Reflecting character and pseudocharacter

Lucia R. Junqueira, Alberto M.E. Levi

Abstract. We say that a cardinal function ϕ reflects an infinite cardinal κ, if given a topological space X with $\phi(X) \geq \kappa$, there exists $Y \in [X]^{\leq\kappa}$ with $\phi(Y) \geq \kappa$. We investigate some problems, discussed by Hodel and Vaughan in *Reflection theorems for cardinal functions*, Topology Appl. **100** (2000), 47–66, and Juhász in *Cardinal functions and reflection*, Topology Atlas Preprint no. 445, 2000, related to the reflection for the cardinal functions character and pseudocharacter. Among other results, we present some new equivalences with CH.

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1. Introduction

The purpose of this paper is to investigate some problems about reflection properties for the cardinal functions χ (character) and ψ (pseudocharacter).

For concepts and notation, our main references are [5] and [7] in general topology and cardinal functions, and [9] in set theory.

The first systematic study of reflection theorems for cardinal functions was done by Hodel and Vaughan in [8], after some studies done by Tkačenko in [17], Juhász in [10], and the result of Hajnal and Juhász in [6] that the weight reflects all infinite cardinals, among others. Hodel and Vaughan give in [8] the following definition:

Definition 1.1 ([8]). Let ϕ be a cardinal function, κ be an infinite cardinal, and $\mathcal{C}$ be a class of topological spaces. We say that ϕ *reflects* κ *for the class* $\mathcal{C}$, if given a space X of $\mathcal{C}$ with $\phi(X) \geq \kappa$, there exists $Y \subseteq X$ with $|Y| \leq \kappa$ and $\phi(Y) \geq \kappa$.

When the class $\mathcal{C}$ is the class of all topological spaces, we simply say that ϕ reflects κ.

The most important result in the literature about reflection for the cardinal function χ is:

Theorem 1.2 ([8, Corollary 3.4]). *χ reflects all infinite cardinals for the class of compact Hausdorff spaces.*

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By the above theorem, if X is compact T_2, and $\chi(Y) = \aleph_0$ for every $Y \in [X]^{\leq \aleph_1}$, then $\psi(X) = \aleph_0$. The next result, stated in [11], extends this for Lindelöf spaces.

Theorem 1.3 ([11, Theorems 1 and 3]). *If X is Lindelöf T_2, and $\chi(Y) = \aleph_0$ for every $Y \in [X]^{\leq \aleph_1}$, then $\psi(X) = \aleph_0$ and $|X| \leq \mathfrak{c}$.*

This theorem is also an extension of the Arhangel'skii theorem in the countable case, which says that $|X| \leq \mathfrak{c}$ when X is Lindelöf T_2 and $\chi(X) = \aleph_0$.

There are two ways to extend this to a true reflection result: replacing "$\psi(X) = \aleph_0$" by "$\chi(X) = \aleph_0$", or replacing "$\chi(Y) = \aleph_0$" by "$\psi(Y) = \aleph_0$". The first way cannot be done in ZFC, because of this surprising result:

Theorem 1.4 ([11, Theorem 2]). CH *is equivalent to the statement: "if X is Lindelöf T_2, and $\chi(Y) = \aleph_0$ for every $Y \in [X]^{\leq \aleph_1}$, then $\chi(X) = \aleph_0$".*

The other way leads us to an open problem.

Problem 1.5 ([11]). *Does X Lindelöf (or even compact) T_2 and $\psi(Y) = \aleph_0$ for every $Y \in [X]^{\leq \aleph_1}$ imply that $\psi(X) = \aleph_0$?*

A partial answer is:

Theorem 1.6 ([11, Theorem 4]). *If X is compact T_2, and $\psi(Y) = \aleph_0$ for every $Y \in [X]^{\leq \mathfrak{c}}$, then $\psi(X) = \aleph_0$.*

From this follows that, under CH, ψ reflects $\aleph_1$ for the class of compact Hausdorff spaces. This is a particular case of this more general result:

Theorem 1.7 ([8]). ($2^\kappa = \kappa^+$) *The cardinal function ψ reflects κ^+ for the class of compact Hausdorff spaces.*

The following problem generalizes Problem 1.5:

Problem 1.8 ([8]). *In* ZFC, *does ψ reflect all infinite cardinals for the class of compact Hausdorff spaces?*

In Section 2 of this paper, we present some reflection results for pseudocharacter, in broader classes than the class of compact Hausdorff spaces. One of this results is used in Section 3 to obtain more equivalences with CH, extending Theorem 1.4. In Section 4 we use the theory of character and convergence spectra developed in [13] to make several remarks that give some partial answers to Problem 1.8.

We conclude this introduction presenting some definitions and results that are used throughout this paper.

Definition 1.9 ([14]). Let $\langle X, \mathcal{T} \rangle$ be a topological space and let M be an elementary submodel of some "large enough" $H(\theta)$. X_M is the new topological space $X \cap M$ with the topology generated by

$$\mathcal{T} \restriction M = \{U \cap M : U \in \mathcal{T} \cap M\}.$$

Definition 1.10 ([7])**.** Given a space X, $L(X)$ is the smallest infinite cardinal κ such that every open cover of X has a subcover of cardinality $\leq \kappa$.

Definition 1.11 ([4])**.** Given a space X, $ll(X)$ is the smallest infinite cardinal κ such that every open cover of X, linearly ordered by inclusion, has a subcover of cardinality $\leq \kappa$.

It is immediate that $ll(X) \leq L(X)$ for every space X. A space X is a Lindelöf space iff $L(X) = \aleph_0$, and it is a linearly Lindelöf space iff $ll(X) = \aleph_0$.

Definition 1.12 ([16])**.** If λ and μ are infinite cardinals, then we say that X is a $[\lambda, \mu]$-*compact space* if every open cover C of X with $|C| \leq \mu$ has a subcover S with $|S| < \lambda$.

The next definition is due to Ofelia T. Alas.

Definition 1.13. For a regular cardinal κ, we say that X is a $L(\kappa)$-*space* if every open cover of X of cardinality κ has a subcover of smaller cardinality.

Definition 1.14 ([10])**.** Let $\langle X, \mathcal{T} \rangle$ be a T_2 space. For each $p \in X$ define

$$\psi_c(p, X) = \min \left\{ |\mathcal{V}| : \mathcal{V} \subseteq \mathcal{T}, p \in \bigcap \mathcal{V}, \{p\} = \bigcap \left\{ \overline{V} : V \in \mathcal{V} \right\} \right\} + \omega.$$

Then define

$$\psi_c(X) = \sup \left\{ \psi_c(p, X) : p \in X \right\}.$$

It is easy to see that $\psi_c(p, X) = \psi(p, X)$ when $p \in X$ and X is a regular space.

Lemma 1.15 ([2, Proposition 2.2(i)])**.** *If X is a T_2 space, then for every $S \subseteq X$ we have*

$$\left|\overline{S}\right| \leq 2^{|S| \psi_c(\overline{S})}.$$

Definition 1.16 ([7])**.** Given a space X,

$$F(X) = \sup \{\kappa : X \text{ has free sequence of length } \kappa\} + \omega.$$

A sequence $\{x_\alpha : \alpha < \kappa\}$ in X is a *free sequence of length* κ if for all $\beta < \kappa$,

$$\overline{\{x_\alpha : \alpha < \beta\}} \cap \overline{\{x_\alpha : \alpha \geq \beta\}} = \emptyset.$$

Proposition 1.17 (see [7])**.** *If X is compact T_2 then $F(X) = t(X)$.*

2. Reflection in broader classes

We first note that the answer to the Problem 1.5 is negative if we replace "compact" by "countably compact".

Example 2.1. Let

$$X = \{\alpha \in \omega_2 : \mathrm{cf}(\alpha) = \omega\} \cup \omega_2$$

with the order topology. Then X is countably compact, every subspace of it of cardinality $\leq \aleph_1$ has countable character (and therefore countable pseudocharacter) but $\psi(X) = \aleph_2$.

The next result extends Theorem 1.6.

Theorem 2.2. *If X is T_2 and of pointwise countable type, and $\psi(Y) = \aleph_0$ for every $Y \in [X]^{\leq \mathfrak{c}}$, then $\psi(X) = \aleph_0$.*

Proof: It is sufficient to prove the result for T_2 compact spaces (Theorem 1.6), since, for any $x \in X$, there is a compact K such that $x \in K \subseteq X$ and $\chi(K, X) = \aleph_0$; then we have

$$\psi(x, X) = \chi(x, X) \leq \chi(x, K)\chi(K, X).$$

Now, assuming that X is compact T_2, choose any $Y \in [X]^{\leq \aleph_1}$, and let M be a countably closed elementary submodel with $X \in M$, $Y \subseteq M$ and $|M| \leq \mathfrak{c}$. It is easy to see that $X \cap M$ is a countably compact space; this implies that

$$\chi(Y) \leq \chi(X \cap M) = \aleph_0,$$

since $\psi(X \cap M) = \aleph_0$. Hence, by Theorem 1.2 we have $\psi(X) = \chi(X) = \aleph_0$. $\square$

We present now some results for pseudocharacter in classes broader than Lindelöf spaces. The next result is due to Ofelia T. Alas.

Lemma 2.3. *Let κ be a regular cardinal, X be a T_2 $L(\kappa)$-space, and $p \in X$ be such that $\psi_c(p, X) = \kappa$. Then, there is $Y \in [X]^\kappa$ such that $p \in Y$ and $\psi(p, Y) = \kappa$.*

Proof: Let $\{U_\alpha : \alpha < \kappa\}$ be a family of open neighborhoods of p in X such that

$$\bigcap_{\alpha<\kappa} \overline{U_\alpha} = \{p\}\,.$$

For every $\alpha < \kappa$ define $P_\alpha = \bigcap \left\{\overline{U_\beta} : \beta \leq \alpha\right\}$.

Then, define

$$I = \left\{ \alpha < \kappa : \bigcap_{\beta<\alpha} P_\beta \nsubseteq P_\alpha \right\},$$

and for each $\alpha \in I$, choose a p_α in $(\bigcap_{\beta<\alpha} P_\beta)\backslash P_\alpha$.

Define $Y = \{p\} \cup \{p_\alpha : \alpha \in I\}$. It is easy to see that $|Y| = \kappa$, so we just have to show that $\psi(p, Y) = \kappa$.

Suppose $\lambda := \psi(p, Y) < \kappa$. Then there is a family $\{W_\gamma : \gamma < \lambda\}$ of open neighborhoods of p in X such that

$$\bigcap_{\gamma<\lambda} W_\gamma \cap Y = \{p\}\,.$$

For each $\gamma < \lambda$, $\bigcap_{\alpha<\kappa} \overline{U_\alpha} \backslash W_\gamma = \emptyset$, then, since X is a $L(\kappa)$-space, there is some $\theta_\gamma < \kappa$ such that $\bigcap_{\alpha<\theta_\gamma} \overline{U_\alpha} \subseteq W_\gamma$.

If $\theta = \sup\{\theta_\gamma : \gamma < \lambda\}$, then we have $\theta < \kappa$ and

$$\bigcap_{\alpha<\theta} \overline{U_\alpha} \subseteq \bigcap_{\gamma<\lambda} W_\gamma,$$

hence

$$Y \cap \bigcap_{\alpha<\theta} \overline{U_\alpha} \subseteq Y \cap \bigcap_{\gamma<\lambda} W_\gamma = \{p\}\,.$$

For $\theta_* \in I$ with $\theta < \theta_* < \kappa$,

$$p_{\theta_*} \in Y \cap \bigcap_{\beta<\theta_*} P_\beta \subseteq Y \cap \bigcap_{\alpha<\theta} \overline{U_\alpha} = \{p\},$$

but $p_{\theta_*} \notin P_{\theta_*}$, a contradiction. □

Corollary 2.4. *If X is a regular $L(\aleph_1)$-space such that $\psi(Y) = \aleph_0$ for every $Y \in [X]^{\leq\aleph_1}$, then $\psi(X) \neq \aleph_1$.*

The next theorem is a reflection result for ψ.

Theorem 2.5. *Let X be a regular $L(\kappa^+)$-space or a T_2 $[\kappa^+, 2^\kappa]$-compact space such that:*

(1) *for every $Y \in [X]^{\leq\kappa}$, $\chi(Y) \leq \kappa^+$;*
(2) *for every $Y \in [X]^{\leq\kappa^+}$, $t(Y) \leq \kappa$ and $\psi(Y) \leq \kappa$.*

Then, $\psi_c(X) \leq \kappa$.

Proof: By (2), we have

$$\overline{Y} = \bigcup \left\{\overline{S} : S \in [Y]^{\leq\kappa}\right\}$$

for every $Y \in [X]^{\leq\kappa^+}$.

Note that $\psi_c(\overline{S}) \leq \kappa$ for every $S \in [X]^{\leq\kappa}$, since, for any $y \in \overline{S}$, we have $\psi_c(y, \overline{S}) \neq \kappa^+$ by (2) and Lemma 2.3, and by (1),

$$\psi_c\left(y, \overline{S}\right) = \psi_c\left(y, \overline{S \cup \{y\}}\right) \leq \chi\left(y, S \cup \{y\}\right) \leq \chi\left(S \cup \{y\}\right) \leq \kappa^+.$$

For every $S \in [X]^{\leq\kappa}$ and every $p \in S$, we will show that there is a family $\mathcal{A}_{p,S}$ of open neighborhoods of p in X such that $|\mathcal{A}_{p,S}| \leq \kappa$ and

$$\bigcap \left\{\overline{\Omega} \cap \overline{S} : \Omega \in \mathcal{A}_{p,S}\right\} = \{p\}\,.$$

Since $\psi(p, \overline{S}) \leq \kappa$, there is a family $\{O_\lambda : \lambda < \kappa\}$ of open neighborhoods of p in X such that $\bigcap\{O_\lambda \cap \overline{S} : \lambda < \kappa\} = \{p\}$. If X is regular, then for each O_λ we can find some open Ω with $p \in \Omega$ and $\overline{\Omega} \subseteq O_\lambda$, and we are done. Now suppose that

X is $[\kappa^+, 2^\kappa]$-compact. For each $\lambda < \kappa$, $\overline{S}\backslash O_\lambda$ is $[\kappa^+, 2^\kappa]$-compact, hence the same occurs with

$$\overline{S}\backslash\{p\} = \bigcup_{\lambda<\kappa} \overline{S}\backslash O_\lambda.$$

By Lemma 1.15, $|\overline{S}\backslash\{p\}| \leq 2^\kappa$, hence if we define, for each $z \in \overline{S}\backslash\{p\}$, an open A_z with $p \in A_z$ and $z \notin \overline{A_z}$, then there is some $Z \in [\overline{S}\backslash\{p\}]^{\leq\kappa}$ such that

$$\bigcap\left\{\overline{A_z} \cap \overline{S} : z \in Z\right\} = \{p\}.$$

Now, suppose $\psi_c(p, X) \geq \kappa^+$ for some $p \in X$. For each $\alpha < \kappa^+$, define $S_\alpha = \{p\} \cup \{p_\beta : \beta < \alpha\}$ and

$$\mathcal{B}_\alpha = \mathcal{A}_{p,S_\alpha} \cup \bigcup_{\beta<\alpha} \mathcal{B}_\beta,$$

and choose some $p_\alpha \in \bigcap\{\overline{\Omega} : \Omega \in \mathcal{B}_\alpha\}\backslash\overline{S_\alpha}$.

Then, define $Y = \bigcup_{\alpha<\kappa^+} S_\alpha$ and $\mathcal{B} = \bigcup_{\alpha<\kappa^+} \mathcal{B}_\alpha$. It is easy to see that $|Y| = \kappa^+$, $|\mathcal{B}| \leq \kappa^+$ and $\bigcap\{\overline{\Omega} \cap \overline{Y} : \Omega \in \mathcal{B}\} = \{p\}$, which implies $\psi_c(p, \overline{Y}) \leq \kappa^+$, since $\overline{\Omega \cap \overline{Y}} \subseteq \overline{\Omega} \cap \overline{Y}$. By (2) and Lemma 2.3, we must have

$$\psi\left(p, \overline{Y}\right) \leq \psi_c\left(p, \overline{Y}\right) \leq \kappa.$$

Let $\{W_\alpha : \alpha < \kappa\}$ be a family of open neighborhoods of p in X such that $\bigcap\{W_\alpha \cap \overline{Y} : \alpha < \kappa\} = \{p\}$. For each $\alpha < \kappa$, we have

$$\bigcap\left\{\left(\overline{\Omega} \cap \overline{Y}\right)\backslash W_\alpha : \Omega \in \mathcal{B}\right\} = \emptyset.$$

Since X is a $L(\kappa^+)$-space and κ^+ is regular, there is some $\delta \in \kappa^+$ such that

$$\{p\} = \bigcap\left\{W_\alpha \cap \overline{Y} : \alpha < \kappa\right\} \supseteq \bigcap\left\{\overline{\Omega} \cap \overline{Y} : \Omega \in \mathcal{B}_\delta\right\} \supseteq \{p_\delta\},$$

which is a contradiction, since $p \neq p_\delta$. $\square$

When X is a regular $L(\kappa^+)$-space in the above theorem, there is another proof, based on some proofs in [3], using elementary submodels and the following lemma, which may be of independent interest.

Lemma 2.6. *If M is a κ-covering elementary submodel, $\langle X, \mathcal{T}\rangle \in M$, $\kappa \cup \{\kappa\} \subseteq M$ and $\psi(X_M) \leq \kappa$, then $\psi(X) \leq \kappa$.*

Proof: The proof is similar to the one in [15, Theorem 3.5]. Fix $x \in X \cap M$. Since $\psi(X_M) \leq \kappa$, there is $\mathcal{B} \in [\mathcal{T} \cap M]^{\leq\kappa}$ such that

$$\bigcap\{V \cap M : V \in \mathcal{B}\} = \{x\}.$$

By κ-covering, there is $\mathcal{B}' \in M$ such that $|\mathcal{B}'| \leq \kappa$, $\mathcal{B} \subseteq \mathcal{B}'$ and we can suppose $\mathcal{B}' \subseteq \{V \in \mathcal{T} : x \in V\}$. Since $\mathcal{B}' \in M$, $\mathcal{B}' \subseteq \mathcal{T}$, $\bigcap\{V \cap M : V \in \mathcal{B}'\} = \{x\}$, and $|\mathcal{B}'| \leq \kappa$, by elementarity we have that $M \models \psi(x, X) \leq \kappa$.

Since this is true for every $x \in X \cap M$, we have

$$M \models \forall x \in X\, (\psi(x, X) \leq \kappa)\,.$$

Thus we have our result by elementarity. $\square$

Corollary 2.7. *If X is such that $\psi(Y) = \aleph_0$ for every $Y \in [X]^{\leq \aleph_1}$, and there is an ω-covering elementary submodel M of cardinality $\aleph_1$ such that $\langle X, \mathcal{T}\rangle \in M$ and X_M is a subspace of X, then $\psi(X) = \aleph_0$.*

3. Some equivalences with CH

We can ask whether it is possible to replace, in Theorem 1.4, "Lindelöf" by "linearly Lindelöf". In this section, we answer this question. Our first result here extends Theorem 1.3.

Theorem 3.1. *If X is a T_2 $[\kappa^+, 2^\kappa]$-compact space, with $\chi(Y) \leq \kappa$ for every $Y \in [X]^{\leq \kappa^+}$, then $\psi_c(X) \leq \kappa$, $L(X) \leq \kappa$ and $|X| \leq 2^\kappa$.*

Proof: First, we have

$$\overline{Y} = \bigcup \{\overline{S} : S \in [Y]^{\leq \kappa}\}$$

for every $Y \in [X]^{\leq \kappa^+}$, since $t(Y) \leq \chi(Y) \leq \kappa$.

We have $\psi_c(X) \leq \kappa$ by Theorem 2.5. Using this and Lemma 1.15, we have that, for any $Y \in [X]^{\leq \kappa}$,

$$\left|\overline{Y}\right| \leq 2^{|Y|\psi_c\left(\overline{Y}\right)} \leq 2^{|Y|\psi_c(X)} \leq 2^\kappa.$$

Now we show that $F(X) \leq \kappa$. Suppose that $(x_\xi)_{\xi<\kappa^+}$ is a free sequence in X. Since X is a $L(\kappa^+)$-space, we can choose some

$$x \in \bigcap_{\eta<\kappa^+} \overline{\{x_\xi : \eta \leq \xi < \kappa^+\}}.$$

Since $x \in \overline{\{x_\xi : \xi < \kappa^+\}}$, then $x \in \overline{\{x_\xi : \xi < \theta\}}$ for some $\theta < \kappa^+$; but $x \in \overline{\{x_\xi : \theta \leq \xi < \kappa^+\}}$, a contradiction.

Let M be a κ-closed elementary submodel, with $X \in M$, $2^\kappa \subseteq M$ and $|M| = 2^\kappa$. Defining $A = X \cap M$, we will show that $L(A) \leq \kappa$ and $A = X$. First, note that $\overline{B} \subseteq A$ for every $B \in [A]^{\leq \kappa}$. In fact, $B \in M$ since M is κ-closed, hence $\overline{B} \in M$, which implies $\overline{B} \subseteq A$.

Now, we show that $L(A) \leq \kappa$. Suppose not. Then there is open cover R of A, with no subcover of cardinality $\leq \kappa$. We will build two sequences $(x_\eta)_{\eta<\kappa^+}$ and

$(R_\eta)_{\eta<\kappa^+}$ such that, for each $\eta<\kappa^+$, $x_\eta \in A$, $R_\eta \in [R]^{\leq\kappa}$ and $\overline{\{x_\xi : \xi < \eta\}} \subseteq \bigcup R_\eta$. For each $\eta < \kappa^+$, proceed as follows: $\overline{\{x_\xi : \xi < \eta\}} \subseteq A$ since $|\eta| \leq \kappa$, and

$$L\left(\overline{\{x_\xi : \xi < \eta\}}\right) \leq \kappa$$

since $|\overline{\{x_\xi : \xi < \eta\}}| \leq 2^\kappa$ and $\overline{\{x_\xi : \xi < \eta\}}$ is $[\kappa^+, 2^\kappa]$-compact; then choose some $R_\eta \in [R]^{\leq\kappa}$ such that $\bigcup_{\xi<\eta} R_\xi \subseteq R_\eta$ and $\overline{\{x_\xi : \xi < \eta\}} \subseteq \bigcup R_\eta$, and some $x \in A \backslash \bigcup R_\eta$.

Now, given any $\eta < \kappa^+$ and any $x \in \overline{\{x_\xi : \xi < \eta\}}$, we have $x \notin \overline{\{x_\xi : \xi \geq \eta\}}$, since

$$\{x_\xi : \xi \geq \eta\} \cap \left(\bigcup R_\eta\right) = \emptyset.$$

Hence, $(x_\eta)_{\eta<\kappa^+}$ is a free sequence, a contradiction.

Finally, suppose that there is some $y \in X \backslash A$. For each $x \in A$, $\psi(x, X) \leq \kappa$, then let $\mathcal{B}_x$ be a family of open neighborhoods of x in X such that $\mathcal{B}_x \in M$, $|\mathcal{B}_x| \leq \kappa$ and $\bigcap \mathcal{B}_x = \{x\}$, which implies $\mathcal{B}_x \subseteq M$. For each $x \in A$, choose some $A_x \in \mathcal{B}_x$ such that $y \notin A_x$. Since $L(A) \leq \kappa$, there is some

$$S \in [\{A_x : x \in A\}]^{\leq\kappa}$$

such that $A \subseteq \bigcup S$; and we have $S \in M$ since $S \subseteq M$ and M is κ-closed. Now, since $X \backslash \bigcup S \neq \emptyset$, by elementarity there is some $z \in M$ such that $z \in X \backslash \bigcup S$, a contradiction. □

Consider the following cardinal function ϕ: $\phi(X)$ is the smallest infinite cardinal κ such that X is $[\kappa^+, 2^\kappa]$-compact and $\chi(Y) \leq \kappa$ for every $Y \in [X]^{\leq\kappa^+}$. The above theorem says that $|X| \leq 2^{\phi(X)}$, hence implies the Arhangel'skii theorem $|X| \leq 2^{L(X)\chi(X)}$, since $\phi(X) \leq L(X)\chi(X)$.

Proposition 3.2. $(2^\kappa > \kappa^+)$ *There is a T_2 space X, with $L(X) = \kappa$, where $\chi(Y) \leq \kappa$ for every $Y \in [X]^{\leq\kappa^+}$, but $\chi(X) > \kappa^+$.*

Proof: Choose any point p in the Cantor cube 2^κ, and define

$$X = ((2^\kappa \backslash \{p\}) \times \{0\}) \cup (2^\kappa \times \{1\}),$$

with the following topology basis:

- the points in $(2^\kappa \backslash \{p\}) \times \{1\}$ are isolated;
- if A is an open neighborhood of $q \neq p$ in 2^κ, then

  $$((A \backslash \{p\}) \times \{0\}) \cup ((A \backslash \{p, q\}) \times \{1\})$$

  is an open neighborhood of $(q, 0)$ in X;
- if A is an open neighborhood of p in 2^κ, and

  $$Z \in [(2^\kappa \backslash \{p\}) \times \{1\}]^{\leq\kappa^+},$$

then $(A \times \{1\})\backslash Z$ is an open neighborhood of $(p, 1)$ in X.

It is easy to show that $L(2^\kappa\backslash\{p\}) = \kappa$. Note that the topology of $(2^\kappa\backslash\{p\})\times\{0\}$ as subspace of X is the same topology as subspace of $2^\kappa \times \{0\}$. Then, $L(X) = \kappa$, since if R is a family of basic open sets of X, with $|R| \leq \kappa$ and $(2^\kappa\backslash\{p\}) \times \{0\} \subseteq \bigcup R$, then

$$\left|(2^\kappa \times \{1\}) \setminus \bigcup R\right| \leq \kappa.$$

If $Y \in [X]^{\leq\kappa^+}$ then $\chi(Y) \leq \kappa$, since $\chi(2^\kappa) = \kappa$ and all points in $Y \cap (2^\kappa \times \{1\})$ are isolated.

Finally, suppose that $\chi(X) \leq \kappa^+$. Let $\{A_\alpha : \alpha < \kappa^+\}$ be a local base for $(p, 1)$ in X. For each $\alpha < \kappa^+$, choose some $x_\alpha \in A_\alpha\backslash\{(p, 1)\}$ (note that $|A_\alpha| = 2^\kappa > \kappa^+$). Now, for every $\alpha < \kappa^+$, we have

$$A_\alpha \nsubseteq (2^\kappa \times \{1\}) \setminus \left\{x_\alpha : \alpha < \kappa^+\right\},$$

a contradiction. □

Theorem 3.3. *For every infinite cardinal κ, the following statements are equivalent in* ZFC*:*

(1) $2^\kappa = \kappa^+$;
(2) *χ reflects κ^+ for the class of T_2 $L(\kappa^+)$-spaces;*
(3) *χ reflects κ^+ for the class $\{X : X$ is T_2 and $ll(X) \leq \kappa\}$;*
(4) *χ reflects κ^+ for the class $\{X : X$ is T_2 and $L(X) \leq \kappa\}$.*

Proof: (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are immediate, and (4) $\Rightarrow$ (1) follows from Proposition 3.2. (1) $\Rightarrow$ (2) follows from Theorem 3.1, since every $L(\kappa^+)$-space is $[\kappa^+, 2^\kappa]$-compact under $2^\kappa = \kappa^+$. □

Corollary 3.4. *The following statements are equivalent in* ZFC*:*

(1) CH;
(2) *χ reflects $\aleph_1$ for the class of T_2 $L(\aleph_1)$-spaces;*
(3) *χ reflects $\aleph_1$ for the class of T_2 linearly Lindelöf spaces;*
(4) *χ reflects $\aleph_1$ for the class of T_2 Lindelöf spaces.*

4. Reflection of pseudocharacter in compact spaces

In this section, we will present some partial answers to the Problem 1.8. As far as we know, the only partial answer in the literature is:

Theorem 4.1 ([1, Theorem 3.16])**.** *ψ reflects all infinite cardinals for the class of dyadic spaces.*

To investigate this issue, we will use some concepts defined in [13].

Definition 4.2 ([13])**.** A transfinite sequence $\langle x_\alpha : \alpha < \kappa\rangle$ is said to converge to a point x in the topological space X (this is denoted by $x_\alpha \to x$) if for every neighbourhood U of x there is an index $\beta < \kappa$ such that $x_\alpha \in U$ whenever $\beta \leq \alpha < \kappa$.

Definition 4.3 ([13]). An infinite subset A of X converges to the point x ($A \to x$) if for every neighbourhood U of x we have $|A \backslash U| < |A|$.

Juhász and Weiss noted in [13]:

- if X is a compactum (an infinite compact Hausdorff space), then $A \to x$ is equivalent to x being the unique complete accumulation point of A;
- if the one-to-one sequence $\langle x_\alpha : \alpha < \kappa \rangle$ converges to x then so does its range $\{x_\alpha : \alpha < \kappa\}$ as a set. Conversely, if $|A| = \kappa$ is a regular cardinal and $A \to x$ then every sequence of order type κ that enumerates A in a one-to-one manner converges to x as well.

Definition 4.4 ([13]). For a non-isolated point p of the space X we let

$$\chi S\,(p, X) = \{\chi(p, Y) : p \text{ is non-isolated in } Y \subseteq X\}$$

and we call $\chi S\,(p, X)$ the *character spectrum of p in X*. Moreover,

$$\chi S\,(X) = \bigcup \{\chi S(x, X) : x \in X \text{ non-isolated}\}$$

is the *character spectrum of X*.

Definition 4.5 ([13]). Fix a topological space X and a point $p \in X$. Then

$$cS\,(p, X) = \{|A| : A \subseteq X \text{ and } A \to p\}$$

is the *convergence spectrum of p in X*. Moreover,

$$cS\,(X) = \bigcup \{cS\,(x, X) : x \in X\}$$

is the *convergence spectrum of X*.

Definition 4.6 ([13]). Fix a topological space X and a point $p \in X$. Then

$$dcS\,(p, X) = \{|D| : D \subseteq X \text{ is discrete and } D \to p\}$$

is the *discrete convergence spectrum of p in X*. Moreover,

$$dcS\,(X) = \bigcup \{dcS\,(x, X) : x \in X\}$$

is the *discrete convergence spectrum of X*.

It is immediate that $dcS(X) \subseteq cS(X)$. Arguments made in [13] show that, if X is a compact Hausdorff space, and $\kappa \in \chi S(X)$, then $\{\kappa, \mathrm{cf}(\kappa)\} \subseteq cS(X)$.

Theorem 4.7. *If κ is an infinite regular cardinal, then ψ reflects κ for the class $\{X : \kappa \in cS(X)\}$.*

Proof: If $\kappa \in cS(X)$, then $\kappa \in cS(p, X)$ for some $p \in X$, hence there is some $A \in [X]^\kappa$ such that $A \to p$. Then, it is easy to see that $\psi(p, A \cup \{p\}) = \kappa$. $\square$

Theorem 4.8 ([13]). *If κ is an infinite cardinal, and X is a compactum with $\chi(X) > 2^\kappa$, then $\kappa^+ \in dcS(X)$.*

Definition 4.9 (see [13]). Given a space X, $\widehat{F}(X)$ is the smallest cardinal κ for which there is no free sequence of size κ in X.

Theorem 4.10 ([13]). *If ϱ is an infinite cardinal with $\varrho = \mathrm{cf}(\varrho) > \omega$, and X is a compactum with $\widehat{F}(X) > \varrho$, then $\varrho \in \chi S(X)$.*

Theorem 4.11 ([13]). *Assume that X is a T_3 space and ϱ, μ are cardinals such that $\widehat{F}(X) \leq \varrho \leq \mathrm{cf}(\mu)$, and moreover $p \in X$ with $\psi(p, X) \geq \mu$. Then either*

(1) *there is a discrete set $D \in [X]^{<\varrho}$ with $p \in \overline{D}$ and $\psi(p, \overline{D}) \geq \mu$, or*
(2) *there is a discrete set $D \in [X]^{\varrho}$ such that $D \to p$.*

Theorem 4.12. *Let κ be an infinite cardinal, and X be a compact Hausdorff space. If ψ does not reflect κ^+ for $\{X\}$, then*

(1) *$\psi(p, X) \neq \kappa^+$ for every $p \in X$;*
(2) *$\kappa^{++} \leq \psi(X) \leq 2^\kappa$;*
(3) *$t(X) \leq \kappa$;*
(4) *$\psi(\overline{D}) \geq \kappa^{++}$ for some discrete $D \in [X]^{\leq\kappa}$.*

Proof: (1) If $\psi(p, X) = \kappa^+$ for some $p \in X$, then we may apply Theorem 4.7, since $\kappa^+ \in \chi S(p, X) \subseteq cS(X)$.

(2) If $\psi(X) = \kappa^+$, then the result is immediate by (1); and if $\psi(X) \leq \kappa$, then the result is immediate by definition. If $\psi(X) > 2^\kappa$, then $\kappa^+ \in dcS(X) \subseteq cS(X)$ by Theorem 4.8.

(3) If $t(X) \geq \kappa^+$ then $\widehat{F}(X) > \kappa^+$, hence $\kappa^+ \in \chi S(X) \subseteq cS(X)$ by Theorem 4.10.

(4) By (3) we have $\widehat{F}(X) \leq \kappa^+$, and by (2) we have $\psi(p, X) \geq \kappa^{++}$ for some $p \in X$. Then we may apply Theorem 4.11, with $\varrho = \kappa^+$ and $\mu = \kappa^{++}$, and Theorem 4.7, to obtain a discrete $D \in [X]^{\leq\kappa}$ with $p \in \overline{D}$ and $\psi(\overline{D}) \geq \kappa^{++}$. $\square$

The above theorem implies Theorem 4.1, since $t(X) = \psi(X)$ when X is a dyadic space.

The item (4) in above theorem shows that if there is some consistent counterexample to the statement "ψ reflects κ^+ for the class of compact Hausdorff spaces", then there is a consistent counterexample X with $d(X) \leq \kappa$.

A specially interesting case is when $\kappa = \aleph_0$, which is Problem 1.5. If X is a consistent counterexample to the above question, then $\aleph_1 \notin cS(X)$. For this X, consider the set

$$\Gamma = (cS(X) \cap REG) \setminus \{\aleph_0\}.$$

If $\Gamma = \emptyset$, then X is what is called in [13] a "K-compactum", and there is an open problem that asks: "Is every K-compactum first countable?". If $\Gamma \neq \emptyset$, then, in the terminology adopted in [13], Γ *omits* $\aleph_1$. The only known example of this is the one-point compactification X of the space constructed in [12], where

$\chi S(X) = cS(X) = \{\aleph_0, \aleph_2\}$. In this case, we do not know if ψ reflects $\aleph_1$ for $\{X\}$.

References

[1] Casarrubias-Segura F., Ramírez-Páramo A., *Reflection theorems for some cardinal functions*, Topology Proc. **31** (2007), 51–65.

[2] Christodoulou S., *Initially κ-compact spaces for large κ*, Comment. Math. Univ. Carolin. **40** (1999), no. 2, 319–325.

[3] Dow A., *An introduction to applications of elementary submodels to topology*, Topology Proc. **13** (1988), 17–72.

[4] Eckertson F.W., *Images of not Lindelöf spaces and their squares*, Topology Appl. **62** (1995), 255–261.

[5] Engelking R., *General Topology*, Heldermann, Berlin, 1989.

[6] Hajnal A., Juhász I., *Having a small weight is determined by the small subspaces*, Proc. Amer. Math. Soc. **79** (1980), 657–658.

[7] Hodel R.E., *Cardinal functions I*, Handbook of Set-theoretic Topology, North-Holland, Amsterdam, 1984, pp. 1–61.

[8] Hodel R.E., Vaughan J.E., *Reflection theorems for cardinal functions*, Topology Appl. **100** (2000), 47–66.

[9] Jech T., *Set Theory*, Springer Monographs in Mathematics, Springer, Berlin, 2003. The third millennium edition, revised and expanded.

[10] Juhász I., *Cardinal functions in topology – ten years later*, Mathematical Centre Tracts, 123, Amsterdam, 1980.

[11] Juhász I., *Cardinal functions and reflection*, Topology Atlas Preprint no. 445, 2000.

[12] Juhász I., Koszmider P., Soukup L., *A first countable, initially ω_1-compact but non-compact space*, Topology Appl. **156** (2009), 1863-1879.

[13] Juhász I., Weiss W.A.R., *On the convergence and character spectra of compact spaces*, Fund. Math. **207** (2010), 179–196.

[14] Junqueira L.R., Tall F.D., *The topology of elementary submodels*, Topology Appl. **82** (1998), 239–266.

[15] Junqueira L.R., *Upwards preservation by elementary submodels*, Topology Proc. **25** (2000), 225–249.

[16] Stephenson R.M., Jr., *Initially κ-compact and related spaces*, Handbook of Set-theoretic Topology, North-Holland, Amsterdam, 1984, pp. 603–632.

[17] Tkačenko M.G., *Chains and cardinals*, Soviet Math. Dokl. **19** (1978), 382–385.

Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa postal 66281, 05314-970 São Paulo, Brasil

E-mail: lucia@ime.usp.br
alberto@ime.usp.br

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