Property (wL) and the reciprocal Dunford-Pettis property in projective tensor products

Ioana Ghenciu

Abstract. A Banach space X has the reciprocal Dunford-Pettis property ($RDPP$) if every completely continuous operator T from X to any Banach space Y is weakly compact. A Banach space X has the $RDPP$ (resp. property (wL)) if every L-subset of X^* is relatively weakly compact (resp. weakly precompact). We prove that the projective tensor product $X \otimes_\pi Y$ has property (wL) when X has the $RDPP$, Y has property (wL), and $L(X,Y^*) = K(X,Y^*)$.

Keywords: the reciprocal Dunford-Pettis property; property (wL); spaces of compact operators; weakly precompact sets

Classification: Primary 46B20, 46B28; Secondary 28B05

1. Introduction

Throughout this paper X, Y, E, and F will denote real Banach spaces. An operator $T : X \to Y$ will be a continuous and linear function. The set of all operators from X to Y will be denoted by $L(X,Y)$, and the compact operators will be denoted by $K(X,Y)$.

In this paper we study weak precompactness and relative weak compactness in spaces of compact operators. Our results are organized as follows. First we give sufficient conditions for subsets of $K(X,Y^*)$ to be weakly precompact and relatively weakly compact. Those results are used to study whether the projective tensor product $X \otimes_\pi Y$ has properties (wL) and the $RDPP$, when X and Y have the respective property.

Finally, we prove that in some cases, if $X \otimes_\pi Y$ has property (wL), then $L(X,Y^*) = K(X,Y^*)$. Our results generalize some results from [17] and [24].

2. Definitions and notations

Our notation and terminology is standard. The unit ball of X will be denoted by B_X, and X^* will denote the continuous linear dual of X. By an operator we understand any bounded linear mapping between Banach spaces. The set of all operators from X to Y will be denoted by $L(X,Y)$, and the subspaces of compact, resp. weakly compact operators will be denoted by $K(X,Y)$, resp. $W(X,Y)$. The operator T is called *completely continuous* (*or Dunford-Pettis*) if T maps weakly

DOI 10.14712/1213-7243.2015.126

convergent sequences to norm convergent sequences. A subset S of X is said to be *weakly precompact* provided that every bounded sequence from S has a weakly Cauchy subsequence [5]. An operator $T : X \to Y$ is called *weakly precompact* (*or almost weakly compact*) if $T(B_X)$ is weakly precompact.

A bounded subset A of X^* is called an *L-subset* of X^* if each weakly null sequence in X tends to 0 uniformly on A; i.e.,

$$\lim_n \sup\{|x^*(x_n)| : x^* \in A\} = 0.$$

The Banach space X has the *reciprocal Dunford-Pettis property* (*RDPP*) if every completely continuous operator T from X to any Banach space Y is weakly compact [25, p. 153]. The space X has the *RDPP* if and only if every L-subset of X^* is relatively weakly compact [27]. Banach spaces with property (V) of Pełczyński, in particular reflexive spaces and $C(K)$ spaces, have the *RDPP* [30]. Emmanuele [20] and Bator [3] showed that $\ell_1 \not\hookrightarrow X$ if and only if every L-subset of X^* is relatively compact. We say that a Banach space X has *property weak* (L) (wL) if every L-subset of X^* is weakly precompact. The space X has the *RDPP* (resp. property (wL)) if and only if any operator $T : Y \to X^*$ such that $T^*|_X$ is completely continuous, is weakly compact (resp. weakly precompact) (by Theorem 4.7 of [23]).

The Banach space X has the *Dunford-Pettis property* (*DPP*) if every weakly compact operator $T : X \to Y$ is completely continuous. The survey article by Diestel [14] is an excellent source of information about classical contributions to the study of the *DPP*.

A topological space S is called *dispersed* (*or scattered*) if every nonempty closed subset of S has an isolated point. A compact Hausdorff space K is dispersed if and only if $\ell_1 \not\hookrightarrow C(K)$ [31].

The Banach-Mazur distance $d(E,F)$ between two isomorphic Banach spaces E and F is defined by $\inf(\|T\|\|T^{-1}\|)$, where the infinum is taken over all isomorphisms T from E onto F. A Banach space E is called an *$\mathcal{L}_\infty$-space* (resp. *$\mathcal{L}_1$-space*) [9, p. 7] if there is a $\lambda \geq 1$ so that every finite dimensional subspace of E is contained in another subspace N with $d(N, \ell_\infty^n) \leq \lambda$ (resp. $d(N, \ell_1^n) \leq \lambda$) for some integer n. Complemented subspaces of $C(K)$ spaces (resp. $L_1(\mu)$ spaces) are $\mathcal{L}_\infty$-spaces (resp. $\mathcal{L}_1$-spaces) [9, Proposition 1.26]. The dual of an $\mathcal{L}_1$-space (resp. $\mathcal{L}_\infty$-space) is an $\mathcal{L}_\infty$-space (resp. $\mathcal{L}_1$- space) [9, Proposition 1.27]. The $\mathcal{L}_\infty$-spaces, $\mathcal{L}_1$-spaces, and their duals have the *DPP* [9, Corollary 1.30].

3. Weakly precompact subsets of spaces of compact operators

We begin by giving sufficient conditions for a subset of $K(X,Y)$ to be weakly precompact and relatively weakly compact. We recall that the dual weak operator topology (w') on $L(X,Y)$ is defined by the functionals $T \longmapsto x^{**}T^*(y^*)$, $x^{**} \in X^{**}$, $y^* \in Y^*$ [26]. In Corollary 3 of [26] it is shown that if (T_n) is a sequence of compact operators such that $T_n \to T$ (w'), where T is a compact operator, then $T_n \to T$ weakly.

If H is a subset of $K(X,Y)$, $x \in X$, $y^* \in Y^*$, and $x^{**} \in X^{**}$, let $H(x) = \{Tx : T \in H\}$, $H^*(y^*) = \{T^*y^* : T \in H\}$, and $H^{**}(x^{**}) = \{T^{**}x^{**} : T \in H\}$.

Theorem 1. *Let H be a bounded subset of $K(X,Y)$ such that*

(i) *$H(x)$ is weakly precompact for each $x \in X$, and*

(ii) *$H^*(y^*)$ is relatively weakly compact for each $y^* \in Y^*$.*

Then H is weakly precompact.

Proof: Let (T_n) be a sequence in H. Let S be the closed linear span of $\{T_n^*y^* : y^* \in Y^*, n \in \mathbb{N}\}$. The compactness of each T_n implies that S is a separable subspace of X^*. Let X_0 be a countable subset of X that separates points of S. Let (x_k) be a sequence in X so that $X_0 = \{x_k : k \in \mathbb{N}\}$. By hypotheses, $\{T_n x_k : n \in \mathbb{N}\}$ is weakly precompact for each k. By diagonalization, we may assume that (T_{n_i}) is a subsequence of (T_n) so that $(T_{n_i}x_k)_i$ is weakly Cauchy for each k. Without loss of generality, we assume that $(T_n x)$ is weakly Cauchy for each $x \in X_0$.

For fixed $y^* \in Y^*$, the sequence $(T_n^*y^*)$ must have a weakly convergent subsequence. Suppose that z_1^* and z_2^* are two weak sequential cluster points of the sequence $(T_n^*y^*)$. Then $z_1^*, z_2^* \in S$. Suppose that $T_{k(n)}^*y^* \xrightarrow{w} z_1^*$, $T_{p(n)}^*y^* \xrightarrow{w} z_2^*$. For each $x \in X_0$,

$$\begin{aligned}
\langle z_1^*, x\rangle &= \lim_n \langle T_{k(n)}^*y^*, x\rangle = \lim_n \langle y^*, T_{k(n)}x\rangle \\
&= \lim_n \langle y^*, T_n x\rangle = \lim_n \langle y^*, T_{p(n)}x\rangle \\
&= \lim_n \langle T_{p(n)}^*y^*, x\rangle = \langle z_2^*, x\rangle.
\end{aligned}$$

Hence $z_1^* = z_2^*$, since X_0 separates points of S. Then $(T_n^*y^*)$ is weakly convergent for all $y^* \in Y^*$. Thus (T_n) is Cauchy in the (w') topology on $K(X,Y)$. Hence for any two subsequences (A_n) and (B_n) of (T_n), $(A_n - B_n) \to 0$ (w'). By Corollary 3 of [26], $(A_n - B_n) \to 0$ weakly; thus (T_n) is weakly Cauchy in $K(X,Y)$. □

Corollary 2. *Let H be a bounded subset of $K(X,Y)$ such that*

(i) *$H^*(y^*)$ is weakly precompact for each $y^* \in Y^*$, and*

(ii) *$H^{**}(x^{**})$ is relatively weakly compact for each $x^{**} \in X^{**}$.*

Then H is weakly precompact.

Proof: Suppose H satisfies the hypotheses. Consider the subset H^* of $K(Y^*, X^*)$. By Theorem 1, H^* is weakly precompact. Let (T_n) be a sequence in H. Without loss of generality, we can assume that (T_n^*) is weakly Cauchy. Hence $(T_n^*y^*)$ is weakly Cauchy for each $y^* \in Y^*$. Therefore (T_n) is Cauchy in the (w') topology on $K(X,Y)$. As in the proof of Theorem 1, (T_n) is weakly Cauchy. □

The following theorem generalizes Theorem 4.9 of [24].

Theorem 3. *Suppose that $L(X,Y) = K(X,Y)$. Let H be a bounded subset of $K(X,Y)$ such that*

(i) *$H(x)$ is relatively weakly compact for each $x \in X$, and*

(ii) $H^*(y^*)$ *is relatively weakly compact for each* $y^* \in Y^*$.

Then H is relatively weakly compact.

Proof: Let (T_n) be a sequence in H. By Theorem 1, H is weakly precompact. Without loss of generality, assume that (T_n) is weakly Cauchy. For each $x \in X$, the sequence $(T_n x)$ has a weakly convergent subsequence and is weakly Cauchy, thus is weakly convergent to Tx, say. Similarly, for each $y^* \in Y^*$, the sequence $(T_n^* y^*)$ has a weakly convergent subsequence and is weakly Cauchy, thus is weakly convergent.

Clearly, the assignment $X \ni x \longmapsto Tx$ is linear and bounded. Hence $T \in L(X,Y)$. For all $y^* \in Y^*$, $x \in X$, $\lim_n \langle T_n^* y^*, x\rangle = \lim_n \langle y^*, T_n x\rangle = \langle T^* y^*, x\rangle$. Then $T_n^* y^* \xrightarrow{w^*} T^* y^*$. Since $(T_n^* y^*)$ is weakly convergent, $T_n^* y^* \xrightarrow{w} T^* y^*$. Hence $T_n \to T$ in the (w') topology of $K(X,Y)$. By Corollary 3 of [26], $T_n \to T$ weakly, and H is relatively weakly compact. $\square$

Remark. If $L(X,Y) = K(X,Y)$, then a subset H of $K(X,Y)$ is relatively weakly compact if and only if conditions (i) and (ii) of the previous theorem hold.

Corollary 4 ([26, Corollary 2])**.** *If X and Y are reflexive and $L(X,Y) = K(X,Y)$, then $K(X,Y)$ is reflexive.*

Proof: Let H be the unit ball of $L(X,Y) = K(X,Y)$. Since X and Y are reflexive, $H(x)$ and $H^*(y^*)$ are relatively weakly compact for all $x \in X$ and $y^* \in Y^*$. By Theorem 3, H is relatively weakly compact, and thus $K(X,Y)$ is reflexive. $\square$

4. Property (wL) and the $RDPP$ in projective tensor products

In this section we consider the property (wL) and the $RDPP$ in the projective tensor product $X \otimes_\pi Y$. We begin by noting that there are examples of Banach spaces X and Y such that $X \otimes_\pi Y$ has property $RDPP$. If $1 < q' < p < \infty$, then $L(\ell_p, \ell_{q'}) = K(\ell_p, \ell_{q'})$ ([33]). Let q be the conjugate of q'. By [26, Corollary 2], $L(\ell_p, \ell_{q'}) \simeq (\ell_p \otimes_\pi \ell_q)^*$ is reflexive. Then $\ell_p \otimes_\pi \ell_q$ is reflexive, and thus has the $RDPP$. Thus the spaces $X = \ell_p$ and $Y = \ell_q$ are as desired.

Observation 1. If X is an infinite dimensional space with the Schur property, then X does not have property (wL).

Since $\ell_1 \hookrightarrow X$, $\ell_1 \hookrightarrow X^*$ ([13], p. 211). All bounded subsets of X^* are L-subsets, and thus there are L-subsets of X^* which fail to be weakly precompact.

Since property (wL) is inherited by quotients, it follows that if X has property (wL), then $\ell_1 \overset{c}{\not\hookrightarrow} X$, and $c_0 \not\hookrightarrow X^*$ [6].

Observation 2. If $T: Y \to X^*$ be an operator such that $T^*|_X$ is compact, then T is compact. To see this, let $T: Y \to X^*$ be an operator such that $T^*|_X$ is compact. Let $S = T^*|_X$. Suppose $x^{**} \in B_{X^{**}}$ and choose a net (x_α) in B_X which is w^*- convergent to x^{**}. Then $(T^* x_\alpha) \xrightarrow{w^*} T^* x^{**}$. Now, $(T^* x_\alpha) \subseteq S(B_X)$, which is a relatively compact set. Then $(T^* x_\alpha) \to T^* x^{**}$. Hence $T^*(B_{X^{**}}) \subseteq \overline{S(B_X)}$, which is relatively compact. Therefore $T^*(B_{X^{**}})$ is relatively compact, and thus T is compact. It follows that if $L(X, Y^*) = K(X, Y^*)$, then $L(Y, X^*) = K(Y, X^*)$.

The following lemma is known [8]; we include proof for the convenience of the reader.

Lemma 5. *Suppose that every operator $T : X \to Y^*$ is completely continuous. If (x_n) is a weakly null sequence in X and (y_n) is a bounded sequence in Y, then $(x_n \otimes y_n)$ is weakly null in $X \otimes_\pi Y$.*

Proof: Suppose that (x_n) is weakly null and $\|y_n\| \leq M$ for all $n \in \mathbb{N}$. Let $T \in L(X, Y^*) \simeq (X \otimes_\pi Y)^*$ ([15, p. 230]). Since T is completely continuous,

$$|\langle T, x_n \otimes y_n \rangle| \leq M \|T x_n\| \to 0.$$

□

Theorem 6. (i) *Suppose that X has the RDPP, Y has property (wL), and $L(X, Y^*) = K(X, Y^*)$. Then $X \otimes_\pi Y$ has property (wL).*

(ii) *Suppose that X has property (wL), Y has the RDPP, and $L(X, Y^*) = K(X, Y^*)$. Then $X \otimes_\pi Y$ has property (wL).*

Proof: (i) We will use Theorem 1. Let H be an L-subset of $(X \otimes_\pi Y)^* \simeq L(X, Y^*) = K(X, Y^*)$. We will verify the conditions (i) and (ii) of this theorem. Let (T_n) be a sequence in H and let $y^{**} \in Y^{**}$. We will show that $\{T_n^* y^{**} : n \in \mathbb{N}\}$ is an L-subset of X^*. Suppose that (x_n) is weakly null in X. For $n \in \mathbb{N}$,

$$|\langle T_n^* y^{**}, x_n \rangle| = |\langle y^{**}, T_n x_n \rangle| \leq \|y^{**}\| \, \|T_n x_n\|.$$

We show that $\|T_n x_n\| \to 0$. Suppose that $\|T_n x_n\| \not\to 0$. Without loss of generality we assume that $|\langle T_n x_n, y_n \rangle| > \epsilon$ for some sequence (y_n) in B_Y and some $\epsilon > 0$. Since $\{T_n : n \in \mathbb{N}\}$ is an L-set and $(x_n \otimes y_n)$ is weakly null in $X \otimes_\pi Y$ (by Lemma 5), $\sup_m |\langle T_m, x_n \otimes y_n \rangle| \to 0$, and so $|\langle T_n, x_n \otimes y_n \rangle| = |\langle T_n x_n, y_n \rangle| \to 0$. This contradiction shows that $\|T_n x_n\| \to 0$. Hence $\{T_n^* y^{**} : n \in \mathbb{N}\}$ is an L-subset of X^*. Therefore this subset is relatively weakly compact [27]. This verifies (ii) of Theorem 1.

It remains to verify (i) of Theorem 1. Let $x \in X$. We show that $\{T_n x : n \in \mathbb{N}\}$ is an L-subset of Y^*. Let (y_n) be a weakly null sequence in Y. For $n \in \mathbb{N}$,

$$|\langle T_n x, y_n \rangle| = |\langle x, T_n^* y_n \rangle| \leq \|x\| \, \|T_n^* y_n\|.$$

An argument similar to the one above shows that $\|T_n^* y_n\| \to 0$. Thus $\{T_n x : n \in \mathbb{N}\}$ is an L-subset of Y^*, hence weakly precompact, for all $x \in X$. We thus verified (i) of Theorem 1. By Theorem 1, (T_n) has a weakly Cauchy subsequence. We proved that H is weakly precompact.

(ii) If $L(X, Y^*) = K(X, Y^*)$, then $L(Y, X^*) = K(Y, X^*)$ (by Observation 2). By (i), $Y \otimes_\pi X$ has property (wL). Since $X \otimes_\pi Y$ is isometrically isomorphic to $Y \otimes_\pi X$, $X \otimes_\pi Y$ has property (wL). □

Theorem 7. *Suppose that X and Y have the RDPP and $L(X,Y^*) = K(X,Y^*)$. Then $X \otimes_\pi Y$ has the RDPP.*

Proof: Let H be an L-subset of $(X \otimes_\pi Y)^* \simeq L(X,Y^*) = K(X,Y^*)$ and let (T_n) be a sequence in H. The proof of Theorem 6 shows that $\{T_n x : n \in \mathbb{N}\}$ is an L-subset of Y^*, and thus relatively weakly compact by [27]. Similarly, $\{T_n^* y^{**} : n \in \mathbb{N}\}$ is an L-subset of X^*, thus relatively weakly compact. Then, by Theorem 3, (T_n) has a weakly convergent subsequence. $\square$

Theorem 7 contains Corollary 4 of [17]. The assumptions that X^* and Y^* are weakly sequentially complete in Corollary 4 of [17] are superfluous.

Corollary 8. *Suppose that $\ell_1 \not\hookrightarrow X$, Y has the RDPP (resp. property (wL)), and $L(X,Y^*) = K(X,Y^*)$. Then $X \otimes_\pi Y$ has the RDPP (resp. property (wL)).*

Proof: If $\ell_1 \not\hookrightarrow X$, then every L-subset of X^* is relatively compact [20], [3]. If Y has the *RDPP* (resp. property (wL)), then $X \otimes_\pi Y$ has the *RDPP* (resp. property (wL)), by Theorem 7 (resp. Theorem 6 (i)). $\square$

The *RDPP* case of the previous result was proved in Theorem 3 of [17]. In Theorem 11 we show that if $X \otimes_\pi Y$ has the *RDPP* (resp. property (wL)), then either $\ell_1 \not\hookrightarrow X$ or $\ell_1 \not\hookrightarrow Y$. Thus, in Theorems 6 and 7 we can suppose without loss of generality that either $\ell_1 \not\hookrightarrow X$ or $\ell_1 \not\hookrightarrow Y$. Hence Theorem 7 is equivalent to Theorem 3 of [17].

Corollary 9. (i) *Suppose that X is a closed subspace of an order continuous Banach lattice and X has property (wL). If Y has the RDPP (resp. property (wL)) and $L(X,Y^*) = K(X,Y^*)$, then $X \otimes_\pi Y$ has the RDPP (resp. property (wL)).*

(ii) *Suppose that X is a Banach space with property (wV^*) and X has property (wL). If Y has the RDPP (resp. property (wL)) and $L(X,Y^*) = K(X,Y^*)$, then $X \otimes_\pi Y$ has the RDPP (resp. property (wL)).*

Proof: If X has property (wL), then $\ell_1 \overset{c}{\not\hookrightarrow} X$ (by Observation 1).

(i) Since X is a subspace of a Banach lattice, $\ell_1 \not\hookrightarrow X$ [36]. Apply Corollary 8.

(ii) Since X has property (wV^*), $\ell_1 \not\hookrightarrow X$ [7]. Apply Corollary 8. $\square$

Corollary 9(i) contains Corollary 5 of [17]. The fact that properties *RDPP* and (wL) are inherited by quotients, immediately implies the following result, which contains Corollary 6 of [17].

Corollary 10. *Suppose that $\ell_1 \not\hookrightarrow E^*$ and F has property RDPP (resp. property (wL)). If $L(E^*,F^*) = K(E^*,F^*)$, then the space $N_1(E,F)$ of all nuclear operators from E to F has the RDPP (resp. property (wL)).*

Proof: It is known that $N_1(E,F)$ is a quotient of $E^* \otimes_\pi F$ [34, p. 41]. Apply Corollary 8. $\square$

Theorem 11. *Suppose that $L(E,F^*) = K(E,F^*)$. The following statements are equivalent:*

(i) *E and F have the RDPP (resp. property (wL)) and either $\ell_1 \not\hookrightarrow E$ or $\ell_1 \not\hookrightarrow F$.*
(ii) *$E \otimes_\pi F$ has the RDPP (resp. property (wL)).*

Proof: (i) $\Rightarrow$ (ii) by Corollary 8.

(ii) $\Rightarrow$ (i) Suppose that $E \otimes_\pi F$ has the *RDPP* (resp. property (wL)). Then E and F have the *RDPP* (resp. property (wL)), since the *RDPP* (resp. property (wL)) is inherited by quotients. Suppose $\ell_1 \hookrightarrow E$ and $\ell_1 \hookrightarrow F$. Hence $L_1 \hookrightarrow E^*$ [29]. Also, the Rademacher functions span ℓ_2 inside of L_1, and thus $\ell_2 \hookrightarrow E^*$. Similarly $\ell_2 \hookrightarrow F^*$. Then $c_0 \hookrightarrow K(E, F^*)$ ([16], [22]), a contradiction with Observation 1. □

The *RDPP* case of the previous result was proved in Theorem 8 of [17].

Observation 3. If $\ell_1 \hookrightarrow E$ and $\ell_1 \hookrightarrow F$, then $c_0 \hookrightarrow K(E, F^*)$ ([16], [22]). More generally, if $\ell_1 \hookrightarrow E$ and $\ell_p \hookrightarrow F^*$, $p \geq 2$, then $c_0 \hookrightarrow K(E, F^*)$ ([16], [22]). Hence $\ell_1 \overset{c}{\hookrightarrow} E \otimes_\pi F$ [6]. By Observation 1, $E \otimes_\pi F$ does not have property (wL).

Observation 4. If E^* has the Schur property, then $\ell_1 \not\hookrightarrow E$. Indeed, if $\ell_1 \hookrightarrow E$, then $L_1 \hookrightarrow E^*$ [29], and E^* does not have the Schur property.

Observation 5. If E^* has the Schur property and F has property (wL), then $L(E, F^*) = K(E, F^*)$. To see this, let $T : F \to E^*$ be an operator. Then T is completely continuous (since E^* has the Schur property). Therefore $T^*(B_{E^{**}})$ is an L-subset of F^*, thus is weakly precompact. Since T^* is weakly precompact, T is weakly precompact, by Corollary 2 of [4]. Then T is compact. By Observation 2, $L(E, F^*) = K(E, F^*)$.

Corollary 12. (i) *Suppose that E^* has the Schur property and F has the RDPP (resp. property (wL)). Then $E \otimes_\pi F$ has the RDPP (resp. property (wL)).*
(ii) [17, Corollary 10] *Suppose that $E = \ell_p$, where $1 < p \leq \infty$, and $F = c_0$. Then $E \otimes_\pi F$ has the RDPP.*
(iii) *Suppose that E is an infinite dimensional $\mathcal{L}_\infty$-space not containing ℓ_1. If F has the RDPP (resp. property (wL)), then $E \otimes_\pi F$ has the RDPP (resp. property (wL)).*

Proof: (i) Since E^* has the Schur property, $\ell_1 \not\hookrightarrow E$ (by Observation 4). By Observation 5, $L(E, F^*) = K(E, F^*)$. Apply Corollary 8.

(ii) By (i), $F \otimes_\pi E$, hence $E \otimes_\pi F$ has the *RDPP*.

(iii) Suppose E is an infinite dimensional $\mathcal{L}_\infty$-space not containing ℓ_1. Then E has the *DPP* by Corollary 1.30 of [9]; thus E^* has the Schur property by Theorem 3 of [14]. Apply (i). □

The *RDPP* case of Corollary 12(i) was proved in Corollary 9 of [17]. Corollary 12(iii) generalizes Corollary 11 of [17]. The hypothesis that F^* is a subspace of an $\mathcal{L}_1$-space in Corollary 11 of [17] is superfluous.

Corollary 13. *Suppose that E and F have the DPP. The following statements are equivalent:*

(i) *E and F have the $RDPP$ (resp. property (wL)) and $\ell_1 \not\hookrightarrow E$ or $\ell_1 \not\hookrightarrow F$;*
(ii) *$E \otimes_\pi F$ has the $RDPP$ (resp. property (wL)).*

Proof: (i)$\Rightarrow$(ii) Suppose that E and F have the DPP and the $RDPP$ (resp. property (wL)). Suppose without loss of generality that $\ell_1 \not\hookrightarrow E$. Then E^* has the Schur property by Theorem 3 of [14]. Apply Corollary 12 (i).

(ii)$\Rightarrow$(i) The proof is the same as the corresponding one in Theorem 11. $\square$

By Theorem 11 (or Corollary 13), the space $C(K_1) \otimes_\pi C(K_2)$ has the $RDPP$ if and only if either K_1 or K_2 is dispersed. The spaces A and H^∞ have the DPP and property (V), hence they have the $RDPP$, and contain copies of ℓ_1 ([10], [11], [12], [35]). Let E, F be A or H^∞. Then $E \otimes_\pi F$ does not have property (wL) (by Observation 3).

Corollary 14. *Suppose that $\ell_1 \not\hookrightarrow E$ and F has the $RDPP$ (resp. property (wL)). If F^* is complemented in a Banach space Z which has an unconditional Schauder decomposition (Z_n), with Z_n having the Schur property for each n, then the following statements are equivalent:*

(i) *$E \otimes_\pi F$ has the $RDPP$ (resp. property (wL));*
(ii) *$L(E, F^*) = K(E, F^*)$.*

Proof: (i)$\Rightarrow$(ii) Suppose $E \otimes_\pi F$ has the $RDPP$ (resp. property (wL)). Since $\ell_1 \not\hookrightarrow E$ and Z_n has the Schur property, $L(E, Z_n) = K(E, Z_n)$ for each n. If $L(E, F^*) \neq K(E, F^*)$, then $c_0 \hookrightarrow K(E, F^*)$ (by Theorem 1 of [18]), a contradiction.

(ii)$\Rightarrow$(i) Apply Corollary 8. $\square$

Next we present some results about the necessity of the conditions $L(E, F^*) = K(E, F^*)$ and $W(E, F^*) = K(E, F^*)$.

A Banach space X has the *approximation property* if for each norm compact subset M of X and $\epsilon > 0$, there is a finite rank operator $T : X \to X$ such that $\|Tx - x\| < \epsilon$ for all $x \in M$. If in addition T can be found with $\|T\| \leq 1$, then X is said to have the *metric approximation property.* For example, $C(K)$ spaces, c_0, ℓ_p for $1 \leq p < \infty$, $L_p(\mu)$ for any measure μ and $1 \leq p < \infty$, and their duals have the metric approximation property [15, p. 238], [34].

A separable Banach space X has an *unconditional compact expansion of the identity (u.c.e.i)* if there is a sequence (A_n) of compact operators from X to X such that $\sum A_n x$ converges unconditionally to x for all $x \in X$ [21]. In this case, (A_n) is called an (u.c.e.i.) of X. A sequence (X_n) of closed subspaces of a Banach space X is called an *unconditional Schauder decomposition* of X if every $x \in X$ has a unique representation of the form $x = \sum x_n$, with $x_n \in X_n$, for every n, and the series converges unconditionally [28, p. 48].

The space X has (Rademacher) *cotype* q for some $2 \le q \le \infty$ if there is a constant C such that for for every n and every $x_1, x_2, \ldots, x_n$ in X,

$$\left(\sum_{i=1}^{n} \|x_i\|^q\right)^{1/q} \le C\left(\int_0^1 \|r_i(t)x_i\|^q dt\right)^{1/q},$$

where (r_n) are the Radamacher functions. A Hilbert space has cotype 2 [1, p. 138]. The dual of $C(K)$, the space $M(K)$, has cotype 2 [1, p. 142].

Theorem 15. *Assume one of the following conditions holds.*

(i) *If $T : E \to F^*$ is an operator which is not compact, then there is a sequence (T_n) in $K(E, F^*)$ such that for each $x \in E$, the series $\sum T_n x$ converges unconditionally to Tx.*
(ii) *Either E^* or F^* has an (u.c.e.i.).*
(iii) *E is an $\mathcal{L}_\infty$-space and F^* is a subspace of an $\mathcal{L}_1$-space.*
(iv) *$E = C(K)$, K a compact Hausdorff space, and F^* is a space with cotype 2.*
(v) *E has the DPP and $\ell_1 \hookrightarrow F$.*
(vi) *E and F have the DPP.*

If $E \otimes_\pi F$ has property (wL), then $L(E, F^*) = K(E, F^*)$.

Proof: Suppose $E \otimes_\pi F$ has property (wL). Then E and F have property (wL).

(i) Let $T : E \to F^*$ be a noncompact operator. Let (T_n) be a sequence as in the hypothesis. By the Uniform Boundedness Principle, $\{\sum_{n\in A} T_n : A \subseteq \mathbb{N}, A \text{ finite}\}$ is bounded in $K(E, F^*)$. Then $\sum T_n$ is wuc and not unconditionally convergent (since T is noncompact). Hence $c_0 \hookrightarrow K(E, F^*)$ [6], and we have a contradiction with Observation 1.

(ii) Suppose that F^* has an (u.c.e.i.) (A_n). Then $A_n : F^* \to F^*$ is compact for each n and $\sum A_n y$ converges unconditionally to y, for each $y \in F^*$. Let $T : E \to F^*$ be a noncompact operator. Hence $\sum A_n T x$ converges unconditionally to Tx for each $x \in E$ and $A_n T \in K(E, F^*)$. Then $c_0 \hookrightarrow K(E, F^*)$ (by (i)), a contradiction.

Similarly, if E^* has an (u.c.e.i.) and $L(E, F^*) \ne K(E, F^*)$, then $c_0 \hookrightarrow K(F, E^*)$.

Suppose (iii) or (iv) holds. It is known that any operator $T : E \to F^*$ is 2-absolutely summing ([32]), hence it factorizes through a Hilbert space. If $L(E, F^*) \ne K(E, F^*)$, then $c_0 \hookrightarrow K(E, F^*)$ (by Remark 3 of [19]), a contradiction.

(v) Suppose that E has the DPP and $\ell_1 \hookrightarrow F$. By Observation 3, $\ell_1 \not\hookrightarrow E$. Then E^* has the Schur property by Theorem 3 of [14]. By Observation 5, $L(E, F^*) = K(E, F^*)$.

(vi) Suppose that E and F have the DPP. If $\ell_1 \hookrightarrow F$, then (v) implies $L(E, F^*) = K(E, F^*)$. If $\ell_1 \not\hookrightarrow F$, then F^* has the Schur property [14]. By the proof of Observation 5, $L(E, F^*) = K(E, F^*)$. ☐

By Theorem 15, if one of the hypotheses (i)-(vi) holds and $L(E, F^*) \neq K(E, F^*)$, then $E \otimes_\pi F$ does not have property (wL). Thus the space $\ell_p \otimes \ell_q$, where $1 < p \leq q' < \infty$ and q and q' are conjugate, does not have property (wL), since the natural inclusion map $i : \ell_p \to \ell_{q'}$ is not compact. Further, the space $C(K) \otimes_\pi \ell_p$, with K not dispersed and $1 < p \leq 2$, does not have property (wL), since $L(C(K), \ell_q) \neq K(C(K), \ell_q)$ (by Corollary 3.11 of [2]), where q is the conjugate of p, $2 \leq q < \infty$.

Theorem 16. *Suppose that F^* is complemented in a Banach space Z which has an unconditional Schauder decomposition (Z_n), and $W(E, Z_n) = K(E, Z_n)$ for all n. If $E \otimes_\pi F$ has property (wL), then $W(E, F^*) = K(E, F^*)$.*

Proof: Let $T : E \to F^*$ be a weakly compact and noncompact operator, $P_n : Z \to Z_n$, $P_n(\sum z_i) = z_n$, and let P be the projection of Z onto F^*. Define $T_n : E \to F^*$ by $T_n x = PP_n T x$, $x \in E$, $n \in \mathbb{N}$. Note that $P_n T$ is compact since $W(E, Z_n) = K(E, Z_n)$. Then T_n is compact for each n. For each $z \in Z$, $\sum P_n z$ converges unconditionally to z; thus $\sum T_n x$ converges unconditionally to Tx for each $x \in E$. Then $\sum T_n$ is wuc and not unconditionally converging. Hence $c_0 \hookrightarrow K(E, F^*)$ [6], and we obtain a contradiction. □

References

[1] Albiac F., Kalton N.J., *Topics in Banach Space Theory*, Graduate Texts in Mathematics, 233, Springer, New York, NY, USA, 2006.

[2] Ansari S.I., *On Banach spaces Y for which $B(C(\Omega), Y) = K(C(\Omega), Y)$*, Pacific J. Math. **169** (1995), 201–218.

[3] Bator E.M., *Remarks on completely continuous operators*, Bull. Polish Acad. Sci. Math. **37** (1989), 409–413.

[4] Bator E.M., Lewis P., *Operators having weakly precompact adjoints*, Math. Nachr. **157** (2006), 99–103.

[5] Bator E.M., Lewis P., Ochoa J., *Evaluation maps, restriction maps, and compactness*, Colloq. Math. **78** (1998), 1–17.

[6] Bessaga C., Pelczynski A., *On bases and unconditional convergence of series in Banach spaces*, Studia Math. **17** (1958), 151–174.

[7] Bombal F., *On V^* sets and Pelczynski's property V^**, Glasgow Math. J. **32** (1990), 109–120.

[8] Bombal F., Villanueva I., *On the Dunford-Pettis property of the tensor product of $C(K)$ spaces*, Proc. Amer. Math. Soc. **129** (2001), 1359–1363.

[9] Bourgain J., *New Classes of $\mathcal{L}_p$-spaces*, Lecture Notes in Math., 889, Springer, Berlin-New York, 1981.

[10] Bourgain J., *New Banach space properties of the disc algebra and H^∞*, Acta Math. **152** (1984), 1–2, 148.

[11] Bourgain J., *H^∞ is a Grothendieck space*, Studia Math. **75** (1983), 193–216.

[12] Delbaen F., *Weakly compact operators on the disk algebra*, J. Algebra **45** (1977), 284–294.

[13] Diestel J., *Sequences and Series in Banach Spaces*, Graduate Texts in Mathematics, 92, Springer, Berlin, 1984.

[14] Diestel J., *A survey of results related to the Dunford-Pettis property*, Contemporary Math. **2** (1980), 15–60.

[15] Diestel J., Uhl J.J., Jr., *Vector Measures*, Math. Surveys 15, American Mathematical Society, Providence, RI, 1977.

[16] Emmanuele G., *A remark on the containment of c_0 in spaces of compact operators*, Math. Proc. Cambr. Philos. Soc. **111** (1992), 331–335.

[17] Emmanuele G., *The reciprocal Dunford-Pettis property and projective tensor products*, Math. Proc. Cambridge Philos. Soc. **109** (1991), 161–166.

[18] Emmanuele G., *On the containment of* c_0 *in spaces of compact operators*, Bull. Sci. Math. **115** (1991), 177–184.

[19] Emmanuele G., *Dominated operators on* $C[0,1]$ *and the* (CRP), Collect. Math. **41** (1990), 21–25.

[20] Emmanuele G., *A dual characterization of Banach spaces not containing* ℓ^1, Bull. Polish Acad. Sci. Math. **34** (1986), 155–160.

[21] Emmanuele G., John K., *Uncomplementability of spaces of compact operators in larger spaces of operators*, Czechoslovak Math. J. **47** (1997), 19–31.

[22] Ghenciu I., Lewis P., *The embeddability of* c_0 *in spaces of operators*, Bull. Polish. Acad. Sci. Math. **56** (2008), 239–256.

[23] Ghenciu I., Lewis P., *The Dunford-Pettis property, the Gelfand-Phillips property and* (L)*-sets"*, Colloq. Math. 106 (2006), 311–324.

[24] Ghenciu I., Lewis P., *Almost weakly compact operators*, Bull. Polish. Acad. Sci. Math. **54** (2006), 237–256.

[25] Grothendieck A., *Sur les applications linéaires faiblement compactes d'espaces du type* $C(K)$, Canad. J. Math. **5** (1953), 129–173.

[26] Kalton N., *Spaces of compact operators*, Math. Ann. **208** (1974), 267–278.

[27] Leavelle T., *Dissertation*, UNT.

[28] Lindenstrauss J., Tzafriri L., *Classical Banach Spaces II*, Springer, Berlin-New York, 1979.

[29] Pełczyński A., *On Banach spaces containing* $L^1(\mu)$, Studia Math. **30** (1968), 231–246.

[30] Pełczyński A., *Banach spaces on which every unconditionally converging operator is weakly compact*, Bull. Acad. Polon. Sci. Math. Astronom. Phys. **10** (1962), 641–648.

[31] Pełczyński A., Semadeni Z., *Spaces of continuous functions (III)*, Studia Math. **18** (1959), 211–222.

[32] Pisier G., *Factorization of Linear Operators and Geometry of Banach Spaces*, CBMS Regional Conf. Series in Math. 60, American Mathematical Society, Providence, RI, 1986.

[33] Pitt H.R., *A note on bilinear forms*, J. London Math. Soc. **11** (1936), 174–180.

[34] Ryan R.A., *Introduction to Tensor Products of Banach Spaces*, Springer, London, 2002.

[35] Saab E., Saab P., *On stability problems of some properties in Banach spaces*, in: K. Sarosz (ed.), Function Spaces, Lecture Notes Pure Appl. Math., 136, Dekker, New York 1992, 367–394.

[36] Tzafriri L., *Reflexivity in Banach lattices and their subspaces*, J. Functional Analysis **10** (1972), 1–18.

Mathematics Department University of Wisconsin-River Falls Wisconsin, 54022, USA

E-mail: ioana.ghenciu@uwrf.edu

(*Received* June 7, 2014, *revised* March 4, 2015)