# A co-ideal based identity-summand graph of a commutative semiring

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*Abstract.* Let $I$ be a strong co-ideal of a commutative semiring $R$ with identity. Let $\Gamma_I(R)$ be a graph with the set of vertices $S_I(R) = \{x \in R \setminus I : \; x + y \in I$ for some $y \in R \setminus I\}$, where two distinct vertices $x$ and $y$ are adjacent if and only if $x + y \in I$. We look at the diameter and girth of this graph. Also we discuss when $\Gamma_I(R)$ is bipartite. Moreover, studies are done on the planarity, clique, and chromatic number of this graph. Examples illustrating the results are presented.

*Keywords:* strong co-ideal; $Q$-strong co-ideal; identity-summand element; identity-summand graph; co-ideal based

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## 1. Introduction

Among the most interesting graphs are the zero-divisor graphs, because these involve both ring theory and graph theory. By studying these graphs, we can gain a broader insight into the concepts and properties that involve both graphs and rings. It was Beck (see [3]) who first introduced the notion of a zero-divisor graph for commutative ring. This notion was later redefined by D. F. Anderson and P. S. Livingston in [1]. In [12], Redmond introduced the zero-divisor graph with respect to a proper ideal. Since then, there has been a lot of interest in this subject and various papers were published establishing different properties of these graphs as well as relations between graphs of various extensions (see [2], [11], [12] and [13]). Recently, such graphs are used to study semirings [5], [6] and [9].

Semirings have proven to be useful in theoretical computer science, in particular for studying automata and formal languages, hence, ought to be in the literature [10] and [14]. From now on let $R$ be a commutative semiring with identity. In [8], the present authors introduced the identity-summand graph, denoted by $\Gamma(R)$, such that vertices are all non-identity identity-summands of $R$ and two distinct vertices are joint by an edge when the sum of them is 1. We use the notation $S(R)$ to refer to the set of elements of $R$ that are identity-summands (we use $S^*(R)$ to denote the set of non-identity identity-summands of $R$), we say that $r \in R$ is an identity-summand of $R$, if there exists $1 \neq a \in R$ such that $r + a = 1$.

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In this paper we will generalize this notion by replacing elements whose sum is identity with elements whose sum lies in some strong co-ideal $I$ of $R$. Indeed, we define an undirected graph $\Gamma_I(R)$ with vertices $S_I(R) = \{x \in R \setminus I : \ x + y \in I$ for some $y \in R \setminus I\}$, where distinct vertices $x$ and $y$ are adjacent if and only if $x + y \in I$. This definition was motivated by [12], [6] and [8]. Here is a brief summary of our paper. We will make an intensive study on identity-summand graph of commutative semirings based on strong co-ideals. In section 2, it is shown that $\Gamma_I(R)$ is connected with $\mathrm{diam}(\Gamma_I(R)) \leq 3$, and if $I$ is a subtractive co-ideal, then $\Gamma_I(R)$ is not complete. We show that if $\Gamma_I(R)$ contains a cycle, then $\mathrm{gr}(\Gamma_I(R)) \leq 4$ and several characterizations of $\Gamma_I(R)$ by girth are given. Also it is proved that if $I$ is a $Q$-strong co-ideal and $\Gamma_I(R)$ and $\Gamma(R/I)$ has a cycle, then $\mathrm{gr}(\Gamma_I(R)) = \mathrm{gr}(\Gamma(R/I))$. In Section 3, it is shown that for a subtractive strong co-ideal $I$ of $R$, $\Gamma_I(R)$ is complete bipartite if and only if there exist two distinct prime strong co-ideals $P_1$ and $P_2$ of $R$ such that $P_1 \cap P_2 = I$. Section 4 is devoted to study chromatic number, clique number and planar property of $\Gamma_I(R)$.

In order to make this paper easier to follow, we recall various notions which will be used in the sequel. For a graph $\Gamma$, we denote by $E(\Gamma)$ and $V(\Gamma)$ the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two distinct vertices. The distance between two distinct vertices $a$ and $b$, denoted by $\mathrm{d}(a,b)$, is the length of the shortest path connecting them (if such a path does not exist, then $d(a,b) = \infty$, also $d(a,a) = 0$). The diameter of graph $\Gamma$, denoted by $\mathrm{diam}(\Gamma)$, is equal to $\sup\{d(a,b) : a, b \in V(\Gamma)\}$. A graph is complete if it is connected with diameter less than or equal to one. We denote the complete graph on $n$ vertices by $K_n$. The girth of a graph $\Gamma$, denoted $\mathrm{gr}(\Gamma)$, is the length of a shortest cycle in $\Gamma$, provided $\Gamma$ contains a cycle; otherwise $\mathrm{gr}(\Gamma) = \infty$. An edge for which the two ends are the same is called a loop at the common vertex. For $r$ a nonnegative integer, an $r$-partite graph is one whose set of vertices can be partitioned into $r$ subsets so that no edge has both ends in any single subset. A complete $r$-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite (i.e., 2-partite) graph with parts of size $m$ and $n$ is denoted by $K_{m,n}$. We will sometimes call $K_{1,n}$ a star graph. We define a coloring of a graph $G$ to be an assignment of colors (elements of some set) to vertices of $G$, one color to each vertex, so that distinct colors are assigned to adjacent vertices. If $n$ colors are used, then the coloring is referred to as an $n$-coloring. If there exists an $n$-coloring of a graph $G$, then $G$ is called $n$-colorable. The minimum $n$ for which a graph $G$ is $n$-colorable is called the chromatic number of $G$, and is denoted by $\chi(G)$. A clique of a graph is its maximal complete subgraph and the maximal number of vertices in any clique of graph $G$, denoted by $w(G)$, is called the clique number of $G$.

A commutative semiring $R$ is defined as an algebraic system $(R, +, \cdot)$ such that $(R, +)$ and $(R, \cdot)$ are commutative semigroups, connected by $a(b+c) = ab + ac$ for all $a, b, c \in R$, and there exists $0, 1 \in R$ such that $r + 0 = r$ and $r0 = 0r = 0$

and $r1 = 1r = r$ for each $r \in R$. In this paper all semirings considered will be assumed to be commutative semirings with non-zero identity.

**Definition 1.1.** Let $R$ be a semiring.

(1) A non-empty subset $I$ of $R$ is called *co-ideal*, if it is closed under multiplication and satisfies the condition $r + a \in I$ for all $a \in I$ and $r \in R$ (so $0 \in I$ if and only if $I = R$). A co-ideal $I$ of $R$ is called *strong* co-ideal provided that $1 \in I$ (in this case, $1 + x \in I$ for every $x \in R$).

(2) A co-ideal $I$ of $R$ is called *subtractive* if $x, xy \in I$ implies $y \in I$ (so every subtractive co-ideal is a strong co-ideal).

(3) If $I$ is a co-ideal of $R$, then the co-rad$(I)$ of $I$, is the set of all $x \in R$ for which $nx \in I$ for some positive integer $n$. This is a co-ideal of $R$ containing $I$ [7].

(4) A proper co-ideal $P$ of $R$ is called *prime* if $x + y \in P$ implies $x \in P$ or $y \in P$. The set of all prime co-ideals of $R$ is denoted by co-Spec$(R)$. A proper co-ideal $I$ of $R$ is called *primary* if $a + b \in I$ implies $a \in I$ or $b \in$ co-rad$(I)$. If $I$ is primary, then co-rad$(I)$ is a prime co-ideal. We say that $I$ is $P$-primary if $I$ is primary and co-rad$(I) = P$ [7].

(5) If $D$ is an arbitrary nonempty subset of $R$, then the set $F(D)$ consisting of all elements of $R$ of the form $d_1 d_2 \ldots d_n + r$ (with $d_i \in D$ for all $1 \leq i \leq n$ and $r \in R$) is a co-ideal of $R$ generated by $D$ [7], [10] and [14].

(6) A semiring $R$ is called *co-semidomain*, if $a + b = 1$ $(a, b \in R)$ implies either $a = 1$ or $b = 1$ [7].

A strong co-ideal $I$ of a semiring $R$ is called a *partitioning strong co-ideal* (= *$Q$-strong co-ideal*) if there exists a subset $Q$ of $R$ such that the following hold.

(1) $R = \bigcup\{qI : q \in Q\}$, where $qI = \{qt : t \in I\}$.

(2) If $q_1, q_2 \in Q$, then $(q_1 I) \cap (q_2 I) \neq \emptyset$ if and only if $q_1 = q_2$.

(3) For each $q_1, q_2 \in Q$, there exists $q_3 \in Q$ such that $q_1 I + q_2 I \subseteq q_3 I$.

Let $I$ be a $Q$-strong co-ideal of a semiring $R$ and let $R/I = \{qI : q \in Q\}$. Then $R/I$ forms a semiring under the binary operations $\oplus$ and $\odot$ defined as follows: $(q_1 I) \oplus (q_2 I) = q_3 I$, where $q_3$ is the unique element in $Q$ such that $(q_1 I + q_2 I) \subseteq q_3 I$, and $(q_1 I) \odot (q_2 I) = q_3 I$, where $q_3$ is the unique element in $Q$ such that $(q_1 q_2) I \subseteq q_3 I$ [7]. If $q_e$ is the unique element in $Q$ such that $1 \in q_e I$, then $q_e I = I$ is the identity of $R/I$. Note that every $Q$-strong co-ideal is subtractive [7]. Throughout this paper we shall assume unless otherwise stated, that $q_0 I$ (resp. $q_e I$) is the zero element (resp. the identity element) of $R/I$. In the following, we give an example of a $Q$-strong co-ideal. One can see another example of $Q$-strong co-ideal in [7].

**Example 1.2.** Let $R$ be the set of all non-negative integers. Define $a + b = \gcd(a, b)$ and $a \times b = \mathrm{lcm}(a, b)$ (take $0 + 0 = 0$ and $0 \times 0 = 0$ ). Then $(R, +, \times)$ is easily checked to be a commutative semiring. Let $I$ be the set of all non-negative odd integers. Then $I$ is a strong co-ideal of $R$. Set $Q = \{0, 1, 2, 4, 8, 16, 32, 64, ...\}$. It is clear that $I$ is a $Q$-strong co-ideal.

## 2. Examples and basic properties of $\Gamma_I(R)$

In this section we study the diameter, girth and cut-point of $\Gamma_I(R)$, when $I$ is a strong co-ideal of the semiring $R$.

**Proposition 2.1.** *Let $I$ be a subtractive co-ideal of a semiring $R$. Then the following hold:*

(1) *if $xy \in I$, then $x, y \in I$ for all $x, y \in R$;*
(2) *$I = \text{co-rad}(I)$;*
(3) *$(I : a) = \{r \in R : r + a \in I\}$ is a subtractive co-ideal of $R$ for all $a \in R$;*
(4) *if $I$ is a $Q$-strong co-ideal of $R$ and $q_eI$ is the identity element in $R/I$, then $q_eI \oplus qI = q_eI$ and $qI \oplus qI = qI$ for all $qI \in R/I$.*

Proof: (1) Observe that $1 + x \in I$ for each $x \in R$. If $xy \in I$, then $y(1 + x) = xy + y \in I$ gives $y \in I$, since $I$ is subtractive. Similarly, $x \in I$.

(2) It suffices to show that $\text{co-rad}(I) \subseteq I$. Let $x \in \text{co-rad}(I)$, so $nx \in I$ for some positive integer $n \in \mathbb{N}$. Thus $nx = x(\underbrace{1 + 1 + \cdots + 1}_{n\ times}) \in I$ gives $x \in I$.

(3) Clearly, $1 \in (I : a)$. If $x, y \in (I : a)$, then $x + a \in I$ and $y + a \in I$, implying $a^2 + ax + ay + xy \in I$. Since $(xy + a)(1 + a)(1 + y)(1 + x) \in I$, $xy + a \in I$ by (1). Thus $xy \in (I : a)$. As $I$ is a co-ideal, $r + x + a \in I$ for each $r \in R$ and so $x + r \in (I : a)$ for each $r \in R$. This shows that $(I : a)$ is a co-ideal of $R$. Now let $xy, x \in (I : a)$. Then $xy + a + y + xa = (x + 1)(y + a) \in I$, which gives $y + a \in I$, and so $y \in (I : a)$, as desired.

(4) Let $q_eI \oplus qI = q'I$, where $q'$ is the unique element in $Q$ such that $q_eI + qI \subseteq q'I$. Since $I$ is co-ideal, $qI + q_eI \subseteq q_eI \cap q'I$, which gives $q_eI = q'I$. Finally, $qI \oplus qI = qI \odot (q_eI \oplus q_eI) = qI \odot q_eI = qI$. □

**Proposition 2.2.** *Let $I$ be a strong co-ideal of a semiring $R$. Then $S_I(R) = \emptyset$ if and only if $I$ is a prime strong co-ideal of $R$.*

Proof: This follows directly from the definitions. □

**Theorem 2.3.** *Let $I$ be a $Q$-strong co-ideal of $R$. Then the following are equivalent:*

(1) *$S_I(R) = \emptyset$;*
(2) *$I$ is a prime co-ideal of $R$;*
(3) *$S^*(R/I) = \emptyset$;*
(4) *$I$ is $P$-primary.*

Proof: (1) $\Leftrightarrow$ (2) follows from Proposition 2.2.

(2) $\Leftrightarrow$ (3) By [[7], Theorem 3.8], $I$ is prime if and only if $R/I$ is co-semidomain. Therefore $I$ is prime if and only if $S^*(R/I) = \emptyset$.

(2) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (2) If $I$ is a $P$-primary strong co-ideal of $R$, then $I = \text{co-rad}(I) = P$ by Proposition 2.1(2) and [7, Proposition 2.2]; hence $I$ is prime. $\square$

Redmond [12] explored the relationship between $\Gamma_I(R)$ and $\Gamma(R/I)$. He gave an example of rings $R, T$ and ideals $I \trianglelefteq R$, $J \trianglelefteq T$, where $\Gamma(R/I) \cong \Gamma(T/J)$ but $\Gamma_I(R) \ncong \Gamma_J(T)$. Here we generalize this concept to the case of semirings.

**Example 2.4.** Let $X = \{a, b, c\}$ and $R = (P(X), \cup, \cap)$ a semiring with $1_R = X$, where $P(X)$ is the set of all subsets of $X$. If $I = \{X, \{a, b\}\}$, then $I$ is a $Q$-strong co-ideal, where $Q = \{q_1 = \{c\}, q_2 = \{a, c\}, q_3 = \{b, c\}, q_e = X\}$. An inspection will show that $q_2I \oplus q_3I = q_eI$ and $S^*(R/I) = \{q_2I, q_3I\}$. Also $S_I(R) = \{\{a\}, \{b\}, \{a, c\}, \{b, c\}\}$. Let $T = \{0, 1, 2, 3, 4, 6, 12\}$. Then $(T, \gcd, \operatorname{lcm})$ (take $\gcd(0, 0) = 0$ and $\operatorname{lcm}(0, 0) = 0$) is a commutative semiring. If $J = \{1, 2\}$, then it easily can be checked that $J$ is a $Q$-strong co-ideal with $Q = \{0, 1, 3, 4, 12\}$, $T/J = \{0J, 1J, 3J, 4J, 12J\}$, $S^*(T/J) = \{3J, 4J\}$ and $S_J(T) = \{3, 4, 6\}$. Thus $\Gamma(R/I) \cong \Gamma(T/J)$, however $\Gamma_I(R) \ncong \Gamma_J(T)$.

The next several results investigate the relationship between $\Gamma(R/I)$ and $\Gamma_I(R)$.

**Proposition 2.5.** *Let $I$ be a $Q$-strong co-ideal of a semiring $R$ and let $x, y \in S_I(R)$ such that $x \in q_1I$ and $y \in q_2I$, for some $q_1, q_2 \in Q$. Then:*

1. *$x$ is adjacent to $y$ in $\Gamma_I(R)$ if and only if $q_1I$ and $q_2I$ are adjacent in $\Gamma(R/I)$ and $q_1 \neq q_2$. In particular, each elements of $q_1I$ are adjacent to each elements of $q_2I$ in $\Gamma_I(R)$.*
2. *If $q_1I \in S^*(R/I)$, then all the distinct elements of $q_1I$ are not adjacent to each other in $\Gamma_I(R)$.*

Proof: (1) Let $x$ be adjacent to $y$ in $\Gamma_I(R)$, so $x + y \in q_eI = I$. Let $q_1I \oplus q_2I = q_3I$, where $q_3$ is the unique element in $Q$ such that $q_1I + q_2I \subseteq q_3I$. Since $x + y \in q_3I \cap q_eI$, $q_3 = q_e$. Thus $q_1I$ is adjacent to $q_2I$ in $\Gamma(R/I)$. We show $q_1 \neq q_2$. Suppose, on the contrary, $q_1 = q_2$. Since $q_1I$ and $q_2I$ are adjacent, we have $I = q_eI = q_1I \oplus q_2I = q_1I \oplus q_1I = q_1I$ by Proposition 2.1(4), a contradiction. Thus $q_1 \neq q_2$. Conversely, let $q_1I$ be adjacent to $q_2I$ in $\Gamma(R/I)$, so $q_1I \oplus q_2I = q_eI$, where $(q_1I + q_2I) \subseteq q_eI$. Then $x + y \in q_1I + q_2I \subseteq q_eI = I$; hence $x$ is adjacent to $y$ in $\Gamma_I(R)$. Now, from above discussion, it is clear that each elements of $q_1I$ are adjacent to each elements of $q_2I$ in $\Gamma_I(R)$.

(2) It is similar to the proof of (1). $\square$

An edge for which the two ends are the same is called a loop at the common vertex.

**Theorem 2.6.** *Let $I$ be a strong co-ideal of a semiring $R$.*

1. *If $I$ is subtractive, then $\Gamma_I(R)$ has no loop.*
2. *If $I$ is a $Q$-strong co-ideal and $\Gamma(R/I) \neq \emptyset$, then $\Gamma(R/I)$ has at least two vertices and has no loop.*
3. *If $I$ is subtractive and $a \in R$ is a vertex of $\Gamma_I(R)$ which is adjacent to every other vertex, then $a + a = a$ and $(I : a)$ is a maximal element of the*

*set $\Delta = \{(I : x) : x \in R \setminus I\}$ with respect to inclusion. Moreover, $(I : a)$ is a prime co-ideal of $R$.*

Proof: (1) Suppose that $a \in R \setminus I$ with $a + a = a(1 + 1) \in I$. Since $I$ is subtractive $a \in I$, which is a contradiction. So $\Gamma_I(R)$ has no loop.

(2) By Proposition 2.1(4), $\Gamma(R/I)$ has no loop, so it has more than one vertex.

(3) Let $a + a \neq a$. As $I$ is subtractive and $a \notin I$, $a + a \notin I$. Since $a$ is adjacent to every other vertex in $\Gamma_I(R)$, $a + a + x \in I$ for each $x \in S_I(R)$. Thus $a + a \in S_I(R)$. Hence $a + a + a = a(1 + 1 + 1) \in I$ gives $a \in I$, a contradiction. So $a + a = a$. Suppose, on the contrary, $(I : a)$ is not maximal. So there is $x \in R \setminus I$ such that $(I : a) \subset (I : x)$. Since $a$ is adjacent to every other vertex in $\Gamma_I(R)$, $x + a \in I$, which gives $x \in (I : a) \subset (I : x)$. So $x + x \in I$, a contradiction by (1).

Let $x + y \in (I : a)$ be such that $x \notin (I : a)$. So $x + a \notin I$. As $(I : a) \subseteq (I : x + a)$ and $(I : a)$ is maximal in $\Delta$, we have $(I : a) = (I : x + a)$. Since $x + y \in (I : a)$, we get $y \in (I : a + x) = (I : a)$. Thus $(I : a)$ is prime. □

Note that the condition that I is subtractive is necessary in Proposition 2.6 (1) as the following example shows.

**Example 2.7.** Let $R = (\{0, 1, 2, 3\}, +, \times)$, where

$$a + b = \begin{cases} 3 & \text{if } a, b \neq 0, \\ b & \text{if } a = 0, \\ a & \text{if } b = 0. \end{cases}$$

and $1 \times 1 = 1,\ 2 \times 1 = 1 \times 2 = 2,\ 3 \times 1 = 1 \times 3 = 3,\ 2 \times 2 = 1,\ 2 \times 3 = 3 \times 2 = 3,\ 3 \times 3 = 3$, moreover $r \times 0 = 0 \times r = 0$ for all $r \in R$. Then $I = \{1, 3\}$ is a strong co-ideal of $R$ which is not subtractive because $3, 3 \times 2 \in I$ but $2 \notin I$. It is easy to see that $S_I(R) = \{2\}$ and $\Gamma_I(R)$ has loop.

**Theorem 2.8.** *Let $I$ be a strong co-ideal of a semiring $R$. Then the following statements hold.*

(1) *$\Gamma_I(R)$ is connected with $\operatorname{diam}(\Gamma_I(R)) \leq 3$.*

(2) *If $I$ is a subtractive co-ideal of $R$ with $|S_I(R)| \geq 3$ then $\Gamma_I(R)$ is not a complete graph. In particular, $\operatorname{diam}(\Gamma_I(R)) = 2$ or $3$.*

Proof: (1) Let $x, y \in S_I(R)$. If $x + y \in I$, then $x, y$ are adjacent and $d(x, y) = 1$. Thus suppose that $x + y \notin I$. By Theorem 2.6(1), $x + x \notin I, y + y \notin I$. As $x, y \in S_I(R)$, $x + a \in I, y + b \in I$ for some $a, b \in R \setminus (I \cup \{x, y\})$. If $a = b$, then $x - a - y$ is a path. If $a \neq b$ and $a + b \in I$, then $x - a - b - y$ is a path. If $a \neq b$ and $a + b \notin I$, then $x - a + b - y$ is a path. Thus $\Gamma_I(R)$ is connected with $\operatorname{diam}\Gamma_I(R) \leq 3$.

(2) Assume that $\Gamma_I(R)$ is complete and let $a, b, c \in S_I(R)$ be distinct elements. Then $a + c, a + b \in I$, so $bc \in (I : a)$, since $(I : a)$ is a strong co-ideal of $R$ by Proposition 2.1(3). If $bc \in I$, then Proposition 2.1(1) gives $b, c \in I$ that is a contradiction. So $bc \notin I$. If $bc = c$, then $c + b = bc + b = b(1 + c) \in I$, implying $b \in I$ by Proposition 2.1, a contradiction. So $bc \neq c$. Since $\Gamma_I(R)$ is complete,

$c(b+1) = bc + c \in I$; hence $c \in I$ which is a final contradiction. Thus $\Gamma_I(R)$ is not complete (so $\mathrm{diam}(\Gamma_I(R)) \neq 1$). Finally, by (1) and Proposition 2.6(1), $\mathrm{diam}(\Gamma_I(R)) = 2$ or 3. □

Note that the condition that I is subtractive is necessary in Theorem 2.8(2), as the following example shows.

**Example 2.9.** Assume that $R = \{0, 1, 2, 3, 4, 5\}$. Define

$$a + b = \begin{cases} 5 & \text{if } a \neq 0,\ b \neq 0, a \neq b, \\ a & \text{if } a = b, \\ b & \text{if } a = 0, \\ a & \text{if } b = 0. \end{cases}$$

and

$$a * b = \begin{cases} 0 & \text{if } a = 0 \text{ or } b = 0, \\ 3 & \text{if } a = b = 2, \\ b & \text{if } a = 1, \\ a & \text{if } b = 1, \\ 5 & \text{otherwise.} \end{cases}$$

Then $(R, +, *)$ is easily checked to be a commutative semiring. An inspection will show that $I = \{1, 5\}$ is a co-ideal of $R$ which is not subtractive because $5 * 2 \in I$, $5 \in I$ but $2 \notin I$. Also $S_I(R) = \{2, 3, 4\}$ and $\Gamma_I(R)$ is a complete graph.

A vertex $x$ of a connected graph $G$ is a cut-point of $G$ if there are vertices $y$ and $z$ of $G$ such that $x$ is in every path from $y$ to $z$ (and $x \neq y, x \neq z$). Equivalently, for a connected graph $G$, $x$ is a cut-point of $G$ if $G - \{x\}$ is not connected.

**Example 2.10.** Let $R = (\{0, 1, 2, 4, 5, 10, 20, 25, 50, 100\}, \mathrm{gcd}, \mathrm{lcm})$ (take $\mathrm{gcd}(0,0) = 0$ and $\mathrm{lcm}(0,0) = 0$) and $I = \{1, 2\}$ be a strong co-ideal of $R$. Observe that $S_I(R) = \{4, 5, 10, 25, 50\}$. It can be easily seen that 4 is a cut-point of $\Gamma_I(R)$.

In the next theorems, we completely characterize the girth of the graph $\Gamma_I(R)$. A cycle graph or a circular graph is a graph that consists of a single cycle, or in other words, some number of vertices connected in a closed chain.

**Theorem 2.11.** *Let $I$ be a strong co-ideal of a semiring $R$.*

(1) *If $\Gamma_I(R)$ contains a cycle, then $\mathrm{gr}(\Gamma_I(R)) \leq 4$.*
(2) *If $I$ is a $Q$-strong co-ideal such that $\Gamma(R/I)$ and $\Gamma_I(R)$ contain a cycle, then $\mathrm{gr}(\Gamma_I(R)) = \mathrm{gr}(\Gamma(R/I))$. Moreover, If $\Gamma(R/I)$ has only two vertices $q_1 I$ and $q_2 I$ with $|q_i I| \geq 2$ $(i = 1, 2)$, then $\mathrm{gr}(\Gamma_I(R)) = 4$.*
(3) *If $I$ is a subtractive co-ideal, then the only cycle graph with respect to $I$ is $K_{2,2}$.*

Proof: (1) It is well-known that for any connected graph $G$, if $G$ contains a cycle, then $\mathrm{gr}(G) \leq 2\mathrm{diam}(G) + 1$. Suppose that $\Gamma_I(R)$ contains a cycle. Hence $\mathrm{gr}(\Gamma_I(R)) \leq 7$. Suppose that $\mathrm{gr}(\Gamma_I(R)) = n$, where $n \in \{5, 6, 7\}$ and let $x_1 - x_2 - \cdots - x_n - x_1$ be a cycle of minimum length. Since $x_1$ is not adjacent to $x_3$,

$x_1 + x_3 \notin I$. If $x_1 + x_3 \neq x_i$ for each $1 \leq i \leq n$, then $x_2 - x_3 - x_4 - x_1 + x_3 - x_2$ is a 4-cycle, that is, a contradiction. Therefore $x_1 + x_3 = x_i$ for some $1 \leq i \leq n$. We split the proof into three cases.

**Case 1**: If $x_1 + x_3 = x_1$ (resp. $x_1 + x_3 = x_3$), then $x_1 - x_2 - x_3 - x_4 - x_1$ (resp. $x_1 - x_2 - x_3 - x_n - x_1$) is a 4-cycle, a contradiction.

**Case 2**: If $x_1 + x_3 = x_2$ (resp. $x_1 + x_3 = x_4$), then $x_2 - x_3 - x_4 - x_2$ (resp. $x_2 - x_3 - x_4 - x_2$) is a 3-cycle, that is, a contradiction.

**Case 3**: If $x_1 + x_3 = x_n$, then $x_2 - x_3 - x_4 - x_n - x_2$ is a 4-cycle, which is a contradiction. Thus, every case leads to a contradiction; hence $\mathrm{gr}(\Gamma_I(R)) \leq 4$.

(2) Assume that $\mathrm{gr}(\Gamma_I(R)) = n$ and let $x_1 - x_2 - \cdots - x_n - x_1$ be a cycle in $\Gamma_I(R)$. Since $I$ is a $Q$-strong co-ideal, there exist unique elements $q_i \in Q$ $(1 \leq i \leq n)$ such that $x_i \in q_i I$. By Proposition 2.5, $q_1 I - q_2 I - \cdots - q_n I - q_1 I$ is a cycle in $\Gamma(R/I)$; thus $\mathrm{gr}(\Gamma(R/I)) \leq \mathrm{gr}(\Gamma_I(R))$. Now suppose that $\mathrm{gr}(\Gamma(R/I)) = m$ and let $q_1 I - q_2 I - \cdots - q_m I - q_1 I$ be a cycle of length $m$ in $\Gamma(R/I)$. Then $q_1 - q_2 - \cdots - q_m - q_1$ is a cycle of length $m$ in $\Gamma_I(R)$ by Proposition 2.5, so $\mathrm{gr}(\Gamma_I(R)) \leq \mathrm{gr}(\Gamma(R/I))$. Thus $\mathrm{gr}(\Gamma_I(R)) = \mathrm{gr}(\Gamma(R/I))$. Let $\Gamma(R/I)$ have only two vertices $q_1 I$ and $q_2 I$; we show that $\mathrm{gr}(\Gamma_I(R)) = 4$. Let $x, y \in S_I(R)$. If $x, y$ are adjacent, then $x \in q_i I$ and $y \in q_j I$, where $i \neq j \in \{1, 2\}$, and if $x, y$ are not adjacent, then either $x, y \in q_1 I$ or $x, y \in q_2 I$ by Proposition 2.5. Also, as $q_1 I$ and $q_2 I$ are adjacent in $\Gamma(R/I)$, every element of $q_1 I$ and $q_2 I$ are adjacent in $\Gamma_I(R)$ by Proposition 2.5. Hence $\Gamma_I(R)$ is complete bipartite with two parts $q_1 I$ and $q_2 I$. Since $|q_i I| \geq 2$ for $i = 1, 2$, $\mathrm{gr}(\Gamma_I(R)) = 4$.

(3) By Theorem 2.8(2), there is no 3-cycle graph. By (1), there are no cycle graph with five or more vertices. So the only cycle graph is $K_{2,2}$. □

Note that the condition that $\Gamma_I(R)$ and $\Gamma(R/I)$ contain cycle in Theorem 2.11(2) is necessary as the following example shows.

**Example 2.12.** Let $R$ and $I$ be as stated in Example 2.4. As we see $\mathrm{gr}(\Gamma(R/I)) = \infty$ and $\mathrm{gr}(\Gamma_I(R) = 4$.

For a graph $G$ and vertex $x \in V(G)$, the degree of $x$, denoted $\deg(x)$, is the number of edges of $G$ incident with $x$.

**Theorem 2.13.** *Let $I$ be a subtractive co-ideal of a semiring $R$. Then the following assertions hold:*

(1) $\mathrm{gr}(\Gamma_I(R)) = \infty$ *if and only if* $\Gamma_I(R)$ *is a star graph,*
(2) $\mathrm{gr}(\Gamma_I(R)) = 4$ *if and only if* $\Gamma_I(R)$ *is bipartite but not a star graph,*
(3) $\mathrm{gr}(\Gamma_I(R)) = 3$ *if and only if* $\Gamma_I(R)$ *contains an odd cycle,*
(4) *if* $\mathrm{gr}(\Gamma_I(R)) = 4$, *then there is no end vertex* (*i.e, vertex with degree* 1) *in* $\Gamma_I(R)$.

Proof: (1) First suppose that $\mathrm{gr}(\Gamma_I(R)) = \infty$ and $\Gamma_I(R)$ is not a star graph. So $|S_I(R)| \geq 4$, because $\Gamma_I(R)$ is not complete by Theorem 2.8(2). Since $\Gamma_I(R)$ is connected, there exists a vertex $x \in S_I(R)$ such that $\deg(x) \geq 2$. As $\Gamma_I(R)$ is not a star graph, there exists a path of the form $a - x - b - c$ in $\Gamma_I(R)$ for some

$a, b, c \in S_I(R)$. If $a$ is adjacent to $c$, then $a - x - b - c - a$ is a cycle in $\Gamma_I(R)$, a contradiction. If $a$ is not adjacent to $c$, then $a + c \notin I$. Since $a + c + x \in I$, $a+c \in S_I(R)$ and $x - a + c - b - x$ is a cycle which is a contradiction. Thus $\Gamma_I(R)$ is a star graph. The other implication is clear.

(2) Let $\mathrm{gr}(\Gamma_I(R)) = 4$. So $\Gamma_I(R)$ is not a star graph by (1). It is known that a graph is bipartite if and only if it contains no odd cycle [[4], Theorem 4.7]. Thus it suffices to show that $\Gamma_I(R)$ has no odd cycle. Assume that $x_1 - x_2 - \cdots - x_n - x_1$ is an odd cycle of minimal length $n$ in $\Gamma_I(R)$. Since $\mathrm{gr}(\Gamma_I(R)) = 4$, $n \geq 5$. As $\mathrm{gr}(\Gamma_I(R)) \neq 3$, $x_2$ is not adjacent to $x_4$, and so $x_2+x_4 \notin I$. Since $x_2+x_4+x_1 \in I$, $x_2 + x_4 \in S_I(R)$. It follows that $x_1 - x_2 + x_4 - x_5 - \cdots - x_n - x_1$ is an odd cycle of length $n - 2$ in $\Gamma_I(R)$, a contradiction. Hence $\Gamma_I(R)$ is a bipartite graph. Conversely, let $\Gamma_I(R)$ be bipartite which is not a star graph. Therefore $\Gamma_I(R)$ has no odd cycle, and so $\mathrm{gr}(\Gamma_I(R)) \neq 3$. By (1), $\mathrm{gr}(\Gamma_I(R)) \neq \infty$. Therefore $\mathrm{gr}(\Gamma_I(R)) = 4$ by Theorem 2.11(1).

(3) If $\mathrm{gr}(\Gamma_I(R)) = 3$, then we are done. Conversely, assume that $\Gamma_I(R)$ has an odd cycle. Let $\mathrm{gr}(\Gamma_I(R)) \neq 3$. If $\mathrm{gr}(\Gamma_I(R)) = 4$, then (2) implies that $\Gamma_I(R)$ is a bipartite graph which is not a star graph. Therefore, by [4, Theorem 4.7], $\Gamma_I(R)$ contains no odd cycle, a contradiction. If $\mathrm{gr}(\Gamma_I(R)) = \infty$, then $\Gamma_I(R)$ is a star graph which contradicts our assumption. Therefore $\mathrm{gr}(\Gamma_I(R)) = 3$.

(4) First we show that if $a - b - c - d$ is a path in $\Gamma_I(R)$ such that the edge $b - c$ is not contained in a 3-cycle and $a, b, c, d$ are vertices, then the vertices $a$ and $d$ are distinct and are adjacent to each other. Clearly $a \neq d$. Assume that $a, d$ are not adjacent. Since $a + b \in I$, $(a + d) + b \in I$; hence $a + d \in S_I(R)$. Thus $a + d - b - c - a + d$ is a 3-cycle, a contradiction.

Now let $a$ be an end vertex in $\Gamma_I(R)$ and $b$ be a vertex in $\Gamma_I(R)$ such that $a$ and $b$ are adjacent. Since $\mathrm{gr}(\Gamma_I(R)) < \infty$, $\Gamma_I(R)$ is not a star graph by (1). By Theorem 2.8(1), $\Gamma_I(R)$ is connected, hence there is a path $a - b - c - d$ in $\Gamma_I(R)$ with $c, d \notin \{a, b\}$, since $\Gamma_I(R))$ has at least 4 elements. As $\mathrm{gr}(\Gamma_I(R)) = 4$, the edge $b - c$ is not contained in a 3-cycle. By the above considerations, $a \neq d$ and $a, d$ are adjacent to each other which is contradiction. □

**Example 2.14.** (1) Let $R = (\{0, 1, 2, 4, 5, 10, 20, 25, 50, 100\}, \gcd, \mathrm{lcm})$ (take $\gcd(0, 0) = 0$ and $\mathrm{lcm}(0, 0) = 0$) and $I = \{1, 2\}$ a strong co-ideal of $R$. Then $S_I(R) = \{4, 5, 10, 25, 50\}$. It can be easily seen that $\Gamma_I(R)$ is a star graph and $\mathrm{gr}(\Gamma_I(R)) = \infty$ (see Example 2.10).

(2) Let $X = \{a, b, c\}$ and $R = (P(X), \cup, \cap)$. Then $I = \{X, \{a, b\}\}$ is a strong co-ideal of $R$, $\Gamma_I(R)$ is a complete bipartite graph and $\mathrm{gr}(\Gamma_I(R)) = 4$.

(3) Let $R = (\{0, 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}, \gcd, \mathrm{lcm})$. Then $I = \{1, 2\}$ is a strong co-ideal of $R$ and $\Gamma_I(R)$ is a graph with odd cycle. It can be easily seen that $\mathrm{gr}(\Gamma_I(R)) = 3$.

## 3. Complete $r$-partite graph

In this section we state some theorems, which characterize the complete bipartite identity-summand graph $\Gamma_I(R)$ with respect to strong co-ideal $I$ of a semiring $R$.

**Theorem 3.1.** *Let $I$ be a strong co-ideal of a semiring $R$. If there exist two prime strong co-ideals $P_1$ and $P_2$ of $R$ such that $I = P_1 \cap P_2$, then $\Gamma_I(R)$ is a complete bipartite graph, and the converse is true when $I$ is a subtractive co-ideal of $R$.*

Proof: We show that $\Gamma_I(R)$ is a complete bipartite graph with two parts $V_1 = P_1 \setminus I$ and $V_2 = P_2 \setminus I$. Let $a, b \in R \setminus I$ with $a + b \in I$; so $a + b \in P_1 \cap P_2$. Since $P_1, P_2$ are prime and $a, b \notin I$, either $a \in P_1 \setminus I, b \in P_2 \setminus I$ or $a \in P_2 \setminus I,\ b \in P_1 \setminus I$.

Let $a, b \in S_I(R)$ be such that $a \in P_2 \setminus I, b \in P_1 \setminus I$. Then $a + b \in P_1 \cap P_2 = I$; hence $a, b$ are adjacent. Now we show that each two elements of $V_i$ are not adjacent. Let $c, d \in V_1$ (so $c, d \notin I$). If $c + d \in I$, then $c + d \in P_2$ gives $c \in P_2$ or $d \in P_2$. As $c, d \in V_1 \subset P_1$, $c \in I$ or $d \in I$, a contradiction. Similarly, each two elements of $V_2$ are not adjacent. So $\Gamma_I(R)$ is complete bipartite with two parts $V_1$ and $V_2$.

Conversely, suppose that $I$ is a subtractive co-ideal and let $V_1, V_2$ be two parts of $\Gamma_I(R)$. Set $P_1 = V_1 \cup I$ and $P_2 = V_2 \cup I$. One can easily see that $I = P_1 \cap P_2$. First we show that $P_1, P_2$ are strong co-ideals of $R$. Let $a, b \in P_1$. If $a, b \in I$, then $ab \in I \subseteq P_1$. So we may assume that $a \notin I$ or $b \notin I$. If $a, b \in V_1$, we have $a + c \in I$ and $b + c \in I$ for each $c \in V_2$, since $\Gamma_I(R)$ is complete bipartite. By Proposition 2.1, $a, b \in (I : c)$ gives $ab \in (I : c)$. If $ab \in I$, then $a \in I$ and $b \in I$ by Proposition 2.1 which is a contradiction. Thus $ab \in S_I(R)$. Since $ab + c \in I$ for each $c \in V_2$, $ab \in V_1$; so $ab \in P_1$. If $a \in V_1$ and $b \in I$, then $a + c, b + c \in I$ for each $c \in V_2$ and $ab \notin I$. As $I$ is subtractive, $ab + c \in I$ by Proposition 2.1, which gives $ab \in V_1$. Now suppose that $a \in P_1$ and $r \in R$; we show that $a + r \in P_1$. If $a \in I$, then $a + r \in I \subseteq P_1$. If $a \in V_1$, then $a + c \in I$ for each $c \in V_2$. Since $I$ is a co-ideal of $R$, $(a + r) + c \in I$ for each $r \in R$. If $a + r \notin I$, then $a + r \in V_1 \subseteq P_1$(because $c \in V_2$ and $\Gamma_I(R)$ is bipartite). If $a + r \in I$, then $a + r \in P_1$. Therefore $P_1$ is a co-ideal of $R$. As $I$ is a strong co-ideal and $1 \in I \subseteq P_1$, $P_1$ is a strong co-ideal of $R$. Similarly, $P_2$ is a strong co-ideal.

Now we claim that $P_1$ is prime. Let $a + b \in P_1$ such that $a, b \notin P_1$; so $a, b \notin I$. If $a + b \in I$, then either $a \in V_1$ and $b \in V_2$ or $a \in V_2$ and $b \in V_1$ which is a contradiction, since $a, b \notin P_1$. Thus $a + b \notin I$. If $a + b \in V_1$, then $a + b + c \in I$ for each $c \in V_2$. We claim that $b + c \notin I$. If $b + c \in I$, then $c \in V_2$ gives $b \in V_1$, a contradiction. Hence $b + c \notin I$. By the similar way, $a + c \notin I$. Since $a + (b + c) \in I$ and $a \notin V_1$, we have $a \in V_2$ and $b + c \in V_1$. Likewise, $b \in V_2$ and $a + c \in V_1$. Because $a + b + c \in I$, $(a + c) + (b + c) \in I$. It shows that two vertices $a + c$ and $b + c$ of $V_1$ are adjacent, a contradiction. Thus $P_1$ is a prime strong co-ideal of $R$. Similarly, $P_2$ is a prime strong co-ideal of $R$. □

**Example 3.2.** Let $X = \{a, b, c\}$, $R = (P(X), \cup, \cap)$ and $I = \{X, \{b, c\}\}$. Consider

$$P_1 = (I : \{a, b\}) = \{\{c\}, \{b, c\}, \{a, c\}, X\},$$
$$P_2 = (I : \{a, c\}) = \{\{b\}, \{a, b\}, \{c, b\}, X\}.$$

An inspection will show that $P_1$ and $P_2$ are prime strong co-ideals of $R$ and $I = P_1 \cap P_2$. It is easy to see that $\Gamma_I(R)$ is a complete bipartite graph with $S_I(R) = \{\{b\}, \{c\}, \{a, b\}, \{a, c\}\}$.

**Theorem 3.3.** *Let $I$ be a subtractive co-ideal of a semiring $R$. If $\Gamma_I(R)$ is complete r-partite, then $\Gamma_I(R)$ is a complete bipartite graph.*

Proof: Let $V_1, V_2, \dots, V_r$ be parts of $\Gamma_I(R)$, $r \geq 3$ and $c_i \in V_i$ for each $i$. Since $\Gamma_I(R)$ is complete $r$-partite, $c_1 + c_3 \in I$ and $c_3 + c_2 \in I$. Since $c_1, c_2 \in (I : c_3)$ and $(I : c_3)$ is a strong co-ideal by Proposition 2.1, $c_1c_2 \in (I : c_3)$. As $c_1, c_2 \notin I$, $c_1c_2 \notin I$ by Proposition 2.1(1). We claim that $c_1c_2 \in V_1$. If not, then $c_1(c_2+1) = c_1c_2+c_1 \in I$ because $c_1 \in V_1$. Since $I$ is subtractive, we get $c_1 \in I$, a contradiction. Therefore $c_1c_2$ and $c_1$ are not adjacent (because $c_1, c_1c_2 \in V_1$ ). As $c_2 \in V_2$, $c_2(c_1+1) = c_1c_2+c_2 \in I$. Since $I$ is subtractive, $c_2 \in I$, a contradiction. Hence $r = 2$. □

The connectivity of a graph $G$, denoted by $k(G)$, is defined to be the minimum number of vertices that are necessary to remove from $G$ in order to produce a disconnected graph.

**Theorem 3.4.** *Let $I$ be a $Q$-strong co-ideal of a semiring $R$. If $\Gamma(R/I)$ is the graph on only two vertices $q_1I, q_2I$, then*

(1) *$\Gamma_I(R)$ is a complete bipartite graph and $k(\Gamma_I(R)) = \min\{\mid q_1I \mid, \mid q_2I \mid\}$;*
(2) *$I = P_1 \cap P_2$, where $P_1 = q_1I \cup I$ and $P_2 = q_2I \cup I$ are prime strong co-ideals of $R$.*

Proof: (1) Since $q_1I$ and $q_2I$ are the only vertices of $\Gamma(R/I)$ and $\Gamma(R/I)$ has no loop, $q_1I \oplus q_2I = I$; so $q_1a + q_2b \in I$ for each $a, b \in I$. Since by Proposition 2.5, all elements of $q_1I$ and $q_2I$ are adjacent and none of elements of $q_iI$ are adjacent together, we get $\Gamma_I(R)$ is a complete bipartite graph. The other statement is clear.

(2) It is clear by the proof of Theorem 3.1. □

**Example 3.5.** Let $R = (P(X), \cup, \cap)$, where $X = \{a, b, c\}$. By Example 2.12, $I = \{X, \{a, b\}\}$ is a $Q$-strong co-ideal of $R$ with $Q = \{q_1 = \{c\}, q_2 = \{a, c\}, q_3 = \{b, c\}, q_e = X\}$ and $S^*(R/I) = \{q_2I, q_3I\}$. Since $\Gamma(R/I)$ has only two vertices, $I = P_1 \cap P_2$, where $P_1 = q_2I \cup I$ and $P_2 = q_3I \cup I$. Moreover $k(\Gamma_I(R)) = 2$.

For every nonnegative integer $r$, the graph $G$ is called $r$-regular if the degree of each vertex of $G$ is equal to $r$.

**Theorem 3.6.** *Let $I$ be a subtractive co-ideal of a semiring $R$, and let $\Gamma_I(R)$ be a finite regular graph. Then $\Gamma_I(R)$ is $K_{n,n}$ for some $n \in \mathbb{N}$.*

Proof: The proof is similar to [8, Theorem 4.8]. □

## 4. Chromatic number, clique number and planar property

In this section we collect some basic properties concerning chromatic number and clique number of the graph $\Gamma_I(R)$.

**Proposition 4.1.** *Let $I$ be a co-ideal of a semiring $R$.*

(1) *If $I$ is a $Q$-strong co-ideal, then $w(\Gamma_I(R)) \leq |Q| - 2$.*
(2) *If $I$ is a subtractive co-ideal with $w(\Gamma_I(R))$ being finite, then $R$ has a.c.c on co-ideals of the form $(I : a)$, where $a \in R$. Moreover, if $(I : a_i)$ and $(I : a_j)$ are distinct maximal elements of $\Delta = \{(I : a) : a \in R \setminus I\}$, then $a_i$ is adjacent to $a_j$ in $\Gamma_I(R)$.*

Proof: (1) If $w(\Gamma_I(R)) = \infty$, then $Q$ must be infinite by Proposition 2.5. Assume that $w(\Gamma_I(R)) = n$ and let $x_1, x_2, \dots, x_n$ be the vertices of the greatest complete subgraph of $\Gamma_I(R)$. Since $I$ is a $Q$-strong co-ideal, there exist unique elements $q_i \in Q$ such that $x_i \in q_i I$ $(1 \leq i \leq n)$. By Proposition 2.5, $q_i \neq q_0, q_e$ and $q_i \neq q_j$ for each $1 \leq i \neq j \leq n$. Thus $w(\Gamma_I(R)) \leq |Q| - 2$.

(2) The proof of the first statement is similar to [8, Lemma 5.1]. Now, if $(I : a_i)$ and $(I : a_j)$ are distinct maximal elements of $\Delta = \{(I : a) : a \in R \setminus I\}$ (partially ordered by inclusion), then by the usual argument, one can show that $(I : a_i)$ and $(I : a_j)$ are prime. We show $a_i + a_j \in I$. If not, then $(I : a_i) \subseteq (I : a_i + a_j)$ and $(I : a_j) \subseteq (I : a_i + a_j)$, and hence $(I : a_i) = (I : a_i + a_j) = (I : a_j)$, a contradiction. □

Note that the condition that $w(\Gamma_I(R))$ is finite is necessary in Proposition 4.1 as the following example shows:

**Example 4.2.** Let $X = \{x_i : i \in \mathbb{N}\}$ and $R = (P(X), \cup, \cap)$. Let $X_2 = \{x_i : i \geq 2\}$ and $I = \{X_2, X\}$. It is clear that $I$ is a subtractive co-ideal of $R$. Set $Y_j = X - \{x_j\}$. Then $A = \{Y_i : i \in \mathbb{N}\}$ is an infinite clique in $\Gamma_I(R)$. An inspection shows that the following chain is infinite:

$$(I : \{x_2\}) \subseteq (I : \{x_2, x_3\}) \subseteq (I : \{x_2, x_3, x_4\}) \subseteq \dots$$

The next theorem does establish a relation between the clique numbers of $\Gamma_I(R)$ and $\Gamma(R/I)$.

**Theorem 4.3.** *Let $I$ be a $Q$-strong co-ideal of a semiring $R$. Then $w(\Gamma_I(R)) = w(\Gamma(R/I))$.*

Proof: Assume that $\{x_i\}_{i \in J}$ is a clique in $\Gamma_I(R)$ and let $q_i$ be the unique element of $Q$ such that $x_i \in q_i I$ $(i \in J)$. Then $\{q_i I\}_{i \in J}$ is a clique in $\Gamma(R/I)$ by Proposition 2.5. Hence $w(\Gamma(R/I)) \geq w(\Gamma_I(R))$. Now, let $\{q_i I\}_{i \in K}$ be a clique in $\Gamma(R/I)$, then $\{q_i\}_{i \in K}$ is a clique in $\Gamma_I(R)$, by Proposition 2.5. Thus $w(\Gamma_I(R)) \geq w(\Gamma(R/I))$. Therefore, $w(\Gamma_I(R)) = w(\Gamma(R/I))$. □

The next several results investigate the relationship between the chromatic number and clique number of the graph $\Gamma_I(R)$.

**Theorem 4.4.** *Let $R$ be a semiring and $I$ be a subtractive co-ideal of $R$. Then the following are equivalent:*

(1) *$\chi(\Gamma_I(R))$ is finite;*
(2) *$w(\Gamma_I(R))$ is finite;*
(3) *the subtractive co-ideal $I$ is a finite intersection of prime co-ideals.*

Proof: The proof is similar to [8, Theorem 5.2]. $\square$

*Remark* 4.5. Let $P, I$ be strong co-ideals of a semiring $R$ with $P$ prime and $I \subseteq P$. Then the non-empty set $\Delta = \{P' \in \text{co-Spec}(R) : I \subseteq P' \subseteq P\}$ has a minimal element $P_1$ with respect to inclusion (by partially ordering $\Delta$ by reverse inclusion and using Zorn's Lemma), so $P_1$ is an element of $\min(I)$, the set of minimal prime strong co-ideals of $R$ containing $I$. Thus if $P$ is a prime strong co-ideal of the commutative semiring $R$ and $P$ contains the strong co-ideal $I$ of $R$, then there exists a minimal prime strong co-ideal $P'$ of $R$ with $I \subseteq P' \subseteq P$.

**Theorem 4.6.** *Let $I$ be a subtractive co-ideal of a semiring $R$.*

(1) *If $\{P_\alpha\}_{\alpha\in\Lambda}$ is the set of all prime strong co-ideals of $R$ containing $I$, then $I = \bigcap_{\alpha\in\Lambda} P_\alpha$.*
(2) *If $P_1, \dots, P_n$ are the only distinct minimal prime strong co-ideals of $R$ containing $I$, then $\bigcap_{i=1}^n P_i = I$ and $I \neq \cap_{1\le i\le n, i\neq j} P_i$, for each $1 \le j \le n$.*

Proof: (1) We need to show that $\cap_{\alpha\in\Lambda} P_\alpha \subseteq I$. Let $x \in \cap_{\alpha\in\Lambda} P_\alpha$ with $x \notin I$. Set $\sum = \{J : J$ *is a subtractive co-ideal of $R$ containing* $I, x \notin J\}$. Since $I \in \sum$, $\sum \neq \emptyset$. An inspection will show that the partially ordered set $(\sum, \subseteq)$ has a maximal element by Zorn's Lemma, say $K$. Since $x \notin K$, $K \neq R$. We show that $K$ is prime. Let $a + b \in K$ such that $a \notin K$. Hence $a \in (K : b)$ and $a \notin K$. As $K \subsetneq (K : b)$ and $(K : b)$ is subtractive by Proposition 2.1, $x \in (K : b)$. Hence $b \in (K : x)$. It is clear that $K \subseteq (K : ax)$. If $(K : ax) \neq K$, then $x \in (K : ax)$. Hence $x(1 + a) = x + ax \in K$. Since $K$ is subtractive, $x \in K$, a contradiction. Therefore $K = (K : ax)$. We claim that $(K : ax) = (K : a) \cap (K : x)$. Let $r \in (K : ax)$. Then $ax \in (K : r)$. By Proposition 2.1(3) and 2.1(1), $a, x \in (K : r)$. Thus $r \in (K : x) \cap (K : a)$ and $(K : ax) \subseteq (K : a) \cap (K : x)$. For the reverse of inclusion let $r \in (K : a) \cap (K : x)$. Then $a, x \in (K : r)$. By Proposition 2.1(3), $ax \in (K : r)$ and so $r \in (K : ax)$. Hence the equality holds. As $b \in (K : a) \cap (K : x)$, $b \in K$. Thus $K$ is prime, which implies $x \in K$, a contradiction, as needed.

(2) By (1) and Remark 4.5, $\bigcap_{i=1}^n P_i = I$. To see the other statement, suppose $I = \bigcap_{1\le i\le n, i\neq j} P_i$ for some $1 \le j \le n$. Since for each $i \neq j$, $P_i \not\subseteq P_j$, there is $x_i \in P_i$ such that $x_i \notin P_j$. As $\sum_{i\neq j} x_i \in \bigcap_{1\le i\le n, i\neq j} P_i \subseteq P_j$, it is clear that $x_i \in P_j$ for some $i \neq j$, that is a contradiction. Thus $I \neq \bigcap_{1\le i\le n, i\neq j} P_i$ for each $1 \le j \le n$. $\square$

**Theorem 4.7.** *Let $I$ be a co-ideal of a semiring $R$.*

(1) *If $I$ is a subtractive co-ideal of $R$ which is not prime, then $w(\Gamma_I(R)) = |\min(I)|$.*

(2) *If $I$ is a $Q$-strong co-ideal with $|\min(I)|$ finite, then each $P \in \min(I)$ is of the form $P = (I : q)$ for some $q \in Q$.*

(3) *If $I$ is a subtractive co-ideal of a semiring $R$, then $\chi(\Gamma_I(R)) = w(\Gamma_I(R))$.*

Proof: (1) First, we prove that $|\min(I)| = \infty$ if and only if $w(\Gamma_I(R)) = \infty$. It suffices to show that $|\min(I)|$ is finite if and only if $w(\Gamma_I(R))$ is finite. Let $|\min(I)|$ be finite. Let $|\min(I)|$ be finite. Then $I$ is a finite intersection of prime co-ideals by Theorem 4.6; so by Theorem 4.4, $w(\Gamma_I(R))$ is finite. Now assume that $w(\Gamma_I(R))$ is finite. Hence by Theorem 4.4, $I = \bigcap_{i=1}^{n} P_i$ for some prime strong co-ideals $P_i$ of $R$. Let $\{Q_\alpha\}_{\alpha\in\Lambda} = \min(I)$. For each $\alpha \in \Lambda$, $I \subseteq Q_\alpha$, so $\bigcap_{i=1}^{n} P_i \subseteq Q_\alpha$ for each $\alpha \in \Lambda$. This implies that $P_i \subseteq Q_\alpha$ for some $1 \leq i \leq n$. Since $Q_\alpha$ is minimal, $P_i = Q_\alpha$. This gives $\Lambda$ is finite, and so $|\min(I)|$ is finite.

Let $|\min(I)| = n$. By Theorem 4.6(2), there exists $x_j \in (\bigcap_{1\leq i\leq n, i\neq j} P_i) \setminus P_j$ for each $1 \leq j \leq n$ . Since each $P_i$ is a strong co-ideal of $R$, $x_i + x_j \in I$; hence $X = \{x_1, x_2, \ldots, x_n\}$ is a clique in $\Gamma_I(R)$. Hence $w(\Gamma_I(R)) \geq n$. Now we show that $w(\Gamma_I(R)) \leq n$. We do this by induction on $n$. If $n = 2$, then $\Gamma_I(R)$ is a complete bipartite graph by Theorem 3.1; hence $w(\Gamma_I(R)) = 2$. Suppose $n > 2$ and the result is true for any integer less than $n$. Let $\{x_1, x_2, \ldots, x_m\}$ be a clique in $\Gamma_I(R)$. Thus $x_1 + x_j \in I = \bigcap_{1\leq i\leq n} P_i$. Without loss of generality, suppose that $x_1 \notin P_1$ and $x_2, x_3, \ldots, x_m \in P_1$ and $x_2, \ldots, x_m \notin \bigcap_{2\leq i\leq n} P_i$. Let $J = \bigcap_{2\leq i\leq n} P_i$. Hence $\{x_2, x_3, \ldots, x_m\}$ is a clique in $\Gamma_J(R)$. By induction hypothesis $m-1 \leq n-1$ and so $m \leq n$.

(2) Let $I$ be a $Q$-strong co-ideal of $R$ with $|\min(I)| = n$. By Theorem 4.6(2), $I = \bigcap_{i=1}^{n} P_i$ where $\min(I) = \{P_1, \ldots, P_n\}$. Then by (1), $w(\Gamma_I(R)) = n$. Let $\{x_1, x_2, \ldots, x_n\}$ be a clique in $\Gamma_I(R)$ where $x_j \in (\bigcap_{1\leq i\leq n, i\neq j} P_i) \setminus P_j$. Since $I$ is a $Q$-strong co-ideal of $R$, there exists unique element $q_j \in Q$ such that $x_j \in q_j I$ for each $1 \leq j \leq n$. As $\{x_1, x_2, \ldots, x_n\}$ is a clique in $\Gamma_I(R)$, $\{q_1, q_2, \ldots, q_n\}$ is a clique in $\Gamma_I(R)$ by Proposition 2.5. Let $x_j = q_j a_j$ for some $a_j \in I$. We show that $q_j \in \bigcap_{1\leq i\leq n, i\neq j} P_i \setminus P_j$. It suffices to show that $q_j \notin P_j$ and there is no $i \neq j$ such that $q_j \notin P_i$. If $q_j \in P_j$, then $x_j = q_j a_j \in P_j$, a contradiction (because $a_j \in I \subseteq P_j$). So $q_j \notin P_j$. Also if $q_j \notin P_i$ for some $i \neq j$, then $q_i + q_j \notin P_i$ and hence $q_i + q_j \notin I$, a contradiction (similarly, as $x_i \notin P_i$, $q_i \notin P_i$). Therefore $q_j \in \bigcap_{1\leq i\leq n, i\neq j} P_i \setminus P_j$. We claim that $(I : q_j) = P_j$. Let $x \in (I : q_j)$. Then $x + q_j \in I$, and so $x + q_j \in P_j$. Since $q_j \notin P_j$, $x \in P_j$. Hence $(I : q_j) \subseteq P_j$. For the reverse of inclusion, let $x \in P_j$. Then $q_j \in \bigcap_{1\leq i\leq n, i\neq j} P_i \setminus P_j$ gives $x + q_j \in I$. Therefore $P_j \subseteq (I : q_j)$ and we have equality.

(3) By Theorem 4.4, $w(\Gamma_I(R)) = \infty$ if and only if $\chi(\Gamma_I(R)) = \infty$. Hence, we assume that $\chi(\Gamma_I(R))$ is finite. It is known that $w(\Gamma_I(R)) \leq \chi(\Gamma_I(R))$. Let $w(\Gamma_I(R)) = n$. By Theorem 4.6, $I = P_1 \cap \cdots \cap P_n$, where for each $i$, $P_i$ is a minimal prime strong co-ideal. By an argument like that in Theorem 4.4 ((3) $\Rightarrow$ (1)), $\chi(\Gamma_I(R)) \leq n$. Therefore $\chi(\Gamma_I(R)) = w(\Gamma_I(R))$. □

A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained by replacing edges of this graph with pairwise internally-disjoint paths. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930, that says that a graph is planar if and only if it contains no subdivision of $K_5$ or $K_{3,3}$ [4]. It is natural to ask for which strong co-ideal $I$ of $R$ the $\Gamma_I(R)$ is planar.

**Proposition 4.8.** *Let $I$ be a $Q$-strong co-ideal of $R$.*

(1) *If $\Gamma_I(R)$ is planar, then for each edge $q_1I$-$q_2I$ of $\Gamma(R/I)$, $|q_iI| \leq 2$ for some $1 \leq i \leq 2$.*
(2) *If $\Gamma_I(R)$ is planar, then $\Gamma(R/I)$ is planar.*

PROOF: (1) Assume that $\Gamma_I(R)$ is planar and $q_1I$ and $q_2I$ are two vertices of $\Gamma(R/I)$ such that $|q_iI| \geq 3$ for each $i = 1, 2$. Let $V_1 = \{x_1, x_2, x_3\} \subseteq q_1I$ and $V_2 = \{y_1, y_2, y_3\} \subseteq q_2I$. As $q_1I$ and $q_2I$ are adjacent in $\Gamma(R/I)$, $x_i$ and $y_j$ are adjacent in $\Gamma_I(R)$ by Proposition 2.5. Then $V_1$ and $V_2$ are two parts of a complete bipartite graph as a subgraph of $\Gamma_I(R)$. Hence $\Gamma_I(R)$ is not planar.

(2) Let $\Gamma_I(R)$ be planar. By Proposition 2.5, two vertices $q_1I$ and $q_2I$ are adjacent in $\Gamma(R/I)$ if and only if $q_1$ and $q_2$ are adjacent in $\Gamma_I(R)$. Hence we can take $\Gamma(R/I)$ as a subgraph of $\Gamma_I(R)$. If $\Gamma(R/I)$ is not planar, then $\Gamma_I(R)$ is not planar, a contradiction. So $\Gamma(R/I)$ is planar. □

The following example shows that the converse of Proposition 4.8 is not true.

**Example 4.9.** Let $X = \{a, b, c, d\}$ and $R = (P(X), \cup, \cap)$. An inspection will show that

$$P_1 = \{Y \subseteq X | b \in Y\},$$
$$P_2 = \{Y \subseteq X | a \in Y\}$$

are prime strong co-ideals of $R$. Let $I = P_1 \cap P_2$. By Theorem 3.1, $\Gamma_I(R)$ is a complete bipartite graph with parts $V_1$ and $V_2$ and $|V_1| = |P_1 \setminus I| = 4$ and $|V_2| = |P_2 \setminus I| = 4$. Hence $K_{3,3}$ is a subgraph of $\Gamma_I(R)$, and so $\Gamma_I(R)$ is not planar.

Set $Q = \{q_0 = \{d, c\}, q_e = X, q_1 = \{b, c, d\}, q_2 = \{d, c, a\}\}$, then $q_0I = \{\{d, c\}, \{c\}, \{d\}, \emptyset\}$, $q_eI = I$, $q_1I = \{\{b, c, d\}, \{b, c\}, \{b, d\}, \{b\}\}$ and $q_2I = \{\{d, c, a\}, \{d, a\}, \{a, c\}, \{a\}\}$. By usual argument, $I$ is a $Q$-strong co-ideal of $R$ and $R/I = \{q_0I, q_eI, q_1I, q_2I\}$. Since $q_1 + q_2 \in I$, $q_1I \oplus q_2I = I$ by Proposition 2.5. Hence $S^*(R/I) = \{q_1I, q_2I\}$. Therefore $\Gamma(R/I)$ is planar.

**Theorem 4.10.** *Let $I$ be a subtractive co-ideal of semiring $R$. If $|\min(I)| \geq 4$, then $\Gamma_I(R)$ is not planar.*

PROOF: If $|\min(I)| \geq 5$, then by Theorem 4.7(1), $w(\Gamma_I(R)) \geq 5$. Hence $\Gamma_I(R)$ is not planar.

If $|\min(I)| = 4$, then Theorem 4.7(1) implies that $w(\Gamma_I(R)) = 4$. Hence there exists $\{x_1, x_2, x_3, x_4\} \subseteq S_I(R)$ such that $\{x_1, \dots, x_4\}$ forms a clique in $\Gamma_I(R)$. Let $x_{ij} = x_ix_j$, where $1 \leq i, j \leq 4$, $i \neq j$. Suppose that $k \in \{1, 2, 3, 4\} \setminus \{i, j\}$. Since $x_i, x_j \in (I : x_k)$, $x_{ij} \in (I : x_k)$. If $x_{ij} \in I$, then $x_i(x_j + 1) = x_{ij} + x_i \in I$, hence $x_i \in I$, which is a contradiction. This implies that $x_{ij} \in S_I(R)$. We claim that

$x_{ij} \notin \{x_1, x_2, x_3, x_4\}$. Assume that $x_{ij} = x_s$ for some $1 \leq s \leq 4$. If $s = i$, then $x_{ij} + x_j \in I$. This implies that $x_i \in I$ which is a contradiction. Similarly, for $s = j$. If $s \neq j$ and $s \neq i$, then $x_{ij} + x_s \in I$; hence $x_s + x_s \in I$. It follows that $x_s \in I$ by Proposition 2.1(1), a contradiction. Therefore $x_{ij} \notin \{x_1, x_2, x_3, x_4\}$. Let $s \neq k$ and $s, k \in \{1, 2, 3, 4\} - \{i, j\}$. Since $x_{ij} + x_s \in I$ and $x_{ij} + x_k \in I$, we have $x_s, x_k \in (I : x_{ij})$; thus $x_{sk} \in (I : x_{ij})$. Set $V_1 = \{x_1, x_{13}, x_3\}$ and $V_2 = \{x_2, x_{24}, x_4\}$. Then $V_1$ and $V_2$ are two parts of a complete 2-partite subgraph of $\Gamma_I(R)$. Therefore $\Gamma(R)$ is not planar. $\square$

In the following example, it is shown that if $|\min(I)| = 3$, then $\Gamma_I(R)$ may be planar.

**Example 4.11.** (1) Let $R = \{p_1^i p_2^j p_3^k p_4^t : \ i \in \{0,1,2,3\},\ j \in \{0,1,2,3\},\ k \in \{0,1\},\ t \in \{0,1\}\ \} \cup \{0\}$ where $p_i$'s are prime integer. Then $(R, \gcd, \mathrm{lcm})$ is a semiring and $I = \{1, p_4\}$ is a subtractive strong co-ideal of $R$. Since for each $1 \leq m, n \leq 3$ where $m \neq n$, $\gcd(p_m, p_n) = 1 \in I$, $\{p_1, p_2, p_3\}$ is a clique in $\Gamma_I(R)$ and $w(\Gamma_I(R)) = 3$. Hence $|\min(I)| = 3$ by Theorem 4.7. Set $V_1 = \{p_1, p_1^2, p_1^3\}$ and $V_2 = \{p_2, p_2^2, p_2^3\}$. Then $K_{3,3}$ is a subgraph of $\Gamma_I(R)$ with two parts $V_1$ and $V_2$. Hence $\Gamma_I(R)$ is not planar.

(2) Let $R = (\{0, 1, 2, 3, 5, 6, 10, 15, 30\}, \gcd, \mathrm{lcm})$. Then $I = \{1\}$ is a subtractive strong co-ideal of $R$ and $S_I(R) = \{2, 3, 5, 6, 10, 15\}$. By drawing $\Gamma_I(R)$, one can see that $w(\Gamma_I(R)) = 3$. Hence $|\min(I)| = 3$ by Theorem 4.7. Also $\Gamma_I(R)$ is planar.

*Remark* 4.12. Let $I$ be a subtractive strong co-ideal of a semiring $R$.

(1) If $|\min(I)| = 1$, then by Theorem 4.6(2), $I$ is a prime strong co-ideal of $R$. Hence $\Gamma_I(R) = \emptyset$ by Proposition 2.3.

(2) If $|\min(I)| = 2$, then $I = P_1 \cap P_2$ for some prime strong co-ideals $P_1$ and $P_2$ by Theorem 4.6. Hence by Theorem 3.1, $\Gamma_I(R)$ is $K_{n,m}$ for some integer $n$ and $m$, where $|P_1 \setminus I| = n$ and $|P_2 \setminus I| = m$. If $n, m \geq 3$, then $K_{3,3}$ is a subgraph of $\Gamma_I(R)$ and so $\Gamma_I(R)$ is not planar.

(3) If $|\min(I)| \geq 4$, then by Theorem 4.10, $\Gamma_I(R)$ is not planar.

(4) If $R$ and $I$ are the semiring and co-ideal as in Example 4.11(2) , then $|\min(I)| = 3$ and $\Gamma_I(R)$ is planar. However there exist a semiring $R$ and a strong co-ideal $I$ of $R$ that have only three minimal prime co-ideals and $\Gamma_I(R)$ is not planar as Example 4.11(1) shows. It is not entirely clear for us for which strong co-ideals $I$ with $|\min(I)| = 3$, the $\Gamma_I(R)$ is planar.

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