$$

Then we have

Lemma 2.3. *Let $i \in \{1,\ldots,n\}$, $h_i \in C([0,1]\times\mathbb{R}^n,\mathbb{R})$ and $f_i \in C([0,1]\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R})$. Then u is a solution of (1.1)–(1.2) if and only if for all $t\in[0,1]$, $T(u)(t) = u(t)$.*

3. Existence of positive solutions

In this section, we will give some preliminary considerations and some lemmas which are essential to establish sufficient conditions for the existence of at least one positive solution for our problem. We make the following additional assumption:

(H1) The functions $h_i : [0,1]\times\mathbb{R}^n \to [0,+\infty[$, $f_i : [0,1]\times\mathbb{R}^n\times\mathbb{R}^n \to [0,+\infty[$, $i\in\{1,\ldots,n\}$ are continuous.

Now, we need some properties of the Green function $G(t,s)$.

Lemma 3.1. *For all $t\in[0,1]$, for all $s\in[0,1]$ we have*

(1) $0 \le G(t,s) \le \varphi(s)$,
(2) $0 \le \frac{\partial G(t,s)}{\partial t} \le 2\varphi(s)$,

where $\varphi(s) = \frac{(1-s)s}{2}$.

Proof: It is easy to see that, if $t \leq s$, $G(t,s) = \frac{1}{2}(1-s)t^2 \geq 0$ and $G(t,s) = \frac{1}{2}(1-s)t^2 \leq \frac{(1-s)s}{2}$. If $s \leq t$, $G(t,s) = \frac{1}{2}(2t - t^2 - s)s = \frac{1}{2}[(1-s)-(1-t)^2]s \geq 0$ and $G(t,s) \leq \frac{1}{2}(1-s)(1-1+s) = \frac{1}{2}(1-s)s$ then the proof of (1) is complete.

We have

$$\frac{\partial G(t,s)}{\partial t} = \begin{cases} (1-s)t & \text{if } \ 0 \leq t \leq s, \\ (1-t)s & \text{if } \ s \leq t \leq 1. \end{cases} \tag{3.1}$$

If $t \leq s$, $\frac{\partial G(t,s)}{\partial t} = (1-s)t \leq (1-s)s$. If $s \leq t$, then $-t \leq -s$ and this implies that $\frac{\partial G(t,s)}{\partial t} = (1-t)s \leq (1-s)s$. We deduce that the proof of (2) is complete. □

Lemma 3.2. *Let $a \in]0,1[$, then for all $(t,s) \in [a,1] \times [0,1]$*

(1) $G(t,s) \geq a^2 \varphi(s)$,

(2) *for all $i \in \{1,\dots,n\}$, $j \in \{1,\dots,p\}$,* $\frac{\partial G(\eta_{j,i},s)}{\partial t} \geq 2\eta_{j,i}\varphi(s)$,

where $\varphi(s) = \frac{(1-s)s}{2}$.

Proof: Let $a \in]0,1[$ and $(t,s) \in [a,1] \times [0,1]$. If $s \leq t$ then

$$\begin{aligned} G(t,s) &= \frac{1}{2}(-s + 2t - t^2)s \\ &= \frac{1}{2}(2t - t^2 - s + t^2 - t^2 + t^2 s - t^2 s)s \\ &\geq \frac{1}{2}t^2(1-s)s + \frac{1}{2}(1-t)[(t-s) + (1-s)t]s \\ &\geq \frac{1}{2}t^2(1-s)s \\ &\geq a^2\varphi(s). \end{aligned}$$

If $s \geq t$ then

$$G(t,s) = \frac{1}{2}(1-s)t^2 \geq \frac{1}{2}(1-s)sa^2 = a^2\varphi(s).$$

Then the proof of (1) is complete. Now we prove the inequality for

$$\frac{\partial G(\eta_{j,i},s)}{\partial t} = \begin{cases} (1-s)\eta_{j,i} & \text{if } \ \eta_{j,i} \leq s \leq 1, \\ (1-\eta_{j,i})s & \text{if } \ 0 \leq s \leq \eta_{j,i}. \end{cases} \tag{3.2}$$

If $\eta_{j,i} \leq s$ then

$$\begin{aligned} \frac{\partial G(t,s)}{\partial t} &= (1-s)\eta_{j,i} \\ &\geq (1-s)s\eta_{j,i} \\ &\geq 2\eta_{j,i}\varphi(s). \end{aligned}$$

If $\eta_{j,i} \geq s$ then
$$\frac{\partial G(\eta_{j,i}, s)}{\partial t} = (1 - \eta_{j,i})s.$$
From $0 < \eta_{1,i} < \eta_{2,i} < \cdots < \eta_{p,i} < \frac{1}{2}$ we deduce that $1 - \eta_{j,i} \geq \eta_{j,i}$, then $\frac{\partial G(\eta_{j,i},s)}{\partial t} \geq 2\eta_{j,i}\varphi(s)$. We conclude that the proofs of (2) and Lemma 3.2 are complete. □

Lemma 3.3. *Suppose that* (H1) *holds and let* $a \in]0,1[$, *then the solution* $u = (u_1, \ldots, u_n)$ *of the problem* (1.1)–(1.2) *is nonnegative and satisfies*
$$\min_{t\in[a,1]} \sum_{i=1}^{n} (u_i(t) + u_i'(t)) \geq \gamma(a)\|u\|_E$$
where
$$\gamma(a) = \frac{\max_{i\in\{1,\ldots,n\}} \gamma_i(a)}{n}$$
and
$$\gamma_i(a) = \frac{a^2 \sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}}{2\left(1 + 2\max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right)}.$$

Proof: Suppose that $u = (u_1, \ldots, u_n)$ is a solution of (1.1)–(1.2), then from Lemma 2.3, (H1) and $G(t,s) \geq 0$ on $[0,1] \times [0,1]$, it is obvious that for all $i \in \{1, \ldots, n\}$ and for all $t \in [0,1]$, $u_i(t) \geq 0$. For all $i \in \{1, \ldots, n\}$ and for all $t \in [0,1]$, we have
$$\begin{aligned}
|u_i(t)| &\leq \int_0^1 H_i(t,s) f_i(s, u(s), u'(s))\, ds + P_i(t) \\
&\leq \left(1 + 2K_i \sum_{j=1}^{p} \beta_{j,i}\right) \int_0^1 \varphi(s) f_i(s, u(s), u'(s))\, ds \\
&\quad + K_i \int_0^1 h_i(u_1(s), \ldots, u_n(s))\, ds \\
&\leq \left(1 + 2\max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right) \int_0^1 \varphi(s) f_i(s, u(s), u'(s))\, ds \\
&\quad + K_i \int_0^1 h_i(u_1(s), \ldots, u_n(s))\, ds.
\end{aligned}$$

This implies that for all $i \in \{1, \ldots, n\}$
$$\|u_i\|_\infty \leq \left(1 + 2\max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right) \int_0^1 \varphi(s) f_i(s, u(s), u'(s))\, ds$$

$$+ K_i \int_0^1 h_i(u_1(s), \dots, u_n(s))\, ds,$$

and for all $i \in \{1, \dots, n\}$ and for all $t \in [0,1]$, we have

$$\begin{aligned}
|u_i'(t)| &\leq \int_0^1 \frac{\partial H_i(t,s)}{\partial t} f_i(s, u(s), u'(s))\, ds + P_i'(t) \\
&\leq 2\left(1 + 2K_i \sum_{j=1}^{p} \beta_{j,i}\right) \int_0^1 \varphi(s) f_i(s, u(s), u'(s))\, ds \\
&\quad + 2K_i \int_0^1 h_i(u_1(s), \dots, u_n(s))\, ds \\
&\leq 2\left(1 + 2 \max_{i \in \{1,\dots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right) \int_0^1 \varphi(s) f_i(s, u(s), u'(s))\, ds \\
&\quad + 2K_i \int_0^1 h_i(u_1(s), \dots, u_n(s))\, ds.
\end{aligned}$$

This implies that for all $i \in \{1, \dots, n\}$

$$\begin{aligned}
\|u_i'\|_\infty &\leq 2\left(1 + 2 \max_{i \in \{1,\dots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right) \int_0^1 \varphi(s) f_i(s, u(s), u'(s))\, ds \\
&\quad + 2K_i \int_0^1 h_i(u_1(s), \dots, u_n(s))\, ds.
\end{aligned}$$

Using Lemma 3.2 we obtain for all $i \in \{1, \dots, n\}$ and for all $t \in [a,1]$

$$\begin{aligned}
u_i(t) &\geq a^2 \int_0^1 \varphi(s) f_i(s, u(s), u'(s))\, ds + a^2 K_i \int_0^1 h_i(s, u_1(s), \dots, u_n(s))\, ds \\
&\geq \frac{a^2}{\left(1 + 2\max_{i \in \{1,\dots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right)} \left[\left(1 + 2 \max_{i \in \{1,\dots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right)\right. \\
&\quad \times \int_0^1 \varphi(s) f_i(s, u(s), u'(s))\, ds + K_i \left(1 + 2 \max_{i \in \{1,\dots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right) \\
&\quad \left. \times \int_0^1 h_i(s, u_1(s), \dots, , u_n(s))\, ds\right] \\
&\geq \frac{a^2}{\left(1 + 2\max_{i \in \{1,\dots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right)} \left[\left(1 + 2 \max_{i \in \{1,\dots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right)\right.
\end{aligned}$$

$$\times \int_0^1 \varphi(s) f_i(s, u(s), u'(s))\, ds + K_i \int_0^1 h_i(s, u_1(s), \ldots, , u_n(s))\, ds\Big]$$
$$\geq \frac{a^2 \sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}}{2\left(1 + 2\max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right)} \|u_i\|_\infty.$$

Then for all $i \in \{1, \ldots, n\}$

$$\min_{t\in[a,1]} u_i(t) \geq \frac{a^2 \sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}}{2\left(1 + 2\max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right)} \|u_i\|_\infty.$$

Similarly for all $i \in \{1, \ldots, n\}$ and for all $t \in [a, 1]$

$$\begin{aligned}
u_i'(t) &\geq 2aK_i \sum_{j=1}^{p} \beta_{j,i}\eta_{j,i} \int_0^1 \varphi(s) f_i(s, u(s), u'(s))\, ds \\
&\quad + 2aK_i \sum_{j=1}^{p} \beta_{j,i}\eta_{j,i} \int_0^1 h_i(s, u_1(s), \ldots, u_n(s))\, ds \\
&\geq a\frac{\sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}}{2\left(1 + 2\max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right)} \left[2\left(1 + 2\max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right)\right. \\
&\quad \times \int_0^1 \varphi(s) f_i(s, u(s), u'(s))\, ds + 2\left(1 + 2\max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right) K_i \\
&\quad \left.\times \int_0^1 h_i(s, u_1(s), \ldots, u_n(s))\, ds\right] \\
&\geq a\frac{\sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}}{2\left(1 + 2\max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right)} \left[2\left(1 + 2\max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right)\right. \\
&\quad \left.\times \int_0^1 \varphi(s) f_i(s, u(s), u'(s))\, ds + 2K_i \int_0^1 h_i(s, u_1(s), \ldots, u_n(s))\, ds\right] \\
&\geq a^2\frac{\sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}}{2\left(1 + 2\max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right)} \|u_i'\|_\infty.
\end{aligned}$$

Then for all $i \in \{1, \ldots, n\}$

$$\min_{t\in[a,1]} u_i'(t) \geq \frac{a^2 \sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}}{2\left(1 + 2\max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right)} \|u_i'\|_\infty.$$

Then for all $i \in \{1, \ldots, n\}$,

$$\begin{aligned}\min_{t\in[a,1]}(u_i(t)+u_i'(t)) &\geq \frac{a^2\sum_{j=1}^p \beta_{j,i}\eta_{j,i}}{2\left(1+2\max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^p \beta_{j,i}\right)}\|u_i\| \\ &= \gamma_i(a)\|u_i\|,\end{aligned}$$

we deduce that

$$\begin{aligned}\min_{t\in[a,1]}\sum_{i=1}^n(u_i(t)+u_i'(t)) &\geq \sum_{i=1}^n \gamma_i(a)\|u_i\| \\ &\geq \max_{j\in\{1,\ldots,n\}}[\gamma_j(a)\times\|u_j\|] \\ &\geq \max_{j\in\{1,\ldots,n\}}\gamma_j(a)\times\max_{j\in\{1,\ldots,n\}}\|u_j\|.\end{aligned}$$

This implies that for all $j \in \{1, \ldots, n\}$,

$$\min_{t\in[a,1]}\sum_{i=1}^n(u_i(t)+u_i'(t)) \geq \max_{i\in\{1,\ldots,n\}}\gamma_i(a)\times\|u_j\|.$$

Then

$$\begin{aligned}\min_{t\in[a,b]}\sum_{i=1}^n(u_i(t)+u_i'(t)) &\geq \frac{\max_{i\in\{1,\ldots,n\}}\gamma_i(a)}{n}\|u\|_E \\ &= \gamma(a)\|u\|_E.\end{aligned}$$

The proof is complete. □

Definition 3.4. We denote by E^+ the following set:

$$E^+ = \{u = (u_1, \ldots, u_n) \in E\,,\ u_i(t) \geq 0\,,\ t \in [0,1], i \in \{1, \ldots, n\}\}.$$

Definition 3.5. Let E be a Banach space. A nonempty closed convex $K \subset E$ is called a cone if it satisfies the following two conditions:

(1) $x \in K$, $\lambda \geq 0$ implies $\lambda x \in K$,
(2) $x \in K$, $-x \in K$ implies $x = 0$.

Remark 3.6. For all $a \in]0,1[$, the set defined by

$$K(a) = \left\{u \in E^+\,,\ \min_{t\in[a,1]}\sum_{i=1}^n(u_i(t)+u_i'(t)) \geq \gamma(a)\|u\|_E\right\}$$

is a cone of E.

Theorem 3.7 (Guo-Krasnosel'skii fixed point theorem [9])**.** *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1 and Ω_2 are two bounded subsets* *of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$ and let $A : K \cap (\overline{\Omega_2} \backslash \Omega_1) \to K$ be a completely continuous operator such that:*

(1) $\|A(u)\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|A(u)\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$ *or*
(2) $\|A(u)\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|A(u)\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then A has a fixed point in $K \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Now, we give the following assumptions.

(H2) For all $i \in \{1, \ldots, n\}$, $h_i \in C([0,1] \times \mathbb{R}_+^n, \mathbb{R}_+)$ and $f_i \in C([0,1] \times \mathbb{R}_+^n \times \mathbb{R}^n, \mathbb{R}_+)$.
(H3) For all $i \in \{1, \ldots, n\}$, there exists $M_i \geq 0$ such that $h_i(t,u) \leq M_i$ for all $t \in [0,1]$ and for all $u \in \mathbb{R}_+^n$.

Theorem 3.8. *Suppose that* (H2) *and* (H3) *hold. Then the problem* (1.1)–(1.2) *has at least one nonnegative solution if for all $i \in \{1, \ldots, n\}$,*

$$\lim_{\|u\|_1+\|v\|_1 \to 0} \min_{t\in[0,1]} \frac{f_i(t,u,v)}{\|u\|_1 + \|v\|_1} = +\infty$$

and

$$\lim_{\|u\|_1+\|v\|_1 \to +\infty} \max_{t\in[0,1]} \frac{f_i(t,u,v)}{\|u\|_1 + \|v\|_1} = 0$$

where $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$.

Proof: **Step 1.** Based on Remark 3.6, there exists $\alpha \in]0,1[$ such that
$K(\alpha) = \left\{u \in E^+ ,\ \min_{t\in[\alpha,1]} \sum_{i=1}^n (u_i(t) + u_i'(t)) \geq \gamma(\alpha)\|u\|_E\right\}$
is a cone of E. By Arzela-Ascoli theorem [10], $T : K(\alpha) \to E$ is a completely continuous mapping.

We will show that $T(K(\alpha)) \subset K(\alpha)$. In fact, for all $t \in [0,1]$, $s \in [0,1]$, $G(t,s) \geq 0$. From (H2), we deduce that for all $i \in \{1, \ldots, n\}$, for all $u \in K(\alpha)$, for all $t \in [0,1]$, $T_i(u)(t) \geq 0$.

For all $i \in \{1, \ldots, n\}$, for all $t \in [0,1]$, we have

$$\begin{aligned}
|T_i(u)(t)| &\leq \int_0^1 H_i(t,s) f_i(s,u(s),u'(s))\,ds + P_i(t) \\
&\leq \left(1 + 2K_i \sum_{j=1}^p \beta_{j,i}\right) \int_0^1 \varphi(s) f_i(s,u(s),u'(s))\,ds \\
&\quad + K_i \int_0^1 h(u_1(s), \ldots, u_n(s))\,ds \\
&\leq \left(1 + 2 \max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^p \beta_{j,i}\right) \int_0^1 \varphi(s) f_i(s,u(s),u'(s))\,ds \\
&\quad + K_i \int_0^1 h(u_1(s), \ldots, u_n(s))\,ds.
\end{aligned}$$

This implies that, for all $i \in \{1, \ldots, n\}$

$$\|T_i(u)\|_\infty \leq \left(1 + 2 \max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right) \int_0^1 \varphi(s) f_i(s, u(s), u'(s))\, ds$$
$$+ K_i \int_0^1 h(u_1(s), \ldots, u_n(s))\, ds$$

and for all $i \in \{1, \ldots, n\}$, for all $t \in [0, 1]$, we have

$$\begin{aligned}
|T_i(u)'(t)| &\leq \int_0^1 \frac{\partial H_i(t,s)}{\partial t} f_i(s, u(s), u'(s))\, ds + P_i'(t) \\
&\leq 2\left(1 + 2K_i \sum_{j=1}^{p} \beta_{j,i}\right) \int_0^1 \varphi(s) f_i(s, u(s), u'(s))\, ds \\
&\quad + 2K_i \int_0^1 h_i(u_1(s), \ldots, u_n(s))\, ds \\
&\leq 2\left(1 + 2 \max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right) \int_0^1 \varphi(s) f_i(s, u(s), u'(s))\, ds \\
&\quad + 2K_i \int_0^1 h_i(u_1(s), \ldots, u_n(s))\, ds.
\end{aligned}$$

This implies that for all $i \in \{1, \ldots, n\}$

$$\|T_i(u)'\|_\infty \leq 2\left(1 + 2 \max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right) \int_0^1 \varphi(s) f_i(s, u(s), u'(s))\, ds$$
$$+ 2K_i \int_0^1 h_i(u_1(s), \ldots, u_n(s))\, ds.$$

Using Lemma 3.2 we obtain, for all $i \in \{1, \ldots, n\}$, for all $t \in [\alpha, 1]$

$$\begin{aligned}
T_i(u)(t) &\geq \alpha^2 \int_0^1 \varphi(s) f_i(s, u(s), u'(s))\, ds + \alpha^2 K_i \int_0^1 h_i(s, u_1(s), \ldots, u_n(s))\, ds \\
&\geq \frac{\alpha^2}{\left(1 + 2\max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right)} \left[\left(1 + 2 \max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right)\right. \\
&\quad \times \int_0^1 \varphi(s) f_i(s, u(s), u'(s))\, ds + K_i \left(1 + 2 \max_{i\in\{1,\ldots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right) \\
&\quad \left. \times \int_0^1 h_i(s, u_1(s), \ldots, , u_n(s))\, ds\right]
\end{aligned}$$

$$\geq \frac{\alpha^2}{\left(1+2\max_{i\in\{1,\dots,n\}} K_i \sum_{j=1}^p \beta_{j,i}\right)} \left[\left(1+2\max_{i\in\{1,\dots,n\}} K_i \sum_{j=1}^p \beta_{j,i}\right)\right.$$
$$\left. \times \int_0^1 \varphi(s) f_i(s,u(s),u'(s))\,ds + K_i \int_0^1 h_i(s,u_1(s),\dots,u_n(s))\,ds\right]$$
$$\geq \frac{\alpha^2}{\left(1+2\max_{i\in\{1,\dots,n\}} K_i \sum_{j=1}^p \beta_{j,i}\right)} \|T_i(u)\|_\infty$$
$$\geq \frac{\alpha^2 \sum_{j=1}^p \beta_{j,i}\eta_{j,i}}{2\left(1+2\max_{i\in\{1,\dots,n\}} K_i \sum_{j=1}^p \beta_{j,i}\right)} \|T_i(u)\|_\infty.$$

Then for all $i \in \{1,\dots,n\}$

$$\min_{t\in[\alpha,1]} T_i(u)(t) \geq \frac{\alpha^2 \sum_{j=1}^p \beta_{j,i}\eta_{j,i}}{2\left(1+2\max_{i\in\{1,\dots,n\}} K_i \sum_{j=1}^p \beta_{j,i}\right)} \|T_i(u)\|_\infty.$$

Similarly for all $i \in \{1,\dots,n\}$, for all $t \in [\alpha,1]$

$$T_i(u)'(t) \geq 2\alpha K_i \sum_{j=1}^p \beta_{j,i}\eta_{j,i} \int_0^1 \varphi(s) f_i(s,u(s),u'(s))\,ds$$
$$+ 2\alpha K_i \int_0^1 h_i(s,u_1(s),\dots,u_n(s))\,ds$$
$$\geq \alpha \frac{\sum_{j=1}^p \beta_{j,i}\eta_{j,i}}{2\left(1+2\max_{i\in\{1,\dots,n\}} K_i \sum_{j=1}^p \beta_{j,i}\right)} \left[2\left(1+2\max_{i\in\{1,\dots,n\}} K_i \sum_{j=1}^p \beta_{j,i}\right)\right.$$
$$\times \int_0^1 \varphi(s) f_i(s,u(s),u'(s))\,ds + 2\left(1+2\max_{i\in\{1,\dots,n\}} K_i \sum_{j=1}^p \beta_{j,i}\right) K_i$$
$$\left. \times \int_0^1 h_i(s,u_1(s),\dots,u_n(s))\,ds\right]$$
$$\geq \alpha \frac{\sum_{j=1}^p \beta_{j,i}\eta_{j,i}}{2\left(1+2\max_{i\in\{1,\dots,n\}} K_i \sum_{j=1}^p \beta_{j,i}\right)} \left[2\left(1+2\max_{i\in\{1,\dots,n\}} K_i \sum_{j=1}^p \beta_{j,i}\right)\right.$$
$$\left. \times \int_0^1 \varphi(s) f_i(s,u(s),u'(s))\,ds + 2K_i \int_0^1 h_i(s,u_1(s),\dots,u_n(s))\,ds\right]$$
$$\geq \alpha^2 \frac{\sum_{j=1}^p \beta_{j,i}\eta_{j,i}}{2\left(1+2\max_{i\in\{1,\dots,n\}} K_i \sum_{j=1}^p \beta_{j,i}\right)} \|T_i(u)'\|_\infty.$$

Then for all $i \in \{1, \dots, n\}$

$$\min_{t\in[\alpha,1]} T_i(u)'(t) \geq \alpha^2 \frac{\sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}}{2\left(1 + 2\max_{i\in\{1,\dots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right)} \|T_i(u)'\|_\infty.$$

Then for all $i \in \{1, \dots, n\}$,

$$\begin{aligned}\min_{t\in[a,1]} (T_i(u)(t) + T_i(u)'(t)) &\geq \frac{\alpha^2 \sum_{j=1}^{p} \beta_{j,i}\eta_{j,i}}{2\left(1 + 2\max_{i\in\{1,\dots,n\}} K_i \sum_{j=1}^{p} \beta_{j,i}\right)} \|T_i(u)\| \\ &= \gamma_i(\alpha)\|T_i(u)\|,\end{aligned}$$

and we deduce that

$$\begin{aligned}\min_{t\in[\alpha,1]} \sum_{i=1}^{n} (T_i(u)(t) + T_i(u)'(t)) &\geq \sum_{i=1}^{n} \gamma_i(\alpha)\|T_i(u)\| \\ &\geq \max_{j\in\{1,\dots,n\}} [\gamma_j(\alpha) \times \|T_j(u)\|] \\ &\geq \max_{j\in\{1,\dots,n\}} \gamma_j(\alpha) \times \max_{j\in\{1,\dots,n\}} \|T_j(u)\|.\end{aligned}$$

This implies that for all $j \in \{1, \dots, n\}$,

$$\min_{t\in[\alpha,1]} \sum_{i=1}^{n} (T_i(u)(t) + T_i(u)'(t)) \geq \max_{i\in\{1,\dots,n\}} \gamma_i(\alpha) \times \|T_j(u)\|.$$

Then

$$\begin{aligned}\min_{t\in[\alpha,1]} \sum_{i=1}^{n} (T_i(u)(t) + T_i(u)'(t)) &\geq \frac{\max_{i\in\{1,\dots,n\}} \gamma_i(\alpha)}{n} \|T(u)\|_E \\ &= \gamma(\alpha)\|T(u)\|_E.\end{aligned}$$

The proof is complete. □

Step 2. Let $i \in \{1, \dots, n\}$ and we have

$$\lim_{\|u\|_1+\|v\|_1\to 0} \min_{t\in[0,1]} \frac{f_i(t,u,v)}{\|u\|_1 + \|v\|_1} = +\infty.$$

Then for all $M > 0$ there exists $R_{i,1} > 0$ such that $\min_{t\in[0,1]} f_i(t,u,v) \geq M(\|u\|_1 + \|v\|_1)$, for $\|u\|_1 + \|v\|_1 \leq R_{i,1}$. This implies that for all $M > 0$, there exists $R_{i,1} > 0$ such that for all $t \in [0,1]$, $f_i(t,u,v) \geq \min_{t\in[0,1]} f_i(t,u,v) \geq M(\|u\|_1 + \|v\|_1)$ for $\|u\|_1 + \|v\|_1 \leq R_{i,1}$. We choose

$$M = \frac{1}{n\,\gamma(\alpha)\,\int_\alpha^1 \varphi(s)\,ds}.$$

Let Ω_1 be an open bounded set in E defined by $\Omega_1 = \{u \in E, \|u\|_E < \min_{i\in\{1,\dots,n\}} R_{i,1}\}$. Then for all $u \in Con(\alpha) \cap \partial\Omega_1$, it yields for all $s \in [\alpha, 1]$,

$$\sum_{i=1}^{n}(u_i(s) + u_i'(s)) \geq \gamma(\alpha) \min_{i\in\{1,\dots,n\}} R_{i,1}.$$

Then for all $i \in \{1, \dots, n\}$, for all $s \in [\alpha, 1]$,

$$f_i(s, u(s), u'(s)) \geq M(\|u(s)\|_1 + \|u'(s)\|_1) \geq M\,\gamma(\alpha) \min_{i\in\{1,\dots,n\}} R_{i,1}.$$

This implies that, for all $i \in \{1, \dots, n\}$, for all $t \in [0, 1]$,

$$|T_i(u)(t)| \geq M \int_{\alpha}^{1} \varphi(s)\, ds\, \gamma(\alpha) \min_{i\in\{1,\dots,n\}} R_{i,1}$$

and for all $i \in \{1, \dots, n\}$, we have

$$\|T_i(u)\| \geq \|T_i(u)\|_\infty \geq \frac{\min_{i\in\{1,\dots,n\}} R_{i,1}}{n}.$$

We deduce that

$$\|T(u)\|_E \geq \min_{i\in\{1,\dots,n\}} R_{i,1} = \|u\|_E.$$

Step 3. Now, let $i \in \{1, \dots, n\}$ and we have

$$\lim_{\|u\|_1+\|v\|_1\to+\infty} \max_{t\in[0,1]} \frac{f_i(t, u, v)}{\|u\|_1 + \|v\|_1} = 0.$$

Then for all $\varepsilon > 0$, there exists $R_0^{f_i} > 0$ such that $\max_{t\in[0,1]} f_i(t, u, v) \leq \varepsilon(\|u\|_1 + \|v\|_1)$ for $\|u\|_1 + \|v\|_1 \geq R_0^{f_i}$. This implies that, for all $\varepsilon > 0$ there exists $R_0^{f_i} > 0$ such that for all $t \in [0, 1]$, $f_i(t, u, v) \leq \max_{t\in[0,1]} f_i(t, u, v) \leq \varepsilon(\|u\|_1 + \|v\|_1)$, for $\|u\|_1 + \|v\|_1 \geq R_0^{f_i}$.

We choose

$$\varepsilon = \frac{1}{4n\left(1 + 2K_i \sum_{j=1}^{p} \beta_{j,i}\right) \int_0^1 \varphi(s)\, ds}.$$

From (H3), for all $i \in \{1, \dots, n\}$ there exist $M_i \geq 0$ such that $0 \leq h_i(s, u_1(s), \dots, u_n(s)) \leq M_i$ for all $s \in [0, 1]$, $u \in E_+$.

Let

$$R_2 = \max\left\{2 \min_{i\in\{1,\dots,n\}} R_{i,1}\,,\ \max_{i\in\{1,\dots,n\}} R_0^{f_i}, 2n\left(\max_{i\in\{1,\dots,n\}} K_i M_i + 1\right),\right.$$
$$\left. 2n\left(2 \max_{i\in\{1,\dots,n\}} K_i M_i + 1\right)\right\}$$

and let Ω_2 be an open bounded set in E defined by $\Omega_2 = \{u \in E, \|u\|_E < R_2\}$. Then for all $u \in K(\alpha) \cap \partial\Omega_2$, we have for all $i \in \{1, \ldots, n\}$ and for all $s \in [0,1]$

$$f_i(s, u(s), u'(s)) \leq \max_{s\in[0,1]} f_i(s, u(s), u'(s)) \leq \varepsilon(\|u(s)\|_1 + \|u'(s)\|_1) \leq \varepsilon R_2.$$

Using Lemma 3.1 we have for all $i \in \{1, \ldots, n\}$, for all $t \in [0,1]$,

$$\begin{aligned}
|T_i(u)(t)| &\leq \int_0^1 H_i(t,s) f_i(s, u(s), u'(s))\, ds + P_i(t) \\
&\leq \varepsilon\, R_2 \left(1 + 2K_i \sum_{j=1}^{p} \beta_{j,i}\right) \int_0^1 \varphi(s)\, ds + \max_{i\in\{1,\ldots,n\}} K_i M_i + 1 \\
&\leq \frac{R_2}{4n} + \frac{R_2}{2n} \leq \frac{R_2}{n}.
\end{aligned}$$

Then for all $i \in \{1, \ldots, n\}$,

$$\|T_i(u)\|_\infty \leq \frac{R_2}{n}\,.$$

$$\begin{aligned}
|T_i(u)'(t)| &\leq \int_0^1 \frac{\partial H_i(t,s)}{\partial t} f_i(s, u(s), u'(s))\, ds + P_i'(t) \\
&\leq 2\varepsilon\, R_2 \left(1 + 2K_i \sum_{j=1}^{p} \beta_{j,i}\right) \int_0^1 \varphi(s) f(s, u(s), u'(s))\, ds \\
&\quad + 2 \max_{i\in\{1,\ldots,n\}} K_i M_i + 1 \\
&\leq \frac{R_2}{2n} + \frac{R_2}{2n} = \frac{R_2}{n}.
\end{aligned}$$

Then for all $i \in \{1, \ldots, n\}$,

$$\|T_i(u)'\|_\infty \leq \frac{R_2}{n}\,.$$

Therefore

$$\|T_i(u)\| \leq \frac{R_2}{n}\,.$$

We deduce that

$$\|T(u)\|_E \leq R_2 = \|u\|_E.$$

Step 4. Let $u \in \overline{\Omega_1}$ then $\|u\| \leq \min_{i\in\{1,\ldots,n\}} R_{i,1} < 2\min_{i\in\{1,\ldots,n\}} R_{i,1} \leq R_2$. This implies that $\|u\| < R_2$, then $u \in \Omega_2$. We deduce that $\overline{\Omega_1} \subset \Omega_2$. By Theorem 3.7, T has at least one fixed point in $K(\alpha) \cap (\overline{\Omega_2} \backslash \Omega_1)$. Then (1.1)–(1.2) has at least one nonnegative solution u.

Theorem 3.9. *Under assumptions of Theorem* 3.8 *and adding the following condition:*

(3.3) *For all* $i \in \{1, \dots, n\}$, *there exist* $t_{0,i} \in]0,1[$ *such that* $f_i(t_{0,i}, x, y) > 0$

for all $x \in \mathbb{R}^n_+$, *for all* $y \in \mathbb{R}^n$. *Then the problem* (1.1)–(1.2) *has at least one positive solution.*

PROOF: Consider the nonnegative solution u for problem (1.1)–(1.2) whose existence is guaranteed by Theorem 3.8. Notice that u_i satisfies for all $i \in \{1, \dots, n\}$, $u_i(t) = \int_0^1 H_i(t,s) f_i(s, u(s), u'(s))\, ds + P_i(t)$. From (H2) and condition (3.3), there exists $t_{0,i} \in [\alpha_i, \beta_i] \subset]0,1[$ such that for all $t \in [\alpha_i, \beta_i]$ and $x \in \mathbb{R}^n_+$, $y \in \mathbb{R}^n$, $f_i(t, x, y) > 0$. Then for all $i \in \{1, \dots, n\}$ and for all $t \in]0,1[$, $u_i(t) = \int_0^1 H_i(t,s) f_i(s, u(s), u'(s))\, ds + P_i(t) \geq \int_{\alpha_i}^{\beta_i} H_i(t,s) f_i(s, u(s), u'(s))\, ds > 0$. The proof is complete. □

4. Nonexistence of positive solutions

In this section, we give some sufficient conditions for the nonexistence of positive solutions. Define the following constants:

$$B_i = (1 + 2K_i \sum_{j=1}^{p} \beta_{j,i}) \int_0^1 \varphi(s)\, ds \text{ and } C(a) = \int_a^1 \varphi(s)\, ds.$$

Theorem 4.1. *Suppose that* (H2) *holds and there exists* $i_0 \in \{1, \dots, n\}$ *such that for all* $t \in [0,1]$, *for all* $x_{i_0} \in]0, +\infty[$, *for all* $i \in \{1, \dots, n\} \backslash \{i_0\}$, x_i, $y_i \in \mathbb{R}$

1. $K_{i_0} h_{i_0}(t, x_1, \dots, x_{i_0}, \dots, x_n) < \frac{x_{i_0}}{2}$,
2. $B_{i_0} f_{i_0}(t, x_1, \dots, x_{i_0}, \dots, x_n, y_1, \dots, y_n) < \frac{x_{i_0}}{2}$.

Then the problem (1.1)–(1.2) *has no positive solution.*

PROOF: Assume, to the contrary, that $u(t)$ is a positive solution of (1.1)–(1.2). We denote by $u(s) = (u_1(s), \dots, u_{i_0}(s), \dots, u_n(s))$ and $u'(s) = (u'_1(s), \dots, u'_{i_0}(s), \dots, u'_n(s))$. Then for all $s \in]0,1[$ we have

$$f_{i_0}(t, u(s), u'(s)) < \frac{u_{i_0}(s)}{4} B_{i_0}^{-1}.$$

Then for all $t \in [0,1]$ and for all $s \in]0,1[$ we have

$$H_{i_0}(t,s) f_{i_0}(s, u(s), u'(s)) < H_{i_0}(t,s) \frac{u_{i_0}(s) B_{i_0}^{-1}}{2}.$$

Multiplying this by $H_{i_0}(t,s)$ and integrating over $[0,1]$ we obtain

$$\int_0^1 H_{i_0}(t,s) f_{i_0}(s,u(s),u'(s))\,ds < \frac{B_{i_0}^{-1}\int_0^1 H_{i_0}(t,s)u_{i_0}(s)\,ds}{2} \leq \frac{\int_0^1 u_{i_0}(s)\,ds}{2}.$$

Since, for all $t \in [0,1]$, we have

$$\begin{aligned} P_{i_0}(t) &\leq K_{i_0}\int_0^1 h_{i_0}(s,u(s))\,ds \\ &\leq \frac{\int_0^1 u_{i_0}(s)\,ds}{2}. \end{aligned}$$

Then for all $t \in [0,1]$,

$$u_{i_0}(t) < \frac{1}{2}\int_0^1 u_{i_0}(s)\,ds + \frac{\int_0^1 u_{i_0}(s)\,ds}{2} = \int_0^1 u_{i_0}(s)\,ds.$$

By mean value theorem there exists $s_0 \in]0,1[$ such that $\int_0^1 u_{i_0}(s)\,ds = u_{i_0}(s_0)$ which is a contradiction. The proof is complete. □

Theorem 4.2. *Suppose that* (H2) *holds, there exist* $a \in]0,1[$ *and* $i_0 \in \{1,\dots,n\}$ *such that for all* $t \in [0,1]$, *for all* $x_{i_0} \in]0,+\infty[$, *for all* $i \in \{1,\dots,n\}\backslash\{i_0\}$, x_i, $y_i \in \mathbb{R}$, *we have*

$$C(a) f_{i_0}(t,x_1,\dots,x_{i_0},\dots,x_n,y_1,\dots,y_n) > x_{i_0}. \tag{4.1}$$

Then the problem (1.1)–(1.2) *has no positive solution.*

Proof: Assume, to the contrary, that $u(t)$ is a positive solution of (1.1)–(1.2). We denote by $u(s) = (u_1(s),\dots,u_{i_0}(s),\dots,u_n(s))$ and $u'(s) = (u_1'(s),\dots,u_{i_0}'(s),\dots,u_n'(s))$. Then for all $s \in]0,1[$ we have

$$f_{i_0}(s,u_1(s),\dots,u_{i_0}(s),\dots,u_n(s),u_1'(s),\dots,u_n'(s)) > u_{i_0}(s)C(a)^{-1}$$

Then for all $t \in [a,1]$ and for all $s \in]0,1[$ we have

$$H_{i_0}(t,s) f_{i_0}(s,u(s),u'(s)) > H_{i_0}(t,s)u_{i_0}(s)C(a)^{-1}.$$

Multiplying this by $H_{i_0}(t,s)$ and integrating over $[0,1]$ we obtain

$$\int_0^1 H_{i_0}(t,s) f_{i_0}(s,u(s),u'(s))\,ds > C(a)^{-1}\int_0^1 H_{i_0}(t,s)u_{i_0}(s)\,ds \geq \int_0^1 u_{i_0}(s)\,ds.$$

Then for all $t \in [a, 1]$

$$\begin{aligned} u_{i_0}(t) &= \int_0^1 H_{i_0}(t,s) f_{i_0}(s, u(s), u'(s))\, ds + P_{i_0}(t) \\ &\geq \int_0^1 H_{i_0}(t,s) f_{i_0}(s, u(s), u'(s))\, ds \\ &> \int_0^1 u_{i_0}(s)\, ds. \end{aligned}$$

By mean value theorem there exists $s_0 \in]a, 1[$ such that $\int_0^1 u_{i_0}(s)\, ds = u_{i_0}(s_0)$, which is a contradiction. The proof is complete. $\square$

Example 4.3. Consider the following system of boundary value problem:

$$(4.2) \qquad \begin{cases} u_1'''(t) + \frac{t^3+1}{\sqrt{(u_1'(t))^2+1}} + e^{-u_2(t)} = 0 \\ u_2'''(t) + e^{-t}\sqrt{|u_1(t)| + |u_2'(t)|} + e^{-u_2'(t)} = 0 \\ u_1(0) = u_1'(0) = u_2(0) = u_2'(0) = 0 \\ u_1'(1) = 2u_1'(\frac{1}{6}) + 3u_1'(\frac{1}{9}) + \int_0^1 \frac{u_2^2(s)}{1+u_2^2(s)+|u_1(s)|}\, ds \\ u_2'(1) = 2u_2'(\frac{1}{6}) + 3u_2'(\frac{1}{9}) + \int_0^1 \frac{|u_1(s)|}{\sqrt{1+u_1^2(s)+|u_2(s)|}}\, ds. \end{cases}$$

Let

$$\begin{aligned} f_1(t, x_1, x_2, y_1, y_2) &= \frac{t^3+1}{\sqrt{y_1^2+1}} + e^{-x_2}, \\ f_2(t, x_1, x_2, y_1, y_2) &= e^{-t}\sqrt{|x_1| + |x_2|} + e^{-y_2}, \\ h_1(t, x_1, x_2) &= \frac{x_2^2}{1 + x_2^2 + |x_1|}, \end{aligned}$$

and

$$h_2(t, x_1, x_2) = \frac{|x_1|}{\sqrt{1 + x_1^2 + |x_2|}}.$$

We can easily show that conditions (H2), (H3) and condition (3.3) are satisfied. Hence, by Theorem 3.9, this problem has at least one positive solution.

Example 4.4. Consider the following system of boundary value problem:

$$(4.3) \qquad \begin{cases} u_1'''(t) + (1 + \frac{128}{9}e^{u_1(t)})^2 + |u_2(t)| + (u_1'(t))^2 = 0 \\ u_2'''(t) + e^{2u_1(t)} + |u_2'(t)|^3 = 0 \\ u_1(0) = u_2(0) = u_1'(0) = u_2'(0) = 0 \\ u_1'(1) = 2u_1'(\frac{1}{6}) + 3u_1'(\frac{1}{9}) + 1 \\ u_2'(1) = 2u_2'(\frac{1}{6}) + 3u_1'(\frac{1}{9}). \end{cases}$$

We denote by $f_1(t, x_1, x_2, y_1, y_2) = (1 + \frac{128}{9}e^{x_1})^2 + |x_2| + y_1^2$, $a = \frac{1}{4}$, $C(a) = \int_a^1 \varphi(s)\, ds = \frac{9}{128}$ and $C(a) f_1(t, x_1, x_2, y_1, y_2) = \frac{9(1+\frac{128}{9}e^{x_1})^2 + |x_2| + y_1^2}{128} > x_1$.

By using Theorem 4.2, the problem (4.3) has no positive solution.

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