Coherent ultrafilters and nonhomogeneity

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Abstract. We introduce the notion of a *coherent P-ultrafilter* on a complete ccc Boolean algebra, strengthening the notion of a P-point on ω, and show that these ultrafilters exist generically under $\mathfrak{c} = \mathfrak{d}$. This improves the known existence result of Ketonen [*On the existence of P-points in the Stone-Čech compactification of integers*, Fund. Math. **92** (1976), 91–94]. Similarly, the existence theorem of Canjar [*On the generic existence of special ultrafilters*, Proc. Amer. Math. Soc. **110** (1990), no. 1, 233–241] can be extended to show that *coherently selective ultrafilters* exist generically under $\mathfrak{c} = \operatorname{cov} \mathcal{M}$.

We use these ultrafilters in a topological application: a coherent P-ultrafilter on an algebra $\mathcal{B}$ is an *untouchable point* in the Stone space of $\mathcal{B}$, witnessing its nonhomogeneity.

Keywords: nonhomogeneity; ultrafilter; Boolean algebra; untouchable point

Classification: 54G05, 06E10

1. Introduction

The article is organized as follows.

In Section 2, we describe the lattice $Part(\mathcal{B})$ of partitions of a complete ccc Boolean algebra $\mathcal{B}$ and see how a given ultrafilter $\mathcal{U}$ on $\mathcal{B}$ interplays with this lattice.

In Section 3, we define *coherent P-ultrafilters* and *coherently selective ultrafilters* on a complete ccc algebra and show that they exist generically, i.e., every filter with a small base can be extended into such ultrafilter, under conditions isolated in [K] and [C].

In Section 4, we recall the homogeneity problem for extremally disconnected compact Hausdorff spaces — the Stone spaces of complete Boolean algebras. We show the relevance of coherent ultrafilters to this question: a coherent P-ultrafilter on a complete ccc Boolean algebra is an *untouchable point* in the corresponding Stone space.

2. The lattice of partitions

Recall that a *partition* of a Boolean algebra is a maximal antichain. We will denote the set of all infinite partitions of an algebra $\mathcal{B}$ by $Part(\mathcal{B})$. We only consider infinite algebras.

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Definition 2.1. Let $\mathcal{B}$ be a Boolean algebra. For two partitions P, Q of $\mathcal{B}$, say that P *refines* Q and write $P \preceq Q$ if for each $p \in P$ there is exactly one $q \in Q$ such that $p \leq q$. For a family $\mathcal{Q}$ of partitions, say that a partition P of $\mathcal{B}$ is a *common refinement* of $\mathcal{Q}$ if $P \preceq Q$ for every $Q \in \mathcal{Q}$.

The minimal possible size of a family of partitions without a common refinement is the *distributivity number* $\mathfrak{h}(\mathcal{B})$ of the algebra $\mathcal{B}$.

Clearly $P \wedge Q = \{p \wedge q; p \in P, q \in Q\} \setminus \{0\}$ is a common refinement of partitions P and Q, and the relation $P \preceq Q$ is easily seen to be a partial order on $Part(\mathcal{B})$; in fact, $P \wedge Q$ is the infimum of $\{P, Q\}$ in $(Part(\mathcal{B}), \preceq)$, making $(Part(\mathcal{B}), \wedge, \{1_{\mathcal{B}}\}, \preceq)$ a semilattice with unit.

Observation 2.2. *For a complete Boolean algebra $\mathcal{B}$, the order $(Part(\mathcal{B}), \preceq)$ is a $\mathfrak{h}(\mathcal{B})$-complete lattice. In particular, $Part(\mathcal{B})$ is complete iff $\mathcal{B}$ is a power algebra.*

Proof: We show that, in fact, every system $\{P_\alpha; \alpha \in \kappa\} \subseteq Part(\mathcal{B})$ has a supremum. Fix any P from the system. For $p \in P$, put $p_0 = p$ and inductively define

$$p_{n+1} = \bigvee \left\{ q \in \bigcup P_\alpha; q \wedge p_n \neq 0 \right\}.$$

Obviously, $p \leq p_n \leq p_{n+1}$ for each $n \in \omega$; put $u(p) = \bigvee \{p_n; n \in \omega\}$. It is easily verified that the set $\bigvee P_\alpha = \{u(p); p \in P\}$ does not depend on the choice of the starting partition P. Clearly $\bigvee P_\alpha$ is a partition refined by every P_α; we show that it is the finest among such partitions, and therefore a supremum.

Let $P_\alpha \preceq R$ for every $\alpha \in \kappa$. It suffices to see that whenever $p \leq r$ for some $p \in P_\alpha$ and $r \in R$, we also have $u(p) \leq r$. This can be shown by induction for every p_n as defined above. Let $p \in P$ and let r be the only member of R such that $p \leq r$. Every $q \in \bigcup P_\alpha$ is below exactly one $r' \in R$, and if $r \neq r'$, then $q \perp p$; hence $p_1 = \bigvee \{q \in \bigcup P_\alpha; q \wedge p \neq 0\} \leq r$. By the same argument, $p_{n+1} \leq r$ for every $n \in \omega$, hence $u(p) \leq r$ and $\bigvee P_\alpha \preceq R$.

We have shown that any system of partitions has a supremum. Hence to have an infimum for a system $\mathcal{Q}$ of size $\kappa < \mathfrak{h}(\mathcal{B})$, we only need to have a lower bound for $\mathcal{Q}$. But this is precisely a common refinement of $\mathcal{Q}$, guaranteed by $\kappa < \mathfrak{h}(\mathcal{B})$.

In the extreme case of an atomic algebra, the set of all atoms is clearly the finest partition, i.e. the smallest element of $Part(\mathcal{B})$. □

Note that for atomless algebras, completeness is actually necessary in the previous observation. The following example shows that in an atomless algebra $\mathcal{B}$ which is not σ-complete, two partitions can always be found that do not have a supremum in $Part(\mathcal{B})$.

Example 2.3. Let $A = \{a_n; n \in \omega\} \subseteq \mathcal{B}$ be a countable subset without a supremum in $\mathcal{B}$; without loss of generality, A is an antichain. Let $\mathcal{C}$ be the completion of $\mathcal{B}$, and consider $c = \bigvee^{\mathcal{C}} A \in \mathcal{C} \setminus \mathcal{B}$. The element $-c \in \mathcal{C}$ can be partitioned into some $\{x_\alpha; \alpha \in \kappa\} = X \subseteq \mathcal{B}$, as $\mathcal{B}$ is dense in $\mathcal{C}$.

Split every $a_n \in A$ into $a_n^0 \vee a_n^1$, put $b_0 = a_0^0$, $b_{n+1} = a_n^1 \vee a_{n+1}^0$ and $B = \{b_n; n \in \omega\}$. Then clearly $\bigvee^{\mathcal{C}} B = \bigvee^{\mathcal{C}} A = c$. Put $P = A \cup X, Q = B \cup X$. Now P, Q are partitions of $\mathcal{B}$, and we show that $\{P, Q\}$ has no supremum in $Part(\mathcal{B})$.

Let $R \in Part(\mathcal{B})$ satisfy $P, Q \preceq R$. Then there must be some $r \in R$ such that $r \geq a_n, b_n$ for all n; but $r \in \mathcal{B}$ cannot be a supremum of a_n, hence r meets some $x \in X$. In fact, we have $x \leq r$, as $X \subseteq P \cap Q$ and $P, Q \preceq R$. Then the partition R_0 which contains $r - x, x \in R_0$ instead of $r \in R$ satisfies $P, Q \preceq R_0 \prec R$. Hence R is not a supremum.

2.1 The structure induced by partitions. Let $\mathcal{B}$ be a complete ccc Boolean algebra. For $P \in Part(\mathcal{B})$, let $\mathcal{B}_P$ be the subalgebra completely generated by $P \subseteq \mathcal{B}$. Denote the inclusion as $e_P : \mathcal{B}_P \subseteq \mathcal{B}$. If $P \preceq Q$, let e_P^Q be the inclusion of $\mathcal{B}_Q$ in $\mathcal{B}_P$. The family $\{\mathcal{B}_P; P \in Part(\mathcal{B})\}$ together with the mappings e_Q^P forms a directed system of complete Boolean algebras indexed by the directed set $(Part(\mathcal{B}), \succeq)$.

Observation 2.4. *In the setting described above,*

(a) *for each $P \in Part(\mathcal{B})$, the algebra $\mathcal{B}_P$ is isomorphic to $P(\omega)$;*
(b) *$\mathcal{B}_{P\wedge Q}$ is completely generated by $\mathcal{B}_P \cup \mathcal{B}_Q$, and $\mathcal{B}_{P\vee Q} = \mathcal{B}_P \cap \mathcal{B}_Q$;*
(c) *$\mathcal{B}_P \cap \mathcal{B}_Q = \{0_{\mathcal{B}}, 1_{\mathcal{B}}\}$ iff $P \vee Q = \{1_{\mathcal{B}}\}$;*
(d) *for $P \preceq Q$, the embedding $e_P^Q : \mathcal{B}_Q \subseteq \mathcal{B}_P$ is regular;*
(e) *for each $P \in Part(\mathcal{B})$, the embedding $e_P : \mathcal{B}_P \subseteq \mathcal{B}$ is regular.*

Lemma 2.5. *The algebra $\mathcal{B}$, with the regular embeddings $e_P : \mathcal{B}_P \to \mathcal{B}$, is a direct limit of the directed system of algebras $\mathcal{B}_P$ and mappings e_P^Q.*

For $P \in Part(\mathcal{B})$, let $\mathcal{J}_P$ be the ideal on $\mathcal{B}$ generated by $P \subseteq \mathcal{B}$. Note that $\mathcal{J}_{P\wedge Q} = \mathcal{J}_P \cap \mathcal{J}_Q$ and $\mathcal{J}_P \subseteq \mathcal{J}_Q$ for $P \preceq Q$. Write $\mathcal{B}/P$ for $\mathcal{B}/\mathcal{J}_P$ and $\mathcal{B}_P/P$ for $\mathcal{B}_P/\mathcal{J}_P$. Whenever $P \preceq Q \in Part(\mathcal{B})$, we have $\mathcal{J}_P \subseteq \mathcal{J}_Q$, hence the algebra $\mathcal{B}/Q$ is a quotient of $\mathcal{B}/P$; denote the quotient mapping by $f_P^Q : \mathcal{B}/P \to \mathcal{B}/Q$. The family of algebras $\mathcal{B}/P$ and mappings f_P^Q for $P, Q \in Part(\mathcal{B})$ forms an inverse system indexed by $(Part(\mathcal{B}), \succeq)$.

Observation 2.6. *In the setting described above,*

(a) *for each $P \in Part(\mathcal{B})$, the quotient $\mathcal{B}_P/P$ is isomorphic to $P(\omega)/fin$;*
(b) *the inclusion $\mathcal{B}_P/P \subseteq \mathcal{B}/P$ is a regular embedding.*

Lemma 2.7. *The algebra $\mathcal{B}$, with the quotient mappings $f_P : \mathcal{B} \to \mathcal{B}/P$, is an inverse limit of the inverse system of algebras $\mathcal{B}/P$ and mappings f_P^Q.*

Employing the Stone duality, we can summarize that

Corollary 2.8. (a) *Every infinite complete ccc algebra is a limit of a directed system of copies of $P(\omega)$. Dually, every infinite ccc extremally disconnected compact space is an inverse limit of an inverse system of copies of $\beta\omega$.*

(b) *Every infinite complete ccc Boolean algebra is an inverse limit of an inverse system of copies of $P(\omega)/fin$. Dually, every infinite ccc extremally disconnected compact space is a direct limit of a system of copies of ω^*.*

For an ultrafilter $\mathcal{U}$ on $\mathcal{B}$ and P a partition of $\mathcal{B}$, let $\mathcal{U}_P = \mathcal{U} \cap \mathcal{B}_P$, which is clearly an ultrafilter on $\mathcal{B}_P$. As $\mathcal{B}_P$ is isomorphic to $P(\omega)$, the ultrafilter $\mathcal{U}_P$ can be viewed as an ultrafilter on ω.

Observation 2.9. *Let $\mathcal{B}$ be a complete atomless ccc algebra, let P, Q be partitions of $\mathcal{B}$, and let $\mathcal{U}$ be an ultrafilter on $\mathcal{B}$. Then*

(a) *$P \cap \mathcal{U} \neq \emptyset$ if and only if $\mathcal{U}_P$ is trivial,*
(b) *$\{P \in Part(\mathcal{B}); \mathcal{U} \cap P = \emptyset\}$ is an open dense subset of $(Part(\mathcal{B}), \preceq)$,*
(c) *$\mathcal{U}_Q = \mathcal{U}_P \cap \mathcal{B}_Q$ for $P \preceq Q$,*
(d) *$\mathcal{B} = \bigcup \{\mathcal{B}_P; P \cap \mathcal{U} = \emptyset\}$.*

3. Coherent ultrafilters

Definition 3.1. Let $\mathcal{B}$ be a complete, atomless, ccc algebra. For a property φ of families of subsets of ω, we say that a subset $X \subseteq \mathcal{B}$ is a *coherent φ-family* on $\mathcal{B}$ if for every partition $P = \{p_n; n \in \omega\}$ of $\mathcal{B}$, the family $\{A \subseteq \omega; \bigvee \{p_n; n \in A\} \in X\}$ of subsets of ω satisfies φ.

For some properties φ, the *coherent* φ is actually no stronger than φ itself. As an easy example, any antichain in $\mathcal{B}$ is a coherent antichain; and any filter $\mathcal{F}$ on $\mathcal{B}$ is a coherent filter, as for every partition P of $\mathcal{B}$, the family $\{A \subseteq \omega; \bigvee \{p_n; n \in A\} \in \mathcal{F}\}$ is a filter on ω. Similarly, every ultrafilter on $\mathcal{B}$ is a coherent ultrafilter, and an ultrafilter that is coherently trivial is a generic ultrafilter on $\mathcal{B}$. We are interested in ultrafilters with special properties, for which the coherent version becomes nontrivial.

It can be seen from the very definition that the ZFC implications between various classes of ultrafilters on ω continue to hold for the corresponding classes of coherent ultrafilters on $\mathcal{B}$. For instance, every coherent selective ultrafilter on $\mathcal{B}$ is a coherent P-ultrafilter on $\mathcal{B}$, as every selective ultrafilter on ω is a P-ultrafilter on ω.

3.1 Coherent P-ultrafilters.

Definition 3.2. An ultrafilter $\mathcal{U}$ on a complete ccc algebra $\mathcal{B}$ is a *coherent P-ultrafilter* if for every partition P of $\mathcal{B}$, the family $\{A \subseteq \omega; \bigvee \{p_n; n \in A\} \in \mathcal{U}\}$ is a P-ultrafilter on ω.

Seeing that the subalgebra $\mathcal{B}_P$ is a copy of $P(\omega)$, we can equivalently characterize coherent P-ultrafilters as follows.

Observation 3.3. *Let $\mathcal{B}$ be a complete ccc algebra. An ultrafilter $\mathcal{U}$ on $\mathcal{B}$ is a coherent P-ultrafilter iff for every pair of partitions P and Q of $\mathcal{B}$ such that $P \preceq Q$, either $\mathcal{U} \cap Q \neq \emptyset$, or there is a set $X \subseteq P$ such that $\bigvee X \in \mathcal{U}$ and for every $q \in Q$, the set $\{p \in X; p \wedge q \neq 0\}$ is finite.*

We show now that coherent P-points consistently exist. The proof is an iteration of the Ketonen argument of [K] for the existence of P-points on ω.

Proposition 3.4. *Let $\mathcal{B}$ be a complete ccc Boolean algebra of size at most $\mathfrak{c}$. Every filter on $\mathcal{B}$ with a base smaller than $\mathfrak{c}$ can be extended to a coherent P-ultrafilter on $\mathcal{B}$ if and only if $\mathfrak{c} = \mathfrak{d}$.*

PROOF: Assume $\mathfrak{c} = \mathfrak{d}$ and let $\mathcal{F} \subseteq \mathcal{B}$ be a filter with a base smaller than $\mathfrak{c}$. We will construct an increasing chain of filters $\mathcal{F}_\alpha$ extending $\mathcal{F}$, eventually arriving at a filter $\bigcup \mathcal{F}_\alpha$, where each $\mathcal{F}_\alpha$ takes care of a pair of partitions, as per 3.3.

Start with $\mathcal{F}_0 = \mathcal{F}$ and enumerate all partition pairs $P \preceq Q$ as (P_α, Q_α), where $\alpha < \mathfrak{d}$ runs through all isolated ordinals. If an increasing chain $(\mathcal{F}_\beta; \beta < \alpha)$ of filters has already been found such that every $\mathcal{F}_\beta$ has a base smaller than $\mathfrak{c}$ and has the P-ultrafilter property 3.3 with respect to the partition pairs $P_\gamma \preceq Q_\gamma$ for $\gamma < \beta$, proceed as follows.

If α is a limit, take for $\mathcal{F}_\alpha$ the filter generated by $\bigcup\{\mathcal{F}_\beta; \beta < \alpha\}$; then $\mathcal{F}_\alpha$ still has a base smaller than $\mathfrak{c} = \mathfrak{d}$. We didn't miss a partition pair here.

If $\alpha = \beta + 1$ is a successor, consider the partition pair $P_\beta \preceq Q_\beta$. If some $q \in Q_\beta$ is compatible with $\mathcal{F}_\beta$, let $\mathcal{F}_\alpha = \mathcal{F}_{\beta+1}$ be the filter generated by $\mathcal{F}_\beta \cup \{q\}$ and be done with (P_β, Q_β). If there is no such q in Q_β, enumerate Q_β as $\{q_n; n \in \omega\}$ and consider the refinement P_β of Q_β. Without loss of generality, every $q_n \in Q_\beta$ is partitioned into infinitely many $p \in P_\beta$; enumerate $\{p \in P; p < q_n\}$ as $\{p_n^m; m \in \omega\}$. Let $\{a_\xi; \xi < \kappa\}$ be a base of $\mathcal{F}_\beta$, for some $\kappa < \mathfrak{c}$.

Now perform the Ketonen construction for this step: for each $\xi < \kappa$, put $f_\xi(n) = \min\{m; a_\xi \wedge p_n^m \neq 0\}$ if there is such an m. The value of $f_\xi(n)$ is defined for infinitely many n, corresponding to those q_n which a_ξ meets. In the missing places, fill the value of $f_\xi(n)$ with the *next* defined value (there must be some). This yields a family $\{f_\xi : \omega \to \omega; \xi < \kappa\}$ of functions which cannot be dominating, as $\kappa < \mathfrak{c} = \mathfrak{d}$. Therefore, there is a function $f : \omega \to \omega$ which is not dominated by any f_ξ; that is, for each ξ, we have $f(n) > f_\xi(n)$ for infinitely many n. We can assume that f is strictly increasing.

Put $a = \bigvee\{p_n^m; n \in \omega, m \leq f(n)\}$. The element a is compatible with $\mathcal{F}_\beta$, because it meets every a_ξ, as witnessed by $f \not\leq^* f_\xi$. Let $\mathcal{F}_\alpha$ be the filter generated by $\mathcal{F}_\beta \cup \{a\}$. This filter obviously extends $\mathcal{F}_\beta$, is generated by fewer than $\mathfrak{c}$ elements, and has the P-ultrafilter property with respect to (P_β, Q_β).

Now every ultrafilter extending $\bigcup\{\mathcal{F}_\alpha; \alpha < \mathfrak{c}\}$ is a coherent P-ultrafilter on $\mathcal{B}$ that extends $\mathcal{F}$, because we have taken care of all possible partition pairs $P \preceq Q$, as requested by 3.3.

The other direction follows from [K] immediately. Being able to extend every small filter $\mathcal{F} \subseteq \mathcal{B}$ into a coherent P-ultrafilter is apparently stronger than being able to extend every small filter $\mathcal{F}$ on ω to a P-point, which itself implies $\mathfrak{c} = \mathfrak{d}$. □

For completeness, we translate the Ketonen argument for the opposite direction into the algebra $\mathcal{B}$, showing how $\mathfrak{d} < \mathfrak{c}$ can break the coherence *anywhere*.

Assume $\mathfrak{d} < \mathfrak{c}$ and let $\{f_\alpha; \alpha < \mathfrak{d}\}$ be a dominating family of functions. Choose any two countable partitions $P \preceq Q$ of $\mathcal{B}$ such that every $q_n \in Q$ is partitioned into countably many $p_n^m \in P$. For each $\alpha < \mathfrak{d}$, put $a_\alpha = \bigcup\{p_n^m; m > f_\alpha(n)\}$. The family $\{a_\alpha; \alpha < \mathfrak{d}\} \cup \{-q_n; n \in \omega\} \subseteq \mathcal{B}$ is centered, and the filter $\mathcal{F}$ that it generates has $\mathfrak{d} < \mathfrak{c}$ generators. No ultrafilter on $\mathcal{B}$ that extends $\mathcal{F}$ can be a coherent P-ultrafilter, as witnessed by $P \preceq Q$.

We have shown that coherent P-ultrafilters consistently exist on complete ccc algebras of size not exceeding the continuum. On the other hand, there is consistently no coherent P-ultrafilter on any complete ccc algebra, as even the classical P-points need not exist [W]. Hence the existence of coherent P-ultrafilters is undecidable in ZFC.

Question 3.5. The consistency we have shown is what [C] calls "generic existence". Under our assumptions, coherent P-ultrafilters not only exist, but every small filter can be enlarged into one.

(a) Is it consistent that P-points exist on ω, but there are no coherent P-ultrafilters on complete atomless ccc algebras?
(b) Is it consistent that a coherent P-ultrafilter exists on a complete atomless ccc algebra $\mathcal{B}$, but it does not exist on another?
(c) Is there a single "testing" algebra $\mathcal{B}$ with the property that if there is a coherent P-ultrafilter on $\mathcal{B}$, then necessarily $\mathfrak{c} = \mathfrak{d}$, and hence P-ultrafilters exist generically?

3.2 Coherent selective ultrafilters. Similarly to coherent P-ultrafilters, the arguments from [K] and [C] on existence of selective ultrafilters on ω can be strengthened to coherent selective ultrafilters on complete ccc algebras.

Definition 3.6. Let $\mathcal{B}$ be a complete ccc algebra. An ultrafilter $\mathcal{U}$ on $\mathcal{B}$ is a coherent selective ultrafilter iff for every pair of partitions P and Q of $\mathcal{B}$ such that $P \preceq Q$, either $\mathcal{U} \cap Q \neq \emptyset$, or there is a set $X \subseteq P$ such that $\bigvee X \in \mathcal{U}$ and for every $q \in Q$, the set $\{p \in X; p \wedge q \neq 0\}$ is at most a singleton.

Proposition 3.7. *Let $\mathcal{B}$ be a complete ccc Boolean algebra of size at most $\mathfrak{c}$. Then every filter $\mathcal{F}$ on $\mathcal{B}$ with a base smaller than $\mathfrak{c}$ can be extended to a coherent selective ultrafilter on $\mathcal{B}$ if and only if $\mathfrak{c} = \mathrm{cov}(\mathcal{M})$.*

Proof: Assume $\mathfrak{c} = \mathrm{cov}(\mathcal{M})$ and let $\mathcal{F}$ be a filter with a base smaller than $\mathfrak{c}$. We will construct an increasing chain of filters extending $\mathcal{F}$. Put $\mathcal{F}_0 = \mathcal{F}$ and enumerate all partition pairs $P \preceq Q$ as $\{(P_\alpha, Q_\alpha); \alpha < \mathrm{cov}(\mathcal{M}) \text{ isolated }\}$.

If an increasing chain $(\mathcal{F}_\beta; \beta < \alpha)$ of filters has been found such that every $\mathcal{F}_\beta$ has a base smaller than $\mathfrak{c}$ and has the selective property with respect to all $\{(P_\gamma, Q_\gamma); \gamma < \beta\}$, proceed as follows.

If α is a limit, take for $\mathcal{F}_\alpha$ the filter generated by $\bigcup\{\mathcal{F}_\beta; \beta < \alpha\}$. Then $\mathcal{F}_\alpha$ still has a base smaller than $\mathfrak{c}$.

If $\alpha = \beta + 1$ is a successor, consider $(P, Q) = (P_\beta, Q_\beta)$. Without loss of generality, both partitions are infinite, and every $q_n \in Q$ is infinitely partitioned into $p_n^m \in P$.

If there is some $q \in Q$ compatible with $\mathcal{F}_\beta$, let $\mathcal{F}_\alpha$ be the filter generated by $\mathcal{F}_\beta \cup \{q\}$. If there is no such $q \in Q$, consider some base $\{a_\xi; \xi < \kappa\}$ of $\mathcal{F}_\beta$, where

$\kappa < \mathfrak{c}$. Every a_ξ intersects infinitely many $q \in Q$: if a_ξ only meets $q_1, \dots, q_n \in Q$, choose a_ξ^i disjoint with q_i, respectively; then $a_\xi \leq \bigvee q_i$ is disjoint with $\bigwedge a_\xi^i$ — a contradiction.

Consider the set $T = \Pi_{n\in\omega} \{p_n^m; m \in \omega\}$; the functions $\varphi \in T$ are the selectors for Q. View T as a copy of the Baire space ω^ω. If no selector for Q is compatible with $\mathcal{F}_\beta$, put $T_\xi = \{\varphi \in T; \bigvee rng(\varphi) \perp a_\xi\}$; then we have $T = \bigcup_{\xi<\kappa} T_\xi$. But the sets T_ξ cannot cover T, as $\kappa < \mathrm{cov}(\mathcal{M})$ and every T_ξ is a nowhere dense subset of T, which is seen as follows.

For a basic clopen subset $[s]$ of T, there is some $n > |s|$ such that a_ξ meets $q_n \in Q$, because a_ξ meets infinitely many q_n. Hence some p_n^m meets a_ξ. Extend s into t so that $t(n) = m$. Then $[t] \subseteq [s]$ is disjoint with T_ξ.

Thus there must be a selector $\varphi \in T$ with $b = \bigvee rng(\varphi)$ compatible with every a_ξ. Let $\mathcal{F}_{\beta+1}$ be the filter generated by $\mathcal{F}_\beta \cup \{b\}$. Iterating this process, we obtain an increasing sequence of filters $(\mathcal{F}_\alpha; \alpha \in \mathfrak{c})$ extending $\mathcal{F} = \mathcal{F}_0$. Now every ultrafilter extending $\bigcup \mathcal{F}_\alpha$ is a coherent selective ultrafilter on $\mathcal{B}$ by 3.6. $\square$

4. Nonhomogeneity

Definition 4.1. A topological space X is *homogeneous* if for every pair of points $x, y \in X$ there is an autohomeomorphism h of X such that $h(x) = y$.

Extremally disconnected compact Hausdorff spaces, which are precisely the Stone spaces of complete Boolean algebras, are long known *not* to be homogeneous. However, the original elegant proof due to Frolík [F] suggests no simple topological property of points to be a reason for this.

If a space X is not homogeneous, then points $x, y \in X$ failing the automorphism property are often called *witnesses of nonhomogeneity*. In large subclasses of the extremally disconnected compacts, such witnesses have been found by isolating a simple topological property that is shared by some, but not all, points in the space.

Definition 4.2. A point x of a topological space X is an *untouchable point* if $x \notin \overline{D}$ for every countable nowhere dense subset $D \subseteq X$ not containing x.

The subclass of extremally disconnected compact spaces where a witness of nonhomogeneity has not been explicitly described yet is currently reduced to the class of ccc spaces of weight at most continuum. In other extremally disconnected compacts, points with even stronger properties have been found. See [S] and [BS] for history and pointers to the development of these questions.

4.1 An application to nonhomogeneity. Via Stone duality, the topic has a Boolean translation: we are looking for ultrafilters on complete ccc Boolean algebras of size (or, equivalently, algebraic density) at most continuum, which are discretely untouchable points in the corresponding Stone spaces. It is in this algebraic form that we actually deal with the question.

Proposition 4.3. *Let $\mathcal{B}$ be a complete ccc algebra. Let $\mathcal{U}$ be a coherent P-ultrafilter on $\mathcal{B}$. Then $\mathcal{U}$ is an untouchable point in the Stone space of $\mathcal{B}$.*

Proof: We assume that $\mathcal{U}$ is not generated by an atom, otherwise there is nothing to prove. Let $R = \{\mathcal{F}_n; n \in \omega\}$ be a countable nowhere dense set in $\mathrm{St}(\mathcal{B})$ such that $\mathcal{F}_n \neq \mathcal{U}$ for all n. Choose some $a_0 \in \mathcal{F}_0$ with $-a_0 \in \mathcal{U}$ and put $R_0 = \{\mathcal{F} \in R; a_0 \in \mathcal{F}\} \subseteq R$. Generally, if $a_i \in \mathcal{B}^+$ for $i < k$ are disjoint elements such that $\bigvee_{i<k} a_i \notin \mathcal{U}$ and $R_i = \{\mathcal{F} \in R; a_i \in \mathcal{F}\}$, consider $\bigcup_{i<k} R_i \subseteq R$. If $\bigcup_{i<k} R_i = R$, we are done, as $\bigvee_{i<k} a_i \notin \mathcal{U}$ guarantees $\mathcal{U} \notin cl(R)$. Otherwise, let n_k be the first index such that $\mathcal{F}_{n_k} \notin \bigcup_{i<k} R_i$ and choose some a_k disjoint with $\bigvee_{i<k} a_i$ such that $a_k \in \mathcal{F}_{n_k}$ and $a_k \notin \mathcal{U}$.

This construction either stops at some k and we are done, or we arrive at an infinite disjoint system $Q = \{a_i; i \in \omega\} \subseteq \mathcal{B}^+$. Again, if $\bigvee Q \notin \mathcal{U}$, we have $\mathcal{U} \notin cl(R)$. Otherwise, we can assume that $\bigvee Q = 1$, so Q is a partition of $\mathcal{B}$. For each $a_i \in Q$, choose an infinite partition P_i of a_i such that $P_i \cap \bigcup R_i = \emptyset$ – this is possible, because $R_i \subseteq R$ is nowhere dense. Now $P = \bigcup P_i \preceq Q$ is a partition pair in $\mathcal{B}$.

As $\mathcal{U}$ is a coherent P-ultrafilter and misses Q, there is some $X \subseteq P$ with $u = \bigvee X \in \mathcal{U}$ such that for every i, the set $\{p \in X; p \leq a_i\}$ is finite. This means that $u \notin \mathcal{F}_n$ for all n: every $\mathcal{F}_n$ is in one particular a_i, so $u \in \mathcal{F}_n$ would mean that $\mathcal{F}_n$ contains one of the finitely many $\{p \leq u; p \leq a_i\}$. But this is in contradiction with $P_i \cap \bigcup R_i = \emptyset$. So $u \in \mathcal{U}$ isolates $\mathcal{U}$ from $cl(R)$. □

In fact, we have proven a slightly stronger statement: $\mathcal{U}$ escapes the closure of any nowhere dense set that can be covered by countably many disjoint open sets.

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References

[BS] Balcar B., Simon P., *On minimal π-character of points in extremally disconnected compact spaces*, Topology Appl. **41** (1991), 133–145.

[C] Canjar R.M., *On the generic existence of special ultrafilters*, Proc. Amer. Math. Soc. **110** (1990), no. 1, 233–241.

[F] Frolík Z., *Maps of extremally disconnected spaces, theory of types, and applications*, in Franklin, Frolík, Koutník (eds.), General Topology and Its Relations to Modern Analysis and Algebra, Proceedings of the Kanpur topological conference (1971), pp. 131–142.

[K] Ketonen J., *On the existence of P-points in the Stone-Čech compactification of integers*, Fund. Math. **92** (1976), 91–94.

[S] Simon P., *Points in extremally disconnected compact spaces*, Rend. Circ. Mat. Palermo (2). Suppl. 24 (1990), 203–213.

[W] Wimmer E.L., *The Shelah P-point independence theorem*, Israel J. Math **43** (1982), no. 1, 28–48.

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