On finite commutative loops which are centrally nilpotent

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Abstract. Let Q be a finite commutative loop and let the inner mapping group $I(Q) \cong C_{p^n} \times C_{p^n}$, where p is an odd prime number and $n \geq 1$. We show that Q is centrally nilpotent of class two.

Keywords: loop; inner mapping group; centrally nilpotent loop

Classification: 20N05, 20D15

1. Introduction

If Q is a loop, then the mappings $L_a(x) = ax$ and $R_a(x) = xa$ are permutations on Q for every $a \in Q$. The permutation group $M(Q) = \langle L_a, R_a : a \in Q\rangle$ is called the *multiplication group* of Q and the stabilizer of the neutral element $e \in Q$ is denoted by $I(Q)$ and we say that $I(Q)$ is the *inner mapping group* of Q. The center $Z(Q)$ of a loop Q contains those elements $a \in Q$ which satisfy the equations $ax \cdot y = a \cdot xy$, $xa \cdot y = x \cdot ay$, $xy \cdot a = x \cdot ya$ and $ax = xa$ for every $x, y \in Q$. The center $Z(Q)$ is an abelian normal subloop of Q and $Z(Q) \cong Z(M(Q))$. If we write $Z_0 = 1$, $Z_1 = Z(Q)$ and $Z_i/Z_{i-1} = Z(Q/Z_{i-1})$, we obtain a series of normal subloops of Q. If Z_{n-1} is a proper subloop of Q and $Z_n = Q$, then Q is *centrally nilpotent of class* n.

In 1946 Bruck [1] showed that Q is centrally nilpotent of class at most two if and only if $N_{M(Q)}(I(Q)) = I(Q) \times Z(M(Q))$ is normal in $M(Q)$. As the core of $I(Q)$ in $M(Q)$ is trivial, it follows that if Q is centrally nilpotent of class at most two, then $I(Q)$ has to be an abelian group. In 1994 Niemenmaa and Kepka [7] managed to show that if Q is a finite loop and $I(Q)$ is abelian, then Q is a centrally nilpotent loop and for some time it was assumed that the converse of Bruck's result would hold: If $I(Q)$ is abelian, then Q is centrally nilpotent of class at most two. However, in 2007 Csörgő [2] gave a construction where Q is a loop of order 128, $I(Q)$ is an elementary abelian group of order 2^6 and Q is centrally nilpotent of class three. In 2008, Drápal and Vojtěchovský [3] gave more examples of loops of nilpotency class three with inner mapping groups which are elementary abelian of order 2^6, 2^9 and 2^{10}.

Now assume that $I(Q)$ is abelian. How does the structure of $I(Q)$ influence the nilpotency class of Q? In particular, we are interested in the following problem: Under which conditions imposed on $I(Q)$ does it follow that Q is centrally

DOI 10.14712/1213-7243.2015.113

nilpotent of class two? Kepka and Niemenmaa [7] have shown that if Q is a finite loop and $I(Q) \cong C_p \times C_p$, then Q is centrally nilpotent of class two (here p is a prime number and C_p denotes the cyclic group of order p). The purpose of this paper is to improve this result in the case that Q is a finite commutative loop and p is an odd prime number. We show that if $I(Q) \cong C_{p^n} \times C_{p^n}$ $(n \geq 1)$, then Q is centrally nilpotent of class two.

2. Connected transversals

Let G be a group, $H \leq G$ and let A and B be two left transversals to H in G. We say that A and B are H-*connected*, if $[A, B] \leq H$. If $A = B$, then A is a *selfconnected* transversal to H in G. We denote by H_G the *core* of H in G (the largest normal subgroup of G contained in H).

Let Q be a loop and write $A = \{L_a : a \in Q\}$ and $B = \{R_a : a \in Q\}$. Then A and B are $I(Q)$-connected transversals in $M(Q)$. Moreover, $M(Q) = \langle A, B\rangle$ and $I(Q)_{M(Q)} = 1$. In 1990, Niemenmaa and Kepka [6, Theorem 4.1] proved the following theorem, which gives the relation between loops and connected transversals.

Theorem 2.1. *A group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H and H-connected transversals A and B such that $H_G = 1$ and $G = \langle A, B\rangle$.*

In the following lemmas we assume that $H \leq G$ and A and B are H-connected transversals in G (that is, $a^{-1}b^{-1}ab \in H$ for every $a \in A$ and $b \in B$) and p is a prime number.

Lemma 2.2. *If $H_G = 1$, then $1 \in A \cap B$ and $N_G(H) = H \times Z(G)$.*

For the proof, see [6, Proposition 2.7]. In Lemmas 2.3–2.8 we further assume that $G = \langle A, B\rangle$.

Lemma 2.3. *If H is cyclic, then $G' \leq H$.*

Lemma 2.4. *If $H \cong C_p \times C_p$, then $G' \leq N_G(H)$.*

Lemma 2.5. *Let G be a finite group and $H \leq G$ an abelian p-group. If $H_G = 1$, then $Z(G) > 1$.*

Lemma 2.6. *If $H_G = 1$ and H is abelian, then the core of $HZ(G)$ in G contains $Z(G)$ as a proper subgroup.*

Lemma 2.7. *If G is finite and $H \cong C_{p^k} \times C_{p^l}$, where p is an odd prime and $k > l \geq 0$, then $H_G > 1$.*

For the proofs, see [4, Theorem 2.2], [7, Lemma 4.2], [8, Theorem 3.2] and [5, Lemma 2.7 and Theorem 3.1].

Lemma 2.8. *If $H > 1$ and $H_G = 1$, then $H \cap H^a > 1$ for each $a \in A \cup B$.*

Proof: Assume that $H \cap H^a = 1$ for some $a \in A$. Then $H \cap H^{a^{-1}} = 1$. If $aH = bH$ for some $b \in B$, then $b^{-1}a \in H$. Now $a^{-1}b^{-1}ab \in H$ and $b = ah$ for some $h \in H$, hence $a^{-1}b^{-1}aa \in H$. Then $b^{-1}a \in H \cap H^{a^{-1}} = 1$. Thus $a = b$ and $a \in A \cap B$.

If $d \in A \cup B$ and $c \in A \cup B$ such that $ad \in cH$, then $c^{-1}ad \in H$. Thus $c^{-1}adaH = c^{-1}aadH = c^{-1}acH = aa^{-1}c^{-1}acH = aH$, hence $a^{-1}c^{-1}ada \in H$. Thus $c^{-1}ad \in H \cap H^{a^{-1}} = 1$ and so $ad = c$.

This means that $aA \subseteq A \cap B$ and $aB \subseteq A \cap B$. If $a^{-1}H = dH$, where $d \in A$, then by Lemma 2.2, $ad \in H \cap A = 1$, and thus $a^{-1} = d \in A$. In fact, $a^{-1} \in A \cap B$. Thus $a^{-1}A \subseteq A \cap B$ and $a^{-1}B \subseteq A \cap B$. Let $f \in A \setminus B$. Now $af \in A \cap B$, hence $a^{-1}(af) = f \in A \cap B$, which is a contradiction. Thus $A = B$.

If $c \in A$, then $a^{-1}c^{-1}ac \in H$. Then $a(a^{-1}c^{-1}ac)a^{-1} = c^{-1}(a^{-1})^{-1}ca^{-1} \in H$, because $a^{-1} \in A = B$. It follows that $a^{-1}c^{-1}ac \in H \cap H^a = 1$, hence $ac = ca$. Thus $a \in Z(A)$ and hence $a \in Z(\langle A \rangle) = Z(G)$. Thus $H \cap H^a = H = 1$, which is a contradiction. $\square$

3. Main results

We shall now consider the situation where G is finite, $A = B$ and $H \cong C_{p^n} \times C_{p^n}$.

Theorem 3.1. *Let p be an odd prime and $H \cong C_{p^n} \times C_{p^n}$, where $n \geq 1$. If A is a selfconnected transversal to H in G and $G = \langle A \rangle$, then $G' \leq N_G(H)$.*

Proof: We proceed by induction on n. If $n = 1$, then our claim follows from Lemma 2.4. If $H_G > 1$, then we consider G/H_G and its subgroup H/H_G. By Lemma 2.7, $H/H_G \cong C_{p^k} \times C_{p^k}$, where $k < n$ and the claim follows by induction.

Thus we may assume that $H_G = 1$. By Lemma 2.2, $N_G(H) = H \times Z(G)$ and from Lemma 2.5, it follows that $Z(G) > 1$. By Lemma 2.6, the core of $HZ(G)$ in G is equal to $KZ(G)$, where $1 < K \leq H$. If $K = H$, then $HZ(G)$ is normal in G and $G' \leq HZ(G) = N_G(H)$. Thus we may assume that K is a proper subgroup of H.

We then consider $G/KZ(G)$ and $HZ(G)/KZ(G)$. By Lemma 2.7, we conclude that $HZ(G)/KZ(G) \cong C_{p^k} \times C_{p^k}$, where $k < n$. Thus by induction,

$$\begin{aligned}(G/KZ(G))' &\leq N_{G/KZ(G)}(HZ(G)/KZ(G)) \\ &= HZ(G)/KZ(G) \times Z(G/KZ(G))\end{aligned}$$

and consequently $G' \leq HM$, where $M/KZ(G) = Z(G/KZ(G))$. Clearly, HM and M are normal in G and $H \cap M = K$.

Then let $a, b \in A$ and write $ab = ch$, where $c \in A$ and $h \in H$. If also $d \in A$, then

$$\begin{aligned} h^d &= (c^{-1}ab)^d = h_1 c^{-1} a h_2 b h_3 = h_1 (c^{-1}ab) h_2^b h_3 \\ &= h_1 h h_2^b h_3 \in H H^b H, \end{aligned}$$

(here $h_1, h_2, h_3 \in H$). Now $HZ(G)$ is normal in HM and HM is normal in G. Thus $H^b \leq HM$, $HZ(G)H^b$ is a subgroup of G and $HH^bH \subseteq HZ(G)H^b$. It follows that $h \in (HZ(G)H^b)^{d^{-1}}$ for every $d \in A$.

We denote by $N(b)$ the intersection $\cap_{g \in G} (HZ(G)H^b)^g$. It is clear that $N(b)$ is normal in G, $h \in N(b)$, $ab \in A(N(b) \cap H)$ and $N(b) \geq KZ(G)$ for every $b \in A$. We write $H = \langle x \rangle \times \langle y \rangle$, where $|x| = |y| = p^n$ and $S = \langle x^p \rangle \times \langle y^p \rangle$. Then let $L = \Pi_{b \in A} N(b)$. Now $A^2 \subseteq A(L \cap H)$ and if $L \cap H \leq S$, then $\langle A \rangle$ is a proper subgroup of G, a contradiction.

Thus we may assume that there exists $b \in A$ such that $HN(b)/N(b)$ is cyclic. By Lemma 2.3, we conclude that $G' \leq HN(b) \leq HZ(G)H^b$ and thus $HZ(G)H^b$ is a normal subgroup of G. If we consider $G/KZ(G)$ and its subgroup $HZ(G)/KZ(G)$, then from Lemma 2.8 it follows that $HZ(G) \cap H^g Z(G) > KZ(G)$ for every $g \in G$. Thus $HZ(G) \cap H^b Z(G) = LZ(G)$, where $K < L \leq H$. Now $LZ(G) \leq Z(HZ(G)H^b) \leq N_G(H) = HZ(G)$. As $Z(HZ(G)H^b)$ is normal in G, we see that the core of $HZ(G)$ in G is larger than $KZ(G)$. But this is a contradiction and the proof is complete. $\square$

If G is the multiplication group and H the inner mapping group of some loop Q, then $G' \leq N_G(H)$ is equivalent with $M(Q)' \leq N_{M(Q)}(I(Q))$, which implies that $N_{M(Q)}(I(Q))$ is normal in $M(Q)$. Thus, by combining the criterion given by Bruck (see the introduction) with Theorems 2.1 and 3.1, we get the following

Corollary 3.2. *If Q is a finite commutative loop and $I(Q) \cong C_{p^n} \times C_{p^n}$, where p is an odd prime number and $n \geq 1$, then Q is centrally nilpotent of class two.*

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(*Received* September 30, 2014)