# Orthosymmetric bilinear map on Riesz spaces

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*Abstract.* Let $E$ be a Riesz space, $F$ a Hausdorff topological vector space (t.v.s.). We prove, under a certain separation condition, that any orthosymmetric bilinear map $T : E \times E \to F$ is automatically symmetric. This generalizes in certain way an earlier result by F. Ben Amor [*On orthosymmetric bilinear maps*, Positivity **14** (2010), 123–134]. As an application, we show that under a certain separation condition, any orthogonally additive homogeneous polynomial $P : E \to F$ is linearly represented. This fits in the type of results by Y. Benyamini, S. Lassalle and J.L.G. Llavona [*Homogeneous orthogonally additive polynomials on Banach lattices*, Bulletin of the London Mathematical Society **38** (2006), no. 3 123–134].

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## 1. Introduction

One of the relevant problems in operator theory is to describe orthogonally additive polynomials via linear operators. This problem can be treated in a different manner, depending on domains and co-domains on which polynomials act. Interest in orthogonally additive polynomials on Banach lattices originates in the work of K. Sundaresan [19] where the space of $n$-homogeneous orthogonally additive polynomials on the Banach lattices $l_p$ and $L_p\,[0,1]$ was characterized. It is only recently that the class of such mappings has been getting more attention. We are thinking here about works on orthogonally additive polynomials and holomorphic functions and orthosymmetric multilinear mappings on different Banach lattices and also $\mathbb{C}^*$-algebras, see for instance [3], [6], [7], [18] and [8], [13], [14], [17]. Proofs of the aforementioned results are strongly based on the representation of these spaces as vector spaces of extended continuous functions. So they are not applicable to general Riesz spaces. That is why we need to develop new approaches. Actually, the innovation of this work consists in making a relationship between orthogonally additive homogeneous polynomials and orthosymmetric multilinear mappings which leads to a constructive proofs of Sundaresan results [19], those of D. Prez-García and I. Villanueva in [18], Y. Benyamini, S. Lassalle and Llavona [3], D. Carando, S. Lassalle and I. Zalduendo in [8], and those of A. Ibort, P. Linares and J.G. Llavona [13].

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A multilinear mapping $T : E^n \longrightarrow F$ is said to be orthosymmetric if $T(x_1, \ldots, x_n) = 0$ whenever $x_1, \ldots, x_n \in E$ satisfy $x_i \perp x_j$ for some $i \neq j$. The main purpose of this paper is to show that orthosymmetric multilinear mappings are symmetric with general conditions on $E$ and $F$. There are many results of this kind in the literature. In this context, we mention some of them. In [7], where orthosymmetric bilinear maps were introduced, the authors showed that any orthosymmetric positive bilinear map between Archimedean Riesz spaces is symmetric. Later, in [10] it is shown that the result remains valid for order bounded orthosymmetric bilinear maps. Recall that a bilinear map $T : E \times E \to F$ is positive if $T(x, y) \geq 0$ whenever $(x, y) \in E^+ \times E^+$, and is order bounded if given $(x, y) \in E^+ \times E^+$ there exists $a \in F^+$ such that $|T(z, w)| \leq a$ for all $(0, 0) \leq (z, w) \leq (x, y) \in E \times E$. Every positive bilinear map is order bounded. Recall also that $T$ is said to be symmetric if $T(x, y) = T(y, x)$. In a recent paper [2, Theorem 13], the result is proved for (r-u) continuous orthosymmetric bilinear operators. Recall that $T : E \times E \to F$ is (r-u) continuous if $x_n, y_n \longrightarrow 0$ (r-u) in $E$ implies that $T(x_n, y_n) \longrightarrow 0$ (r-u) in $F$. Finally, in a preprint [11] of the first and second author together with M. Meyer the result is obtained for any continuous orthosymmetric bilinear map defined on a relatively uniformly complete Riesz space with values in a Hausdorff t.v.s. The purpose of this paper is to extend the result in [11] to a general domain $E$. Actually, the relative uniform completeness condition on the domain will be removed and substituted by a separation condition. Namely, $(T(E^n))'$, the topological dual of the vector space generated by the range of $T$, separates points. Recall that $(T(E^n))'$ separates points if for every two vectors $x, y$ in the vector space generated by the range of $T$ such that $x \neq y$ there is some $f \in (T(E^n))'$ with $f(x) \neq f(y)$. This replaces $T$ positive, $T$ order bounded and domain uniformly complete and co-domain Riesz space in [7], [10] and [2] respectively. In the second section of this paper this result is applied to show under the same requirements that each of such mappings $T$ is factorized by a linear operator $S : \prod_{i=1}^{n} E \to F$ such that $T(x_1, \ldots, x_n) = S(x_1 \cdots x_n)$ for $x_1, \ldots, x_n \in E$ and $\prod_{i=1}^{n} E = \{x_1 \cdots x_n, x_i \in E\}$ (the multiplication under consideration is the $f$-algebra multiplication of $E^u$, the universal completion of $E$). The paper concludes with an application to orthogonally additive polynomials.

## 2. Preliminaries

We take it for granted that the reader is familiar with the notion of Riesz spaces (or vector lattices) and operators between them. For terminology, notations and concepts not explained or proved in this paper we refer the reader to the standard monographs [1], [15].

A Riesz space $E$ is called Archimedean if for each non-zero $a \in E$ the set $\{na, n = \pm 1, \pm 2, \ldots\}$ has no upper bound in $E$. In order to avoid unnecessary repetition, we will assume throughout the paper that all Riesz spaces under consideration are Archimedean. The relatively uniform topology on Riesz spaces plays a key role in the context of this work. Let us recall the definition of the *relatively uniform convergence*. Let $E$ be an Archimedean Riesz space and an element $u \in E^+$. A sequence $(x_n)_n$ of elements of $E$ *converges u-uniformly* to an element $x \in E$ whenever, for every $\epsilon > 0$, there exists a natural number $N_\epsilon$ such that $|x_n - x| \leq \epsilon u$ holds for all $n \geq N_\epsilon$. This will be denoted $x_n \to x(u)$. The element $u$ is called the *regulator of the convergence*. The sequence $(x_n)_n$ *converges relatively uniformly* to $x \in E$, whenever $x_n \to x(u)$ for some $u \in E^+$. We shall write $x_n \to x(\text{r-u})$ if we do not want to specify the regulator. *Relatively uniform limits* are unique if and only if $E$ is Archimedean [15, Theorem 63.2]. A nonempty subset $D$ of $E$ is said to be *relatively uniformly closed* if every relatively uniformly convergent sequence in $D$, has its limit also in $D$. We emphasize that the regulator does not need to be an element of $D$. The relatively uniformly closed subsets are the closed sets of the relatively uniform topology in $E$. The notion of *relatively uniform Cauchy sequence* is defined in the obvious way. A Riesz space $E$ is said to be *relatively uniformly complete* whenever every relatively uniformly Cauchy sequence has a (unique) limit. For more details we refer the reader to [15].

Next, we discuss linear operators on Riesz spaces. Let $E$ and $F$ be Riesz spaces with positive cones $E^+$ and $F^+$ respectively, and let $T$ be an operator from $E$ into $F$. $T$ is said to be order bounded if for each $x \in E^+$ there exists $y \in F^+$ such that $|T(z)| \leq y$ in $F$ whenever $|z| \leq x$ in $E$. The operator $T$ is said to be positive if $T(x) \in F^+$ for all $x \in E^+$. Every positive operator is of course order bounded. The set $\mathcal{L}_b(E)$ of all order bounded operators on $E$ is an ordered vector space with respect to the pointwise operations and order. The positive cone of $\mathcal{L}_b(E)$ is the subset of all positive operators. An element $T$ in $\mathcal{L}_b(E)$ is referred to as an orthomorphism if, for all $x, y \in E$, $|T(x)| \wedge |y| = 0$ whenever $|x| \wedge |y| = 0$. Under the ordering and operations inherited from $\mathcal{L}_b(E)$, the set $Orth(E)$ of all orthomorphisms on $E$ is an Archimedean Riesz space. The absolute value in $Orth(E)$ is given by $|T|(x) = |T(x)|$ for all $x \in E^+$. More details about order bounded operators and orthomorphisms can be found in [1], [12].

The following paragraph deals with $f$-algebras. The Riesz space $E$ is said to be a Riesz algebra if there exists an associative multiplication in $E$ with the usual algebra properties such that $xy \in E^+$ for all $x, y \in E^+$. We say that the Riesz algebra is semiprime if 0 is the only nilpotent element. The Riesz algebra $E$ is said to be an $f$-algebra whenever $x \wedge y = 0$ implies $xz \wedge y = zx \wedge y = 0$ for all $z \in E^+$. It follows that multiplication in an $f$-algebra is an orthomorphism. If $E$ is a Riesz space then the Riesz space $Orth(E)$ is an $f$-algebra with respect to the composition as multiplication. Moreover the identity map on $E$ is the multiplicative unit of $Orth(E)$. In particular, the $f$-algebra $Orth(E)$ is semiprime and commutative. We end this paragraph with a remarkable result due to A.C. Zaanen [1, Theorem 2.62] that if $E$ is an $f$-algebra with unit element, then the mapping $\pi : x \to \pi_x$ from $E$ into $Orth(E)$ is a Riesz and algebra isomorphism, where $\pi_x(y) = xy$ for all $y \in E$. For more details about $f$-algebras we refer the reader to [1], [4], [12].

We end this section recalling that a Dedekind complete Riesz space $E$ is said to be *universally complete* whenever every set of pairwise disjoint positive elements has a supremum. Every Archimedean Riesz space $E$ has a unique (up to a Riesz isomorphism) universally completion denoted $E^u$, i.e., there exists a unique universal complete Riesz space such that $E$ can be identified with an order dense Riesz subspace of $E^u$. Moreover $E^u$ is furnished with a multiplication, under which $E^u$ is an $f$-algebra with unit element. For more details see [1].

## 3. Main results

The following definition turns out to be useful in the sequel. We shall say that a mapping $T$ from a Riesz space $E$ into a Hausdorff t.v.s. $F$ is continuous, if $x_n \to 0$ (r-u) in $E$ implies $T(x_n) \to 0$ in $F$.

The main result of this paper is strongly based on the following theorem [10].

**Theorem 1.** *Let $E$ be a relatively uniformly complete Riesz space, $F$ be a Hausdorff t.v.s. (not necessarily Riesz space) and let $\varphi : E \times E \to F$ be a continuous orthosymmetric bilinear map. Then $\varphi$ is symmetric.*

In what follows, the relative uniform completeness condition on the domain will be removed and substituted by a separation condition. First let us prove the following useful proposition.

**Proposition 2.** *Let $E$ be a Riesz space, $F$ be a Hausdorff t.v.s., and let $T : E^n \to F$ be a continuous orthosymmetric multilinear map such that $(T(E^n))'$ separates points. If $\sigma \in S(n)$ is a permutation then*

$$T(x_1, \dots, x_n) = T(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

*for all $x_1, \dots, x_n \in E$.*

Proof: We begin by proving the case where $n = 2$. Let $x, y \in E^+$ and put $e = x + y$. The principal ideal generated by $e$ is denoted $E_e$. It follows from the Kakutani Theorem [16, Theorem 2.1.3] that $E_e$ is uniformly dense in $C(K)$-space for a compact Hausdorff space $K$. Denote $T_e$ the restriction of $T$ to $E_e \times E_e$. We claim that for any $f \in (T(E \times E))'$, $f \circ T_e$ can be extended to a uniformly continuous orthosymmetric bilinear map from $C(K) \times C(K)$ to $\mathbb{R}$. To see this let $(u, v) \in C(K) \times C(K)$. From the uniform density of $E_e \times E_e$ in $C(K) \times C(K)$ there exists a sequence $((u_n, v_n))_n \in E_e \times E_e$ such $(u_n, v_n) \to (u, v)$(r-u). From bilinearity and continuity of $f \circ T_e$ there exists $c > 0$ such that $|f \circ T_e(u_m, v_m) - f \circ T_e(u_n, v_n)| \leq c(\|u_m - u_n\| + \|v_m - v_n\|)$. Consequently, $f \circ T_e(u, v)$ will be defined as the limit of the Cauchy sequence $f \circ T_e(u_n, v_n)$. Denoting $G_e : C(K) \times C(K) \to \mathbb{R}$ the extension of $f \circ T_e$ and applying the above theorem it is easy to see that $G_e(x, y) = G_e(y, x)$ and then $f \circ T_e(x, y) = f \circ T_e(y, x)$ for all $f \in (T(E \times E))'$. Since $(T(E \times E))'$ separates points, it follows that $T(x, y) = T(y, x)$. For the general case where $n \geq 2$, let $i, j \in \{1, \dots, n\}$ such that $i \neq j$, $x_1, \dots, x_n \in E$, and define the mapping $\phi$ as follows:

$$\begin{aligned}\phi : E \times E &\longrightarrow F\\ (x, y) &\longmapsto T(x_1, \ldots, \underbrace{x}_{i}, \ldots, \underbrace{y}_{j}, \ldots, x_n)\end{aligned}$$

It is easily seen that $\phi$ is a continuous orthosymmetric bilinear map, then by the preceding case $\phi$ is symmetric. So

$$T(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = T(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n).$$

Since $S(n)$ is generated by transpositions, we obtain

$$T(x_1, \ldots, x_n) = T(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

for all $\sigma \in S(n)$. □

As an immediate application to the above proposition we obtain the following result.

**Proposition 3.** *Let $E$ be a Riesz space, $F$ be a Hausdorff t.v.s., and let $T : E^n \to F$ be a continuous orthosymmetric multilinear map such that $(T(E^n))'$ separates points. Then*

$$T(\pi_1(x_1), \ldots, \pi_n(x_n)) = T(x_1, \ldots, x_{n-1}, \pi_1 \ldots \pi_n(x_n))$$

*for all $x_1, \ldots, x_n \in E$ and $\pi_1, \ldots, \pi_n \in Orth(E)$.*

Proof: It is sufficient to prove that if $i \leq j \in \{1, \ldots, n\}$ then

$$T(x_1, \ldots, \pi(x_i), \ldots, x_n) = T(x_1, \ldots, \pi(x_j), \ldots, x_n)$$

for all $x_1, \ldots, x_n \in E$ and $\pi \in Orth(E)$.

Let $i \leq j \in \{1, \ldots, n\}$, $\pi \in Orth(E)$ and define the mapping $\phi$ as follows

$$\begin{aligned}\phi : E \times E &\longrightarrow F\\ (x, y) &\longmapsto T(x_1, \ldots, \underbrace{\pi(x)}_{i}, \ldots, \underbrace{y}_{j}, \ldots, x_n).\end{aligned}$$

Since $(T(E^n))'$ separates points, it is not hard to verify that as a consequence $(\phi(E \times E))'$ separates points. Now it is straightforward to show that $\phi(x, y) = \phi(y, x)$ because $\phi$ is a continuous orthosymmetric bilinear map such that $(\phi(E \times E))'$ separates points. Therefore

$$T(x_1, \ldots, \underbrace{\pi(x)}_{i}, \ldots, \underbrace{y}_{j}, \ldots, x_n) = T(x_1, \ldots, \underbrace{\pi(y)}_{i}, \ldots, \underbrace{x}_{j}, \ldots, x_n)$$

on the other hand $T$ is symmetric (Proposition 2), so

$$T(x_1, \ldots, \underbrace{\pi(y)}_{i}, \ldots, \underbrace{x}_{j}, \ldots, x_n) = T(x_1, \ldots, \underbrace{x}_{i}, \ldots, \underbrace{\pi(y)}_{j}, \ldots, x_n).$$

Consequently,

$$T(x_1, \ldots, \underbrace{\pi(x)}_{i}, \ldots, \underbrace{y}_{j}, \ldots, x_n) = T(x_1, \ldots, \underbrace{x}_{i}, \ldots, \underbrace{\pi(y)}_{j}, \ldots, x_n)$$

for all $x, y \in E$. In particular, we have

$$T(x_1, \ldots, \pi(x_i), \ldots, x_j, \ldots, x_n) = T(x_1, \ldots, x_i, \ldots, \pi(x_j), \ldots, x_n)$$

and the proof is finished. $\square$

The previous results will be used to characterize orthogonally additive homogeneous polynomials. Let $E$ be a Riesz space and let $F$ be a t.v.s. A map $P : E \to F$ is called a homogeneous polynomial of degree $n$ (or an $n$-homogeneous polynomial) if $P(x) = \varphi(x, \ldots, x)$, where $\varphi$ is an $n$-multilinear map from $E^n$ into $F$. A homogeneous polynomial of degree $n$; $P : E \to F$ is said to be orthogonally additive if $P(x+y) = P(x)+P(y)$ where $x, y \in E$ are orthogonal (i.e. $|x| \wedge |y| = 0$). We denote by $P_0(^nE, F)$ the set of $n$-homogeneous orthogonally additive polynomials from $E$ to $F$ (continuous in the case that $E$ is equipped with the relatively uniform topology and $F$ a Hausdorff topology). The interest in orthogonally additive polynomials on Banach lattices originates in the work of K. Sundaresan [19], where the space of $n$-homogeneous orthogonally additive polynomials on $L^p$ and $l^p$ was characterized. More precisely, K. Sundaresan proved that every $n$-homogeneous orthogonally additive polynomial $P : L^p \to \mathbb{R}$ is determined by some $g \in L^{\frac{p}{p-n}}$ via the formula $P(f) = \int f^n g d_\mu$ for all $f \in L^p$. After that, D. Prez-García and I. Villanueva in [18], D. Carando, S. Lassalle and I. Zalduendo in [9] proved the following analogous result for $C(X)$ spaces: Let $Y$ be a Banach space, let $P : C(X) \to Y$ be an orthogonally additive $n$-homogeneous polynomial and let $\varphi : (C(X))^n \to Y$ be its unique associated symmetric multilinear operator. Then there exists a linear operator $S : C(X) \to Y$ such that $\|S\| = \|\varphi\|$ and there exists a finitely additive measure $\mu : \sum \to Y^{**}$ such that for every $f \in C(X)$, we have $P(f) = S(f^n) = \int_X f^n g d_\mu$. Here $\sum$ is the Borel $\sigma$-algebra on $X$. By using heavily the representation of Riesz spaces as vector spaces of extended continuous functions, Y. Benyamini , S. Lassalle and Llavona [3] have proven a result analogous to that of K. Sundaresan for the classes of Banach lattices of functions of order continuous Köthe function space, whose dual is given by integrals. Very recently the first and the second author together with M. Meyer show the analogous result for uniformly complete Riesz spaces, that is, if $E$ is a uniformly complete Riesz space, $F$ is a Hausdorff t.v.s. and $P \in P_0(^nE, F)$ then there exists $S : \prod_{i=1}^n E \to F$ such that $P(x) = S(x^n)$ for every $x \in E$, where $\prod_{i=1}^n E = \{\pi_{i=1}^n x_i : x_i \in E\}$ is the Riesz subspace of $E^u$, the universal completion of $E$. Here the multiplication under consideration is the $f$-algebra multiplication of $E^u$, see [1] for more details. It is now relevant to investigate if the same result still holds true in the case where the domain is just a Riesz space. Indeed, we show in the following section that if $E$ is a Riesz space, $F$ is a Hausdorff t.v.s. and $P \in P_0(^nE, F)$, then under additional condition that $(\varphi(E^n))'$ separates points, there exists $S : \prod_{i=1}^n E \to F$ such that $P(x) = S(x^n)$ for every $x \in E$.

The following theorem which is somehow an extension of the well known Kantorovich theorem [1, Theorem 1.10] will be of great use next.

**Theorem 4.** *Let $E$ be a Riesz space, $F$ be a Hausdorff t.v.s., and let $T : E^+ \to F$ be a continuous additive mapping. Then $T$ has a unique extension to a continuous operator from $E$ to $F$. Moreover the extension, denoted by $T$ again, is given by*

$$T(x) = T(x^+) - T(x^-)$$

*for all $x \in E$.*

Proof: Consider the mapping $S : E \to F$ defined by $S(x) = T(x^+) - T(x^-)$. Obviously $S$ is the only possible linear extension of $T$ to all of $E$. The additivity of $S$ can be proved as in [1, Theorem 1.10]. To show that $S$ is homogeneous, let $\lambda \in \mathbb{R}^+$, $x \in E$ and take a sequence of nonnegative rational numbers $(\lambda_n)$ such that $\lambda_n \to \lambda$ in $\mathbb{R}$, then $S(\lambda_n x) = \lambda_n S(x)$ for all $n \in \mathbb{N}$.

We claim that

$$\lambda_n S(x) \longrightarrow \lambda S(x), \text{ and } \mathrm{S}(\lambda_\mathrm{n}\mathrm{x}) \longrightarrow \mathrm{S}(\lambda\mathrm{x}).$$

From the continuity of the scalar multiplication $\mathbb{R} \times F \to F$ it is easily verified that $\lambda_n S(x) \longrightarrow \lambda S(x)$.

By definition, $S(\lambda_n x) = T((\lambda_n x)^+) - T((\lambda_n x)^-)$. As $(\lambda_n x)^+ \to (\lambda x)^+$(r-u) and $(\lambda_n x)^- \to (\lambda x)^-$(r-u), it follows from the continuity of $T$ that $S(\lambda_n x) \to S(\lambda x)$.

Since $F$ is a Hausdorff space we deduce that

$$\lambda S(x) = S(\lambda x) \quad for\ \lambda \in \mathbb{R}^+,\ x \in E.$$

Finally, if $\lambda \in \mathbb{R}^-$ and $x \in E$, since $0 = S(y - y) = S(y) + S(-y)$ it follows that

$$S(y) = -S(-y),$$

then

$$S(\lambda x) = S(-(-\lambda)x)) = -S((-\lambda)x)) = -(-\lambda)S(x) = \lambda S(x).$$

So, $S$ is homogeneous. It remains to show the continuity of $S$. To this end let $x \in E$ and $(x_n)$ be a sequence in $E$ such that $x_n \to x$ (r-u), then it follows that $x_n^+ \to x^+$(r-u) and $x_n^- \to x^-$(r-u). Writing that $S(x_n) = T(x_n^+) - T(x_n^-)$ we see that continuity of $S$ follows immediately from the continuity of $T$, and the proof is finished. □

Now we are able to announce the main result of this paper.

**Theorem 5.** *Let $E$ be an Archimedean Riesz space, $F$ be a Hausdorff t.v.s., and let $T : E^n \to F$ be a continuous orthosymmetric multilinear map such that $(T(E^n))'$ separates points. Then there exists a linear operator $S : \prod_{i=1}^n E \to F$ such that*

$$T(x_1, \ldots, x_n) = S(x_1 \cdots x_n).$$

Proof: Let $0 \leq x, y \in E$ such that $x^n = y^n$. From the fact that $E^u$ is a unital $f$-algebra, it follows that $x = y$ [4, Proposition 2]. This implies that we can define a mapping $S$ on the positive cone of $\prod_{i=1}^n E$ by putting $S(x^n) = T(x, \ldots, x)$ for all $0 \leq x \in E$. We claim that $S$ is additive. To see this, let $0 \leq x, y \in E$, then there exists a unique $0 \leq z$ such that $x^n + y^n = z^n$ [5, Lemma 3.1]. Therefore

$$S(x^n + y^n) = S(z^n) = T(z, \ldots, z).$$

Now, put $e = x + y + z$. We denote by $I_e$ the principal ideal generated by $e$. $I_e$ is a uniformly complete Riesz space with strong order unit, then from [12, Remark 19.5] there exists $\pi_x, \pi_y, \pi_z \in Orth(I_e)$ such that $x = \pi_x(e), y = \pi_y(e), z = \pi_z(e)$. Now by Proposition 3,

$$S(z^n) = T(\pi_z(e), \ldots, \pi_z(e)) = T(e, \ldots, \pi_z^n(e)).$$

On the other hand, applying [5, Theorem 2.2] to the mapping

$$\begin{aligned} \phi : E \times \cdots \times E &\longrightarrow \prod_{i=1}^n E \\ (x_1, \ldots, x_n) &\longmapsto x_1 x_2 \cdots x_n \end{aligned}$$

we get

$$\phi(\pi_x^n(e), \ldots, e) = \phi(\pi_x(e), \pi_x(e), \ldots, \pi_x(e)).$$

So

$$\pi_x^n(e) e^{n-1} = (\pi_x(e))^n = x^n.$$

Analogously, we have

$$\pi_y^n(e) e^{n-1} = (\pi_y(e))^n = y^n \ \ \textit{and} \ \ \pi_z^n(e) e^{n-1} = (\pi_z(e))^n = z^n.$$

Therefore

$$(\pi_x^n(e) + \pi_y^n(e)) e^{n-1} = x^n + y^n = z^n = \pi_z^n(e) e^{n-1}.$$

And thus

$$(\pi_x^n(e) + \pi_y^n(e) - \pi_z^n(e)) e^{n-1} = 0.$$

Since $\pi_x^n(e) + \pi_y^n(e) - \pi_z^n(e) \in \{e\}$, where $\{e\}$ is the band generated by $e$, and since $\{e\}$ is a ring ideal of $E^u$, we deduce that $(\pi_x^n(e) + \pi_y^n(e) - \pi_z^n(e)) e^{n-2} \in \{e\}$. On the other hand, we have that $(\pi_x^n(e) + \pi_y^n(e) - \pi_z^n(e)) e^{n-2} e = 0$, so that

$(\pi_x^n(e)+\pi_y^n(e)-\pi_z^n(e))e^{n-2} \in \{e\}^d$ (because $E^u$ is semiprime). Consequently, $(\pi_x^n(e)+\pi_y^n(e)-\pi_z^n(e))e^{n-2}=0$ and by iteration we deduce that

$$\pi_x^n(e)+\pi_y^n(e)=\pi_z^n(e)$$

so we have that

$$\begin{aligned} S(z^n) &= T(e,\dots e,\pi_z^n(e)) \\ &= T(e,\dots,e,\pi_x^n(e)+\pi_y^n(e)) \\ &= T(e,\dots,e,\pi_x^n(e))+T(e,\dots,e,\pi_y^n(e)) \\ &= T(\pi_x(e),\dots,\pi_x(e))+T(\pi_y(e),\dots,\pi_y(e)) \\ &= S(x^n)+S(y^n). \end{aligned}$$

It follows that $S$ is additive on $\prod_{i=1}^n E^+$. Since $\prod_{i=1}^n E^+$ is the positive cone of the Riesz space $\prod_{i=1}^n E$, then by the preceding theorem, $S$ can be extended in a unique way to a continuous operator from $\prod_{i=1}^n E$ to $F$. This extension is also denoted by $S$. Now, to complete the proof it suffices to show that for all $x_1,\dots,x_n \in E^+$

$$T(x_1,\dots,x_n)=S(x_1\cdots x_n).$$

Put $e=x_1+\cdots+x_n$, then there exists $\pi_{x_i} \in Orth(I_e)$ such that $x_i=\pi_{x_i}(e)$ so

$$T(x_1,\dots,x_n)=T(\pi_{x_1}(e),\dots,\pi_{x_n}(e))=T(e,\dots,\pi_{x_1}\cdots\pi_{x_n}(e)).$$

On the other hand $Orth(I_e)$ is a uniformly complete semiprime $f$-algebra then there exists $\pi \in Orth(I_e)$ such that $\pi_{x_1}\cdots\pi_{x_n}=\pi^n$ therefore

$$\begin{aligned} T(x_1,\dots,x_n) &= T(e,\dots,\pi^n(e)) \\ &= S(\pi(e)^n) \\ &= S(\pi_{x_1}\cdots\pi_{x_n}(e)e\cdots e) \\ &= S(\pi_{x_1}(e)\cdots\pi_{x_n}(e)) \\ &= S(x_1\cdots x_n) \end{aligned}$$

and the proof is finished. □

In what follows, we intend to apply the above theorem to the orthogonally additive polynomials case. To do this, we will need the following lemma.

**Lemma 6.** *Let $E$ be a Riesz space, $F$ a Hausdorff t.v.s., and let $P \in P_0({}^nE,F)$ whose associated symmetric multilinear map $\varphi$ satisfies that $(\varphi(E^n))'$ separates points. Then, $\varphi$ is orthosymmetric.*

Proof: We begin by proving the case where $n=2$. Let $x,y \in E$ such that $|x|\wedge|y|=0$ and put $e=|x|+|y|$. The principal ideal generated by $e$ is denoted $E_e$. It follows from the Kakutani Theorem [16, Theorem 2.1.3] that $E_e$ is uniformly dense in $C(K)$-space for a compact Hausdorff space $K$. Denote $\varphi_e$ the restriction of $\varphi$ to $E_e \times E_e$. Using the same technique of Proposition 2 we show that for any $f \in (\varphi(E\times E))'$, $f \circ \varphi_e$ can be extended to a uniformly continuous orthosymmetric bilinear map from $C(K) \times C(K)$ to $\mathbb{R}$, denote $G_e$ this extension. Now it is not hard to see that $P_e$, the homogenous polynomial associated to $G_e$, is orthogonally additive since $P$ is orthogonally additive. Applying [7, Lemma 4.1] it follows that $G_e$ is orthosymmetric, thus $G_e(x, y) = 0$ and then $f \circ \varphi_e(x, y) = 0$ for all $f \in (\varphi(E \times E))'$. Since $(\varphi(E \times E))'$ separates points, it follows that $\varphi(x, y) = 0$. The general case is left to the reader. □

**Corollary 7.** *Let $E$ be a Riesz space, $F$ a Hausdorff t.v.s., and let $P \in P_0(^nE, F)$ whose associated symmetric multilinear map $\varphi$ satisfies that $(\varphi(E^n))'$ separates points. Then there exists a linear operator $S : \prod_{i=1}^n E \to F$ such that*

$$P(x) = S(x^n).$$

We end this paper by giving some remarks. First of all we point out that our approach works in the special case where $\psi$ is any orthosymmetric multilinear map with co-domain $\mathbb{R}[X]$, the Hausdorff t.v.s. of all real polynomials, while [2, Theorem 13] and [3, Theorem 2.3] do not apply for such co-domain since $\mathbb{R}[X]$ is neither a Riesz space nor a Banach lattice. Note that the result of [7] and [11] also fails for such maps. Now let $E$ be the Archimedean Riesz space of piecewise linear functions on $[0, 1]$, with only finite number of discontinuities. Then $E$ is not uniformly complete so the result in [9] does not apply but Theorem 2 in the current paper holds. It is easily seen that for every multilinear map $T$ such that $T(E^n) = \mathbb{R}[X]$ we have $(T(E^n))'$ separates points. Finally note that our approach fails for non continuous orthosymmetric multilinear mappings as it is shown in the following example.

**Example 8.** Let $E$ be the Riesz space of all real valued functions $f$ on $[0, 1]$ satisfying that there is a finite subset $\{x_0, \dots, x_n\}$ of $[0, 1]$ such that $0 = x_0 < x_1 < \cdots < x_n = 1$ and $f$ is linear on each interval $[x_{i-1}, x_i)$, i.e., $f(x) = m_i(f)x + b_i(f)$ for all $x \in [x_{i-1}, x_i)$. Now define

$$\begin{aligned} \varphi : E \times E &\longrightarrow \mathbb{R} \\ (f, g) &\longmapsto m_0(f)b_0(g) \end{aligned}$$

It is easily checked that $\varphi$ is well defined, orthosymmetric and bilinear map. On the other hand, note that $\varphi(E^2) = \mathbb{R}$ so $(\varphi(E^2))'$ separates points but $\varphi$ is not symmetric. Now it is a routine to verify that $\varphi$ is neither positive nor order bounded, so since $E$ and $\mathbb{R}$ are normed Riesz spaces it follows that $\varphi$ is not continuous. From the preceding example we emphasize that the condition $(\varphi(E^2))'$ separates points has no kind of relation with $\varphi$ being positive, order bounded or order continuous.

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