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Σ_s-products revisited

Reynaldo Rojas-Hernández

Abstract. We show that any Σ_s-product of at most $\mathfrak{c}$-many $L\Sigma(\leq \omega)$-spaces has the $L\Sigma(\leq \omega)$-property. This result generalizes some known results about $L\Sigma(\leq \omega)$-spaces. On the other hand, we prove that every Σ_s-product of monotonically monolithic spaces is monotonically monolithic, and in a similar form, we show that every Σ_s-product of Collins-Roscoe spaces has the Collins-Roscoe property. These results generalize some known results about the Collins-Roscoe spaces and answer some questions due to Tkachuk [*Lifting the Collins-Roscoe property by condensations*, Topology Proc. **42** (2012), 1–15]. Besides, we prove that if X is a simple Lindelöf Σ-space, then $C_p(X)$ has the Collins-Roscoe property.

Keywords: Σ_s-product; Lindelöf Σ-space; $L\Sigma(\leq \omega)$-space; monotonically monolithic space; Collins-Roscoe space; function space; simple space

Classification: Primary 54C35, 54B10, 54D99

1. Introduction

Lindelöf Σ-property is important in topology, functional analysis and descriptive set theory. One of many equivalent definitions says that X is a Lindelöf Σ-space if and only if there exists a second countable space M and an upper semicontinuous compact-valued onto map $\varphi : M \to X$.

Given a class $\mathcal{K}$ of compact spaces, Kubiś, Okunev and Szeptycki introduced and studied in [7] the class $L\Sigma(\mathcal{K})$ of spaces X for which there exists a second countable space M and an upper semicontinuous onto map $\varphi : M \to X$ such that $\varphi(x)$ belongs to the class $\mathcal{K}$ for any $x \in M$. If $\mathcal{K}$ consists of compact spaces of weight at most ω then the class $L\Sigma(\mathcal{K})$ is denoted in [7] by $L\Sigma(\leq \omega)$. Compact spaces from the class $L\Sigma(\leq \omega)$ were studied (under a different name) by Tkachuk in [12] and Tkachenko in [11].

A compact space X is a *Gul'ko compact space* if $C_p(X)$ is a Lindelöf Σ-space. Molina-Lara and Okunev proved in [8] that every Gul'ko compact space of cardinality at most $\mathfrak{c}$ is an $L\Sigma(\leq \omega)$-space. Tkachuk proved in [17] that, for any space X for which $C_p(X)$ has the Lindelöf Σ-property, $C_p(X)$ is an $L\Sigma(\leq \omega)$-space if and only if $|X| \leq \mathfrak{c}$.

The concept of Σ_s-product was introduced in [10] by Sokolov who proved that a compact space X is a Gul'ko compact space if and only if X embeds into a Σ_s-product of real lines. We establish that if $\mathcal{K}$ is a class of compact spaces

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closed with respect to finite unions, closed subspaces and countable products, then any Σ_s-product of at most $\mathfrak{c}$-many $L\Sigma(\mathcal{K})$-spaces has the $L\Sigma(\mathcal{K})$-property. In particular, any Σ_s-product of at most $\mathfrak{c}$-many $L\Sigma(\leq\omega)$-spaces has the $L\Sigma(\leq\omega)$-property. We use this statement to give another proof of the mentioned theorems on $L\Sigma(\leq\omega)$-spaces.

The above results show that Σ_s-products are useful in the study of the Lindelöf Σ-property in general and in function spaces (see also [15] and [14]). We also will use Σ_s-products to study monotone monolithicity and the Collins-Roscoe-property.

Tkachuk introduced in [16] the concept of a monotonically monolithic space. Collins-Roscoe spaces where studied in [3] by Collins and Roscoe (under a different name). Gruenhage proved in [5] that every Collins-Roscoe space is monotonically monolithic. Tkachuk gave in [18] an example of a monotonically monolithic space which does not have the Collins-Roscoe property.

Gruenhage established in the paper [5] that every Gul'ko compact space X has the Collins-Roscoe property. Also, Tkachuk proved in [18] that, if X is a Lindelöf Σ-space which can be condensed into some Σ_s-product of real lines, then the space X has the Collins-Roscoe property. It was proved in [15] that every Σ_s-product of second countable spaces has the Collins-Roscoe property. In the same paper Tkachuk posed the following question: is it true that every Σ_s-product of Collins-Roscoe spaces has the Collins-Roscoe property? We give a positive answer to this question and use this result to give a different proof of the above results about the Collins-Roscoe property. We also prove that any Σ_s-product of monotonically monolithic spaces is a monotonically monolithic space, answering another question in [15].

Besides, answering a question in [8], Tkachuk proved in [17] that if X is a simple Lindelöf Σ-space and $|X|\leq\mathfrak{c}$, then $C_p(X)$ is a Lindelöf Σ-space. He also asked whether the condition $|X|\leq\mathfrak{c}$ can be omitted in his result. In virtue of results in [18], if the answer to Tkachuk's question is positive, then $C_p(X)$ must have the Collins-Roscoe property when X is a simple Lindelöf Σ-space. We do not know if the answer to Tkachuk's question is positive, but in the last part of this paper, we show that $C_p(X)$ has the Collins-Roscoe property when X is a simple Lindelöf Σ-space.

2. Terminology and notation

All spaces in this article are assumed to be Tychonoff. We use terminology and notation as in [2] and [4]. The symbol ω denotes the set of all natural numbers (always considered with the discrete topology) and $\mathfrak{c}$ is the cardinal 2^ω.

For a subset A of a topological space X let $\mathrm{cl}_X(A)$ be the closure of A in X. If there is no possibility of confusion, we will simply write $\mathrm{cl}(A)$ instead of $\mathrm{cl}_X(A)$.

We denote by $C_p(X)$ the space of all real-valued continuous functions with the topology of pointwise convergence; that is, the topology of subspace of the space $\mathbb{R}^X$ of all functions from X to $\mathbb{R}$ equipped with the Tychonoff product topology.

Let $\mathcal{C}$ be a cover of a space X. A family $\mathcal{N}$ of subsets of X is called a *network with respect to* $\mathcal{C}$ if for every element C of $\mathcal{C}$ and any neighborhood U of C, there is an element N of $\mathcal{N}$ such that $C \subset N \subset U$.

If $X = \prod\{X_t : t \in T\}$ is a topological product, $t \in T$ and $E \subset T$, then p_t and p_E denote the natural projections onto X_t and $\prod\{X_t : t \in E\}$, respectively.

Let $X = \prod\{X_t : t \in T\}$ be a topological product and suppose that $a \in X$ is fixed. Given any $x \in X$, $A \subset X$ and $E \subset T$, let $\mathrm{supp}_a(x) = \{t \in T : x(t) \neq a(t)\}$, $\mathrm{supp}_a(x, E) = \mathrm{supp}_a(x) \cap E$ and $\mathrm{supp}_a(A, E) = \bigcup\{\mathrm{supp}_a(x, E) : x \in A\}$. If there is no possibility of confusion we simply write $\mathrm{supp}(x)$, $\mathrm{supp}(x, E)$ and $\mathrm{supp}(A, E)$ instead of $\mathrm{supp}_a(x)$, $\mathrm{supp}_a(x, E)$ and $\mathrm{supp}_a(A, E)$, respectively.

We denote by υX the Hewitt realcompactification of the space X.

3. $L\Sigma(\leq \omega)$-property in Σ_s-products

The aim of this section is to show that the class of $L\Sigma(\mathcal{K})$ is closed under Σ_s-products of at most $\mathfrak{c}$-many factors, when $\mathcal{K}$ has some nice properties. We give some applications of this result when $\mathcal{K}$ is the class of all metrizable compact spaces.

The following notion was introduced by Sokolov [10].

Definition 3.1. Given a family of spaces $\{X_t : t \in T\}$, let $X = \prod\{X_t : t \in T\}$ and fix a point $a \in X$. Suppose that $s = \{T_n : n \in \omega\}$ is a sequence of subsets of T. Given $x \in X$ denote by Ω_x the set $\{n \in \omega : |\mathrm{supp}(x, T_n)| < \omega\}$. The subspace $Z = \{x \in X : T = \bigcup\{T_n : n \in \Omega_x\}\}$ of X is called the Σ_s*-product of the family* $\{X_t : t \in T\}$ *centered at* a.

Remark 3.2. Given a product $X = \prod\{X_t : t \in T\}$, a fixed point a in X, and a sequence $s = \{T_n : n \in \omega\}$ of subsets of T, let us observe that:

(a) if x is an element of the Σ_s-product of the family $\{X_t : t \in T\}$ centered at a, it follows from $T = \bigcup\{T_n : n \in \Omega_x\}$ that $\mathrm{supp}(x) = \bigcup\{\mathrm{supp}(x, T_n) : n \in \Omega_x\}$ and hence $|\mathrm{supp}(x)| \leq \omega$;

(b) if s^* is a sequence of subsets of T with $s \subset s^*$, then the Σ_s-product of the family $\{X_t : t \in T\}$ centered at a is contained in the Σ_{s^*}-product of the family $\{X_t : t \in T\}$ centered at a.

We will use the following characterization of $L\Sigma(\mathcal{K})$-spaces [7]: given a class $\mathcal{K}$ of compact spaces, a space X is an $L\Sigma(\mathcal{K})$-space if and only if there is a cover $\mathcal{C}$ of X such that $\mathcal{C} \subset \mathcal{K}$ and a countable network $\mathcal{N}$ with respect to $\mathcal{C}$.

We are ready to prove the main result of this section.

Theorem 3.3. *Let $\mathcal{K}$ be a class of compact spaces closed with respect to finite unions, closed subspaces and countable products. Then any Σ_s-product of at most $\mathfrak{c}$-many $L\Sigma(\mathcal{K})$-spaces is also an $L\Sigma(\mathcal{K})$-space.*

Proof: Consider a family $\{X_t : t \in T\}$ of $L\Sigma(\mathcal{K})$-spaces with $|T| \leq \mathfrak{c}$. Let $X = \prod\{X_t : t \in T\}$, fix a point $a \in X$ and consider a sequence $s = \{T_n : n \in \omega\}$ of subsets of T. Denote by Z the Σ_s-product of the family $\{X_t : t \in T\}$ centered at a. We can assume that $T \cap \omega = \emptyset$. Given $t \in T$, choose a cover $\mathcal{C}_t$ of X_t such that $\mathcal{C}_t \subset \mathcal{K}$ and a countable network $\mathcal{N}_t$ with respect to $\mathcal{C}_t$. We can assume that $\mathcal{C}_t$ and $\mathcal{N}_t$ are closed under finite intersections, $\{a(t)\} \in \mathcal{C}_t \cap \mathcal{N}_t$, and $a(t) \in C_t \cap N_t$ for every $C_t \in \mathcal{C}_t$ and $N_t \in \mathcal{N}_t$. Choose an enumeration $\{N_{t,m} : m \in \omega\}$ of $\mathcal{N}_t$.

Pick $n \in \omega$. Denote by Y_n the σ-product in $\prod\{X_t : t \in T_n\}$ centered at $p_{T_n}(a)$. Let $\mathcal{C}_n$ be the family of all sets of the form $\prod\{C_t : t \in T_n\}$ for which $C_t \in \mathcal{C}_t$ for $t \in G$ and $C_t = \{a(t)\}$ for $t \in T_n \setminus G$, where G is a finite subset of T_n. It is clear that $\mathcal{C}_n$ is a cover of Y_n and $\mathcal{C}_n \subset \mathcal{K}$. Since $|T_n| \le \mathfrak{c}$, we can find a countable family $\mathcal{B}_n$ of subsets of T_n such that for any finite set $F \subset T_n$ there exists a pairwise disjoint family $\{B_t : t \in F\} \subset \mathcal{B}_n$ such that $t \in B_t$ for each $t \in F$. Given a finite pairwise disjoint family $\mathcal{F} \subset \mathcal{B}_n$ and $u : \mathcal{F} \to \omega$ let $N_{\mathcal{F},u,n} = \prod\{N_t : t \in T_n\}$ where $N_t = N_{t,u(B)}$ if $t \in B$ for some $B \in \mathcal{F}$ and $N_t = \{a(t)\}$ if $t \in T_n \setminus \bigcup \mathcal{F}$. Let $\mathcal{N}_n = \{Y_n \cap N_{\mathcal{F},u,n} : \mathcal{F}$ is a finite pairwise disjoint subfamily of $\mathcal{B}_n$ and $u : \mathcal{F} \to \omega\}$. Let us observe that $\mathcal{N}_n$ is a countable family of subsets of Y_n.

Claim 1. The family $\mathcal{N}_n$ is a network with respect to $\mathcal{C}_n$.

Choose $C \in \mathcal{C}_n$ and suppose that $C \subset U \cap Y_n$ for some open set U in $\prod\{X_t : t \in T_n\}$. Choose a finite set $G \subset T_n$ such that $C = \prod\{C_t : t \in T_n\}$ where $C_t \in \mathcal{C}_t$ for $t \in G$ and $C_t = \{a(t)\}$ for $t \in T_n \setminus G$. By [4, Theorem 3.2.10] we can find a finite set $F \subset T_n$ and open sets U_t in X_t for $t \in F$ in such a way that $C \subset \bigcap\{p_t^{-1}(U_t) : t \in F\} \subset U$ (here p_t denotes the projection from $\prod\{X_t : t \in T_n\}$ onto X_t). We can assume that $G \subset F$. By the choice of $\mathcal{B}_n$ we can find a pairwise disjoint family $\mathcal{F} = \{B_t : t \in F\} \subset \mathcal{B}_n$ such that $t \in B_t$ for each $t \in F$. Given $t \in F$, since $C_t \subset U_t$, we can find $m_t \in \omega$ such that $C_t \subset N_{t,m_t} \subset U_t$. Define a function $u : \mathcal{F} \to \omega$ by $u(B_t) = m_t$ for each $B_t \in \mathcal{F}$. Then $C \subset N_{\mathcal{F},u,n} \subset U$ and hence $C \subset N_{\mathcal{F},u,n} \cap Y_n \subset U \cap Y_n$, where $N_{\mathcal{F},u,n} \cap Y_n \in \mathcal{N}_n$. So we have proved Claim 1.

Now we are ready to show that Z is an $L\Sigma(\mathcal{K})$-space. Let $\mathcal{A} = \{A \subset \omega : T = \bigcup\{T_n : n \in A\}\}$. Consider the family $\mathcal{C}$ of all sets of the form $\bigcap\{p_{T_n}^{-1}(C_n) : n \in A\}$ where $C_n \in \mathcal{C}_n$ for each $n \in A$ and $A \in \mathcal{A}$, and the family $\mathcal{N}$ of all sets of the form $Z \cap \bigcap\{p_{T_n}^{-1}(N_n) : n \in B\}$ where $N_n \in \mathcal{N}_n$ for $n \in B$ and B is a finite subset of ω. Observe that $\mathcal{N}$ is a countable family of subsets of Z.

Claim 2. $\mathcal{C}$ is a family of subsets of Z, $\mathcal{C} \subset \mathcal{K}$ and $\mathcal{C}$ is a cover of Z.

Pick $C \in \mathcal{C}$. Choose $A \in \mathcal{A}$ and $C_n \in \mathcal{C}_n$, for each $n \in A$, for which $C = \bigcap\{p_{T_n}^{-1}(C_n) : n \in A\}$. Pick $x \in C$. Given $n \in A$, it follows from $p_{T_n}(x) \in C_n \subset Y_n$ that $n \in \Omega_x$. Hence $A \subset \Omega_x$. Since $A \in \mathcal{A}$, we have the equalities $T = \bigcup\{T_n : n \in A\} = \bigcup\{T_n : n \in \Omega_x\}$, that is, $x \in Z$. Hence $C \subset Z$. On the other hand, $C_n = \prod\{C_t^n : t \in T_n\}$ where $C_t^n \in \mathcal{C}_t$ for $t \in G_n$ and $C_t^n = \{a(t)\}$ for $t \in T_n \setminus G_n$, where G_n is a finite subset of T_n. Observe that $C = \prod\{C_t : t \in T\}$ where $C_t = \bigcap\{C_t^n : t \in T_n$ and $n \in A\}$. It is clear that C is a compact space; consider the set $G = \bigcup\{G_n : n \in \omega\}$ and note that if $t \in T \setminus G$ we can choose $n_t \in A$ for which $t \in T_{n_t}$ and hence $C_t = C_t^{n_t} = \{a(t)\}$. It follows that $p_G : C \to \prod\{C_t : t \in G\}$ is a homeomorphism, so $C \in \mathcal{K}$.

Now we will prove that $\mathcal{C}$ is a cover of Z. Pick $x \in Z$. It is clear that $\Omega_x \in \mathcal{A}$. Given $n \in \Omega_x$ we know that $p_{T_n}(x) \in Y_n$ and hence we can choose $C_n \in \mathcal{C}_n$ such that $p_{T_n}(x) \in C_n$. Let $C = \bigcap\{p_{T_n}^{-1}(C_n) : n \in \Omega_x\}$. It is clear that $x \in C$ and $C \in \mathcal{C}$. Therefore $\mathcal{C}$ is a cover of Z.

Claim 3. The family $\mathcal{N}$ is a network with respect to the cover $\mathcal{C}$.

Pick $C \in \mathcal{C}$ and let U be an open set in X with $C \subset U \cap Z$. Choose $A \in \mathcal{A}$ and $C_n \in \mathcal{C}_n$ for each $n \in A$, in such a way that $C = \bigcap\{p_{T_n}^{-1}(C_n) : n \in A\}$. It follows from $C_n \in \mathcal{C}_n$ that $C_n = \prod\{C_t^n : t \in T_n\}$ where $C_t^n \in \mathcal{C}_t$ for $t \in G_n$ and $C_t^n = \{a(t)\}$ for $t \in T_n \setminus G_n$, where G_n is a finite subset of T_n. Let us observe that $C = \prod\{C_t : t \in T\}$ where $C_t = \bigcap\{C_t^n : t \in T_n \text{ and } n \in A\}$. By [4, Theorem 3.2.10] there is an open set U_t in X_t, for each $t \in T$, and a finite subset F of T such that $U_t = X_t$ for $t \in T \setminus F$ and $C \subset \prod\{U_t : t \in T\} \subset U$. Given $t \in T$, since $C_t = \bigcap\{C_t^n : t \in T_n \text{ and } n \in A\} \subset U_t$, we can find a finite subfamily $\mathcal{D}_t$ of $\{C_t^n : t \in T_n \text{ and } n \in A\} \subset \mathcal{C}_t$ such that $C_t \subset D_t \subset U_t$ where $D_t = \bigcap \mathcal{D}_t$. Since $\mathcal{C}_t$ is closed under finite intersections, $D_t \in \mathcal{C}_t$. Given $n \in A$ let $D_n = \prod\{D_t^n : t \in T_n\}$ where $D_t^n = D_t \in \mathcal{C}_t$ for $t \in G_n$ and $D_t^n = \{a(t)\}$ for $t \in T_n \setminus G_n$. It is clear that $C_n \subset D_n$. Also, observe that $D_n \in \mathcal{C}_n$ and $D_n \subset U_n$ where $U_n = \prod\{U_t : t \in T_n\}$. By Claim 1 we can choose $N_n \in \mathcal{N}_n$ such that $D_n \subset N_n \subset U_n$. Since $A \in \mathcal{A}$, there exists a finite subset $B \subset A$ such that $F \subset \bigcup\{T_n : n \in B\}$. Finally note that

$$C \subset \bigcap_{n \in B} p_{T_n}^{-1}(C_n) \subset \bigcap_{n \in B} p_{T_n}^{-1}(D_n) \subset \bigcap_{n \in B} p_{T_n}^{-1}(N_n) \subset \bigcap_{n \in B} p_{T_n}^{-1}(U_n) = \prod_{t \in T} U_t \subset U.$$

Hence $C \subset N \subset U \cap Z$ where $N = Z \cap \bigcap\{p_{T_n}^{-1}(N_n) : n \in B\} \in \mathcal{N}$. We have proved Claim 3.

It follows from Claims 2 and 3 that Z is an $L\Sigma(\mathcal{K})$-space. $\square$

The classes of compact spaces and second countable spaces are closed with respect to finite unions, closed subspaces and countable products. Therefore we have the following corollaries.

Corollary 3.4. *Any Σ_s-product of at most $\mathfrak{c}$-many Lindelöf Σ-spaces is a Lindelöf Σ-space.*

Corollary 3.5. *Any Σ_s-product of at most $\mathfrak{c}$-many $L\Sigma(\leq \omega)$-spaces is an $L\Sigma(\leq \omega)$-space.*

It was proved in [10] that a compact space X is Gul'ko compact if and only if X embeds into a Σ_s-product of real lines. Since the real line is clearly an $L\Sigma(\leq \omega)$-space and the $L\Sigma(\leq \omega)$-property is inherited by closed subspaces, we have the following consequence.

Corollary 3.6 ([8])**.** *Every Gul'ko compact space of cardinality $\leq \mathfrak{c}$ is an $L\Sigma(\leq \omega)$-space.*

Recall that a compact space X is *Eberlein compact* if X is homeomorphic to a subspace of $C_p(K)$ for some compact space K. It is well known that every Eberlein compact space is a Gul'ko compact space. This shows that we have the following corollary.

Corollary 3.7 ([12])**.** *Let X be an Eberlein compact space of cardinality not exceeding continuum. Then X is an $L\Sigma(\leq \omega)$-space.*

Now we will prove a result about the $L\Sigma(\leq \omega)$-property in function spaces (see [17, Theorem 2.10]).

Corollary 3.8. *If X is a space such that $|X| \leq \mathfrak{c}$ and $C_p(X)$ is a Lindelöf Σ-space, then $C_p(X)$ is an $L\Sigma(\leq \omega)$-space.*

Proof: Because of [9, Theorem 3.5] and [13, Theorem 2.3] both υX and $C_p(\upsilon X)$ are Lindelöf Σ-spaces. Apply [17, Proposition 2.8] to see that $|C_p(\upsilon X)| \leq \mathfrak{c}$. It follows from [9, Corollary 2.11] that $C_p(C_p(\upsilon X))$ is a Lindelöf Σ-space. Now we can apply [14, Corollary 4.12] to see that the space $C_p(\upsilon X)$ can be condensed in a Σ_s-product of real lines. Since $|C_p(\upsilon X)| \leq \mathfrak{c}$, the space $C_p(\upsilon X)$ can be condensed in a Σ_s-product Z of at most $\mathfrak{c}$-many copies of the real line. Because of Corollary 3.5 the space Z is an $L\Sigma(\leq \omega)$-space. Apply [8, Corollary 2.2] and [8, Lemma 2.3] to conclude that $C_p(\upsilon X)$ is an $L\Sigma(\leq \omega)$-space. The space $C_p(X)$, being a continuous image of $C_p(\upsilon X)$, is also an $L\Sigma(\leq \omega)$-space. □

4. Monotone monolithicity and the Collins-Roscoe property in Σ_s-products

In this section we prove that the classes of monotonically monolithic spaces and Collins-Roscoe spaces are closed under Σ_s-products. We apply these results to prove some known results about monotone monolithicity and the Collins-Roscoe property.

First we will deal with monotonically monolithic spaces. The following concepts were introduced by Tkachuk [16].

Definition 4.1. Given a subset A of a space X we say that a family $\mathcal{N}$ of subsets of X is an *external network* of A in X if for each $x \in A$ and each open subset U of X with $x \in U$ there is $N \in \mathcal{N}$ such that $x \in N \subset U$.

Definition 4.2. We say that a space X is *monotonically monolithic* if to each $A \subset X$ we can assign an external network $\mathcal{N}(A)$ of $\mathrm{cl}(A)$ in X such that:

(a) $|\mathcal{N}(A)| \leq \max\{|A|, \omega\}$;
(b) if $A \subset B \subset X$, then $\mathcal{N}(A) \subset \mathcal{N}(B)$;
(c) if $\{A_\alpha : \alpha < \gamma\}$ is a family of subsets of X with $A_\alpha \subset A_\beta$ for $\alpha < \beta < \gamma$, then $\mathcal{N}(\bigcup\{A_\alpha : \alpha < \gamma\}) = \bigcup\{\mathcal{N}(A_\alpha) : \alpha < \gamma\}$.

The following equivalence of monotone monolithicity, which turned out to be very useful, was obtained by Gruenhage [5] and Guo and Junnila [6].

Theorem 4.3. *A space X is monotonically monolithic if and only if to each finite subset F of X we can assign a countable collection $\mathcal{N}(F)$ of subsets of X such that, for each subset A of X, the family $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ is an external network of $\mathrm{cl}(A)$ in X.*

Given a sequence s of subsets of a set T, we define a relation $\sim$ on T as follows: we say that $t_1 \sim t_2$ if, for every $E \in s$, we have $t_1 \in E$ if and only if $t_2 \in E$.

Lemma 4.4. *Suppose that s is a sequence of subsets of a set T, which is closed under complements and finite intersections. Assume that $H_1, \dots, H_n \in [T]^{<\omega}$ is a family of non-empty sets such that if $t_i \in H_i$ and $t_j \in H_j$ then $t_i \sim t_j$ if and only if $i = j$. Then we can find a disjoint family $\{E_1, \dots, E_n\} \subset s$ such that $H_i \subset E_i$ for $i = 1, \dots, n$.*

Proof: If $n = 2$, for $t_1 \in H_1$ and $t_2 \in H_2$ we can find $E \in s$ such that $t_1 \in E$ and $t_2 \in T \setminus E$. Let $E_1 = E$ and $E_2 = T \setminus E$. Then, $\{E_1, E_2\}$ satisfies the required conditions. For $n > 2$ take $H_1, \dots, H_n$ as in the Lemma. For every $i, j \leq n$ with $i \neq j$, take a disjoint family $\{E_{ij}, E_{ij}^*\} \subset s$ such that $H_i \subset E_{ij}$ and $H_j \subset E_{ij}^*$. Now take $E_i = \bigcap\{E_{ij} \cap E_{ji}^* : j \leq n \text{ and } j \neq i\}$ for $i = 1, \dots, n$. Then the family $\{E_1, \dots, E_n\} \subset s$ is pairwise disjoint and $H_i \subset E_i$ for $i = 1, \dots, n$. $\square$

We are ready to show that monotone monolithicity is closed under Σ_s-products.

Theorem 4.5. *Every Σ_s-product of monotonically monolithic spaces is monotonically monolithic.*

Proof: Suppose that X_t is monotonically monolithic and fix the respective operator $\mathcal{N}_t$ as in Theorem 4.3 and $\mathcal{N}_t(\emptyset) = \emptyset$ for every $t \in T$. Let $X = \prod\{X_t : t \in T\}$ and fix a point $a \in X$. Suppose that $s = \{T_n : n \in \omega\}$ is a sequence of subsets of T. We must prove that the Σ_s-product Z of the family $\{X_t : t \in T\}$ centered at a is monotonically monolithic. Since monotone monolithicity is a hereditary property, by Remark 3.2 (b), we can assume that the family s is closed under complements and finite intersections. Let $\mathcal{E}(s) = \{\{E_1, \dots, E_n\} \in [s]^{<\omega} : E_i \cap E_j = \emptyset$ for $i \neq j\}$.

We shall construct a monotonic monolithicity operator in Z. Pick a finite set $F \subset Z$. Given a set $E \subset T$ because of Remark 3.2(a) the set $\mathrm{supp}(F, E)$ is countable. Let $\mathcal{N}_E(F)$ be the family of all sets of the form $\prod\{N_t : t \in E\}$, where $N_t \in \mathcal{N}_t(p_t(F))$ if $t \in G$ and $N_t = \{a(t)\}$ if $t \in E \setminus G$ for some finite subset G of $\mathrm{supp}(F, E)$. Notice that $\mathcal{N}_E(F)$ is countable. Finally, let

$$\mathcal{N}(F) = \left\{ Z \cap \bigcap_{E \in \mathcal{F}} p_E^{-1}(N_E) : \mathcal{F} \in \mathcal{E}(s) \text{ and } N_E \in \mathcal{N}_E(F) \text{ for every } E \in \mathcal{F} \right\}.$$

Since $\mathcal{E}(s)$ and $\mathcal{N}_E(F)$ for each $E \in s$ are countable, the family $\mathcal{N}(F)$ is also countable. We shall prove that the operator $\mathcal{N}$ satisfies the conditions in Theorem 4.3.

Claim. For every $A \subset Z$ the family $\bigcup\{\mathcal{N}(F) : F \in [A]^{<\omega}\}$ is an external network of $\mathrm{cl}_Z(A)$ in Z.

Pick $A \subset Z$, $x \in \mathrm{cl}_Z(A)$ and an open set U in Z with $x \in U$. We shall prove that there exist $F \in [A]^{<\omega}$ and $N \in \mathcal{N}(F)$ such that $x \in N \subset U$. Choose a finite set $H \subset T$ and a family $\{W_t : t \in H\}$ such that W_t is open in X_t for every $t \in H$ and $x \in W \subset U$ for $W = Z \cap \bigcap\{p_t^{-1}(W_t) : t \in H\}$. We can assume that $a(t) \notin W_t$ if $x(t) \neq a(t)$. Let $\{H_1, \ldots, H_n\}$ be a partition of H such that if $t_i \in H_i$ and $t_j \in H_j$ then $t_i \sim t_j$ if and only if $i = j$. By Lemma 4.4 we can obtain a pairwise disjoint family $\{E_1^*, \ldots, E_n^*\} \in \mathcal{E}(s)$ such that $H_i \subset E_i^*$ for $i = 1, \ldots, n$.

Take $i \in \{1, \ldots, n\}$ and fix $t_i \in H_i$. It follows from $x \in Z$ that $T = \bigcup\{T_m : m \in \Omega_x\}$. Hence we can find $T_{m_i} \in s$ such that $t_i \in T_{m_i}$ and $|\,\mathrm{supp}(x, T_{m_i})| < \omega$. Let $E_i = E_i^* \cap T_{m_i}$, $G_i = \mathrm{supp}(x, E_i)$ and $K_i = G_i \cup H_i$. Using the definition of $\sim$ we can see that $K_i \subset E_i$. For every $t \in G_i \setminus H_i$ let $W_t = X_t \setminus \{a(t)\}$. Notice that $a(t) \notin W_t$ and $x(t) \in W_t$ for every $t \in G_i$. Let $B_i = A \cap \bigcap\{p_t^{-1}(W_t) : t \in G_i\}$. Then $x \in \mathrm{cl}_Z(B_i)$; observe that $G_i \subset \mathrm{supp}(z, E_i)$ for every $z \in B_i$ and pick $t \in G_i$. Then $x(t) \in \mathrm{cl}_{X_t}(p_t(B_i))$. By the choice of $\mathcal{N}_t$, the family $\bigcup\{\mathcal{N}_t(p_t(F)) : F \in [B_i]^{<\omega}\}$ is an external network for $\mathrm{cl}_{X_t}(p_t(B_i))$ in X_t. Then we can choose a non-empty finite set $F_t \subset B_i$ and $N_t \in \mathcal{N}_t(p_t(F_t))$ such that $x(t) \in N_t \subset W_t$. Let $F_i = \bigcup\{F_t : t \in G_i\} \subset B_i$. For $t \in E_i \setminus G_i$ let $N_t = \{a(t)\}$. Choose $N_{E_i} = \prod\{N_t : t \in E_i\}$. It follows from $F_i \subset B_i$ that G_i is a finite subset of $\mathrm{supp}(F_i, E_i)$. Hence $N_{E_i} \in \mathcal{N}_{E_i}(F_i)$. Note that $x \in p_{E_i}^{-1}(N_{E_i}) \subset \bigcap\{p_t^{-1}(W_t) : t \in K_i\}$.

Finally, it is clear that $\mathcal{F} = \{E_1, \ldots, E_n\} \in \mathcal{E}(s)$ and $F = \bigcup\{F_i : i = 1, \ldots, n\}$ is a finite subset of A. Let $N = Z \cap \bigcap\{p_E^{-1}(N_E) : E \in \mathcal{F}\}$. Since $N_{E_i} \in \mathcal{N}_{E_i}(F_i) \subset \mathcal{N}_{E_i}(F)$, for every $E_i \in \mathcal{F}$, we conclude that $N \in \mathcal{N}(F)$. It is clear that $x \in N$. Besides, for $K = \bigcup\{K_i : i = 1, \ldots, n\}$ we have $H \subset K$ and

$$\bigcap_{i=1}^{n} p_{E_i}^{-1}(N_{E_i}) \subset \bigcap_{i=1}^{n} \bigcap_{t \in K_i} p_t^{-1}(W_t) = \bigcap_{t \in K} p_t^{-1}(W_t) \subset \bigcap_{t \in H} p_t^{-1}(W_t).$$

So $N = Z \cap \bigcap\{p_{E_i}^{-1}(N_{E_i}) : i = 1, \ldots, n\} \subset Z \cap \bigcap\{p_t^{-1}(W_t) : t \in H\} = W \subset U$. $\square$

Let $\{X_t : t \in T\}$ be a family of spaces, a a point in $\prod\{X_t : t \in T\}$ and $s = \{T_n : n \in \omega\}$ a sequence of subsets of T. Observe that if $T_n = T$, for each $n \in \omega$, then the Σ_s-product of the family $\{X_t : t \in T\}$ centered at a coincides with the σ-product of the family $\{X_t : t \in T\}$ centered at a. On the other hand, if $T = \{t_n : n \in \omega\}$ is countable and $T_n = \{t_n\}$, for each $n \in \omega$, then the Σ_s-product of the family $\{X_t : t \in T\}$ centered at a coincides with the countable product $\prod\{X_t : t \in T\}$. This gives the following corollaries.

Corollary 4.6 ([1])**.** *Every σ-product of monotonically monolithic spaces is monotonically monolithic.*

Corollary 4.7 ([16])**.** *Every countable product of monotonically monolithic spaces is monotonically monolithic.*

We finish this section by proving that every Σ_s-product of a family of Collins-Roscoe spaces shares this property. First, we recall the definition [3].

Definition 4.8. Given a space X, assume that for every point $x \in X$ a countable family $\mathcal{G}(x)$ of subsets of X is chosen. Say that $\{\mathcal{G}(x) : x \in X\}$ is a *Collins-Roscoe collection* if for any $x \in X$ and each open set U in X which contains x we can find an open set V such that $x \in V \subset U$ and for any $y \in V$ there exists a set $P \in \mathcal{G}(y)$ with $x \in P \subset U$. If a space X has a Collins-Roscoe collection then we will say that X has the *Collins-Roscoe property.*

Remark 4.9. Let $X = \prod\{X_t : t \in F\}$ be a finite product. Suppose that for each $t \in F$ the family $\{\mathcal{G}_t(x_t) : x_t \in X_t\}$ is a Collins-Roscoe collection for X_t. For each $x \in X$ let $\mathcal{G}(x)$ be the family of all sets of the form $\prod\{G_t : t \in F\}$, where $G_t \in \mathcal{G}_t(x(t))$ for each $t \in F$. Then $\{\mathcal{G}(x) : x \in X\}$ is a Collins-Roscoe collection for the space X.

Gruenhage established in [5] the following equivalence of the Collins-Roscoe property which turned out to be very useful.

Theorem 4.10. *A collection $\{\mathcal{G}(x) : x \in X\}$ of countable families of subsets of a space X is a Collins-Roscoe collection if and only if for any set $A \subset X$, the family $\bigcup\{\mathcal{G}(x) : x \in A\}$ contains an external network for* $\mathrm{cl}(A)$.

Theorem 4.11. *Every Σ_s-product of Collins-Roscoe spaces has the Collins-Roscoe property.*

PROOF: Let $\{\mathcal{G}_t(x_t) : x_t \in X_t\}$ be a Collins-Roscoe collection in X_t, for every $t \in T$. Suppose that X, a, s, Z and $\mathcal{E}(s)$ are as in the proof of Theorem 4.5. We must prove that Z has the Collins-Roscoe property. Since the Collins-Roscoe property is inherited by arbitrary subspaces, because of Remark 3.2(b), we can assume that the family s is closed under complements and finite intersections. We shall construct a Collins-Roscoe collection in Z. Pick $x \in Z$. Given $E \subset T$, by Remark 3.2(a) the set $\mathrm{supp}(x, E)$ is countable. Let $\mathcal{G}_E(x)$ be the family of all sets of the form $\prod\{G_t : t \in E\}$, where $G_t \in \mathcal{G}_t(x(t))$ for $t \in F$, $G_t = \{a(t)\}$ for $t \in E \setminus F$ and F is a finite subset of $\mathrm{supp}(x, E)$. Note that the family $\mathcal{G}_E(x)$ is countable. Finally, let

$$\mathcal{G}(x) = \left\{ Z \cap \bigcap_{E \in \mathcal{F}} p_E^{-1}(G_E) : \mathcal{F} \in \mathcal{E}(s) \ \text{ and } \ G_E \in \mathcal{G}_E(x) \ \text{ for every } \ E \in \mathcal{F} \right\}.$$

Since $\mathcal{E}(s)$ and $\mathcal{G}_E(x)$, for each $E \in s$, are countable, the family $\mathcal{G}(x)$ is also countable. By Theorem 4.10 it is sufficient to prove the following claim.

Claim. For every $A \subset Z$ the family $\bigcup\{\mathcal{G}(x) : x \in A\}$ is an external network of $\mathrm{cl}_Z(A)$ in Z.

Pick $A \subset Z$, $x \in \mathrm{cl}_Z(A)$ and an open set U in Z with $x \in U$. We shall prove that there exist $z \in A$ and $G \in \mathcal{G}(z)$ with $x \in G \subset U$. Choose a finite set $H \subset T$ and a family $\{W_t : t \in H\}$ such that W_t is open in X_t for every $t \in H$ and

$x \in W \subset U$ for $W = Z \cap \bigcap\{p_t^{-1}(W_t) : t \in H\}$. We can assume that $a(t) \notin W_t$ if $x(t) \neq a(t)$. Let $\{H_1, \ldots, H_n\}$ be a partition of H such that if $t_i \in H_i$ and $t_j \in H_j$ then $t_i \sim t_j$ if and only if $i = j$. Applying Lemma 4.4 we can obtain a pairwise disjoint family $\{E_1^*, \ldots, E_n^*\} \in \mathcal{E}(s)$ such that $H_i \subset E_i^*$ for $i = 1, \ldots, n$.

Given $i \in \{1, \ldots, n\}$, pick any $t_i \in H_i$. Since $x \in Z$, we have $T = \bigcup\{T_m : m \in \Omega_x\}$ and so we can fix $T_{m_i} \in s$ such that $t_i \in T_{m_i}$ and $|\operatorname{supp}(x, T_{m_i})| < \omega$. Let $E_i = E_i^* \cap T_{m_i}$, $F_i = \operatorname{supp}(x, E_i)$ and $K_i = H_i \cup F_i$. By the definition of $\sim$, we have $K_i \subset E_i$; observe that also $\{E_1, \ldots, E_n\} \in \mathcal{E}(s)$. For every $t \in F_i \setminus H_i$ let $W_t = X_t \setminus \{a(t)\}$. Note that $a(t) \notin W_t$ and $x(t) \in W_t$ for every $t \in F_i$.

Let $F = \bigcup\{F_i : i = 1, \ldots, n\}$, $W^* = Z \cap \bigcap\{p_t^{-1}(W_t) : t \in F\}$ and $B = A \cap W^*$. It follows from $x \in W^*$ that $x \in \operatorname{cl}_Z(B)$. Let $X_F = \prod\{X_t : t \in F\}$; it is clear that $p_F(x) \in \operatorname{cl}_{X_F}(p_F(B)) \cap p_F(W^*)$. Let us observe that $F_i \subset \operatorname{supp}(z, E_i)$ for every $z \in B$ and $i = 1, \ldots, n$. For each $y \in X_F$, let $\mathcal{G}_F(y)$ be the family of all sets of the form $\prod\{G_t : t \in F\}$ where $G_t \in \mathcal{G}_t(y(t))$ for each $t \in F$. It follows from Remark 4.9, that the family $\{\mathcal{G}_F(y) : y \in X_F\}$ is a Collins-Roscoe collection in X_F. By Theorem 4.10 the family $\bigcup\{\mathcal{G}_F(y) : y \in p_F(B)\}$ is an external network of $\operatorname{cl}_{X_F}(p_F(B))$ in X_F. Since $p_F(x) \in \operatorname{cl}_{X_F}(p_F(B)) \cap p_F(W^*)$, there are $z \in B$ and $G_t \in \mathcal{G}_t(z(t))$, for each $t \in F$, such that $p_F(x) \in \prod\{G_t : t \in F\} \subset p_F(W^*) = \prod\{W_t : t \in F\}$. It follows that $p_t(x) \in G_t \subset W_t$ for any $t \in F_i$ and $i = 1, \ldots, n$.

Given $i \in \{1, \ldots, n\}$ let $G_{E_i} = \prod\{G_t : t \in E_i\}$, where $G_t = \{a(t)\}$ for $t \in E_i \setminus F_i$ and G_t is as above if $t \in F_i$. Observe that $p_{E_i}(x) \in G_{E_i}$ and $p_{K_i}(G_{E_i}) \subset \prod\{W_t : t \in K_i\}$. Since $z \in B$, $F_i \subset \operatorname{supp}(z, E_i)$ and so $G_{E_i} \in \mathcal{G}_{E_i}(z)$. We know that $\mathcal{F} = \{E_1, \ldots, E_n\} \in \mathcal{E}(s)$, so $G = Z \cap \bigcap\{p_E^{-1}(G_E) : E \in \mathcal{F}\} \in \mathcal{G}(z)$. It is clear that $x \in G$. For $K = \bigcup\{K_i : i = 1, \ldots, n\}$ we have $H \subset K$ and

$$\bigcap_{i=1}^{n} p_{E_i}^{-1}(G_{E_i}) \subset \bigcap_{i=1}^{n} p_{K_i}^{-1}(p_{K_i}(G_{E_i})) \subset \bigcap_{i=1}^{n} p_{K_i}^{-1}\left(\prod\{W_t : t \in K_i\}\right) = \bigcap_{t \in K} p_t^{-1}(W_t).$$

Therefore $G = Z \cap \bigcap_{i=1}^{n} p_{E_i}^{-1}(G_{E_i}) \subset Z \cap \bigcap\{p_t^{-1}(W_t) : t \in H\} = W \subset U$. □

Any second countable space has countable network weight and hence has the Collins-Roscoe property. As an immediate consequence, we obtain the following corollary.

Corollary 4.12 ([15]). *Any Σ_s-product of second countable spaces has the Collins-Roscoe property.*

Recall that a compact space X is Gul'ko compact if and only if X embeds into a Σ_s-product of real lines. Since the Collins-Roscoe property is inherited by arbitrary subspaces, we have the following result.

Corollary 4.13 ([5]). *Any Gul'ko compact space is a Collins-Roscoe space.*

The following results are particular cases of Theorem 4.11.

Corollary 4.14 ([18]). *Every σ-product of Collins-Roscoe spaces has the Collins-Roscoe property.*

Corollary 4.15 ([18]). *Every countable product of Collins-Roscoe spaces has the Collins-Roscoe property.*

Recall that a family $\mathcal{A}$ of subsets of a space X is T_0-*separating* if for any distinct points $x, y \in X$ there exists $A \in \mathcal{A}$ such that $A \cap \{x, y\}$ is a singleton. A family $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \omega\}$ of subsets of X is called *weakly σ-point-finite* if for any point $x \in X$ we have the equality $\mathcal{U} = \bigcup\{\mathcal{U}_n :$ the family $\mathcal{U}_n$ is point-finite at $x\}$.

Corollary 4.16 ([18]). *Suppose that X is a Lindelöf Σ-space and there exists a weakly σ-point-finite T_0-separating family of cozero subsets of X. Then the space X has the Collins-Roscoe property.*

Proof: Suppose that a family $\mathcal{U}$ of cozero subsets of X is weakly σ-point-finite and T_0-separating. For any $U \in \mathcal{U}$ take a continuous function $f_U : X \to [0,1]$ such that $U = f_U^{-1}((0,1])$; then the diagonal product of the family $\{f_U : U \in \mathcal{U}\}$ condenses X onto a subset Y of a Σ_s-product Z of real lines. It follows from Theorem 4.11 that Z is a Collins-Roscoe space. Since the Collins-Roscoe property is inherited by arbitrary subspaces, Y has the Collins-Roscoe property. Now we can apply [15, Theorem 3.1] to see that X has the Collins-Roscoe property. □

5. Simple spaces and the Collins-Roscoe property

Say that X is a *simple* space if X has at most one non-isolated point.

Recall that space X is Lindelöf Σ if and only if there is a compact cover $\mathcal{C}$ of X and a countable family $\mathcal{N}$ of subsets of X which is a network with respect to $\mathcal{C}$.

Theorem 5.1. *If X is a simple Lindelöf Σ-space, then there exists a topology τ^* on the set X such that $\tau(X) \subset \tau^*$, the space $X^* = (X, \tau^*)$ is Lindelöf Σ and $C_p(X^*)$ is also Lindelöf Σ.*

Proof: Since X is a Lindelöf Σ-space, we can fix a compact cover $\mathcal{C}$ of the space X for which there exists a closed countable network $\mathcal{N}$ with respect to $\mathcal{C}$. We can assume that X is uncountable. Denote by p the unique non-isolated point of X. We can assume that $\mathcal{N}$ is closed under finite intersections and $p \in C \cap N$ for each $C \in \mathcal{C}$ and $N \in \mathcal{N}$. Let $\mathcal{F} = \{A \subset X : p \in A \text{ and } X = \bigcup\{N \in \mathcal{N} : |N \setminus A| < \omega\}\}$.

Claim 1. $\tau(p, X) \subset \mathcal{F}$.

Let U be a neighborhood of p in X. We need to show that $X = \bigcup\{N \in \mathcal{N} : |N \setminus U| < \omega\}$. Pick a point $x \in X$ and choose $C \in \mathcal{C}$ with $x \in C$. Clearly, $F = C \setminus U$ is a finite set and $U \cup F \in \tau(C, X)$. Therefore we can choose $N \in \mathcal{N}$ with $C \subset N \subset U \cup F$. It follows that $x \in N$ and $N \setminus U \subset F$ is a finite set. Hence $X = \bigcup\{N \in \mathcal{N} : |N \setminus U| < \omega\}$.

Claim 2. $\mathcal{F}$ is a filter.

It is clear that $\mathcal{F}$ is closed under supersets and does not contain the empty set. We shall prove that $\mathcal{F}$ is closed under finite intersections. Pick $A_1, A_2 \in \mathcal{F}$. First observe that $p \in A_1 \cap A_2$. Given $x \in X$ there exist $N_1, N_2 \in \mathcal{N}$ for which $x \in N_1 \cap N_2$, $|N_1 \setminus A_1| < \omega$ and $|N_2 \setminus A_2| < \omega$. Let $N = N_1 \cap N_2$. Then $x \in N$ and since $\mathcal{N}$ is closed under finite intersections, $N \in \mathcal{N}$. Besides $N \setminus (A_1 \cap A_2) \subset (N_1 \setminus A_1) \cup (N_2 \setminus A_2)$ is a finite set. This shows that $X = \bigcup\{N \in \mathcal{N} : |N \setminus (A_1 \cap A_2)| < \omega\}$. Hence $A_1 \cap A_2 \in \mathcal{F}$. It follows from an induction argument that $\mathcal{F}$ is closed under finite intersections.

Let $T = X \setminus \{p\}$. Define a topology τ^* on X as follows: any point in T is declared isolated and $\mathcal{F}$ is the system of open neighborhoods of p. Denote by X^* the space X endowed with the topology τ^*. Because of Claim 2 the topology τ^* is well defined, and because of Claim 1 the identity map $i : X^* \to X$ is continuous.

Claim 3. X^* is a Lindelöf Σ-space.

For each $x \in X^*$ let $C_x = \bigcap\{N \in \mathcal{N} : x \in N\}$; consider the families $\mathcal{C}^* = \{C_x : x \in X\}$ and $\mathcal{N}^* = \mathcal{N}$. First, we will prove that $\mathcal{C}^*$ is a compact cover of X^*. It is clear that $\mathcal{C}^*$ covers X^*. Pick $C_x \in \mathcal{C}$. Notice that $p \in C_x$. Let $\mathcal{U} \subset \tau^*$ be an open cover of C_x. Choose $U \in \mathcal{U}$ with $p \in U$. Since $U \in \mathcal{F}$, there exists $N \in \mathcal{N}$ such that $x \in N$ and $|N \setminus U| < \omega$. Notice that $C_x \subset N$ and hence $C_x \setminus U$ is finite. It follows that C_x can be covered by a finite subfamily of $\mathcal{U}$. Thus C_x is compact. We have proved that $\mathcal{C}^*$ is a compact cover of X^*. Now we will prove that $\mathcal{N}^*$ is a network with respect to $\mathcal{C}^*$. Pick $C_x \in \mathcal{C}^*$ and take $U \in \tau^*$ with $C_x \subset U$. It follows from $p \in C_x$ that $U \in \mathcal{F}$ and so there exists $N_x \in \mathcal{N}$ such that $x \in N_x$ and $|N_x \setminus U| < \omega$. For each $y \in N_x \setminus U$ choose $N_y \in \mathcal{N}$ such that $C_x \subset N_y \subset X \setminus \{y\}$ and let $N = N_x \cap \bigcap\{N_y : y \in N_x \setminus U\}$. It follows that $N \in \mathcal{N} = \mathcal{N}^*$ and $C_x \subset N \subset U$. This concludes the proof of this claim.

Claim 4. $C_p(X^*)$ is a Lindelöf Σ-space.

Consider the set $Q = \{f \in C_p(X^*, 2) : f(p) = 0\}$. Let $\{N_n : n \in \omega\}$ be a numeration of $\mathcal{N}$ and let $s = \{T_n : n \in \omega\}$ where $T_n = N_n \cap T$ for each $n \in \omega$. It is clear that Q is homeomorphic to the Σ_s-product in 2^T centered at zero. It follows from [15, Theorem 3.2] that Q has the Lindelöf Σ-property. Since $C_p(X^*, 2)$ is a union of two subspaces homeomorphic to Q, the space $C_p(X^*, 2)$ also has the Lindelöf Σ-property. By Claim 3 the space X^* is Lindelöf Σ; being zero-dimensional, it embeds in $C_p(C_p(X^*, 2))$ which, together with Okunev's theorem [9, Corollary 2.11], implies that $C_p(X^*)$ is a Lindelöf Σ-space. □

Corollary 5.2. *If X is a simple Lindelöf Σ-space, then $C_p(X)$ has the Collins-Roscoe property.*

Proof: By Theorem 5.1 there exists a topology τ^* on the set X such that $\tau(X) \subset \tau^*$, the space $X^* = (X, \tau^*)$ is Lindelöf Σ and $C_p(X^*)$ is also Lindelöf Σ. It follows from [18, Corollary 2.15] that $C_p(X^*)$ has the Collins-Roscoe property. Since the identity map $i : X^* \to X$ is a condensation, $C_p(X) \subset C_p(X^*)$. Hence $C_p(X)$ has the Collins-Roscoe property. □

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Departamento de Matemáticas, Facultad de Ciencias, Universidad Autónoma del Estado de México, Instituto Literario No. 100, Col. Centro, C.P. 50000, Toluca, Estado de México, Mexico

E-mail: satzchen@yahoo.com.mx

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