Module-valued functors preserving the covering dimension

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Abstract. We prove a general theorem about preservation of the covering dimension dim by certain covariant functors that implies, among others, the following concrete results.

(i) If G is a pathwise connected separable metric NSS abelian group and X, Y are Tychonoff spaces such that the group-valued function spaces $C_p(X,G)$ and $C_p(Y,G)$ are topologically isomorphic as topological groups, then $\dim X = \dim Y$.
(ii) If free precompact abelian groups of Tychonoff spaces X and Y are topologically isomorphic, then $\dim X = \dim Y$.
(iii) If R is a topological ring with a countable network and the free topological R-modules of Tychonoff spaces X and Y are topologically isomorphic, then $\dim X = \dim Y$.

The classical result of Pestov [*The coincidence of the dimensions dim of l-equivalent spaces*, Soviet Math. Dokl. **26** (1982), no. 2, 380–383] about preservation of the covering dimension by l-equivalence immediately follows from item (i) by taking the topological group of real numbers as G.

Keywords: covering dimension; topological group; function space; topology of pointwise convergence; free topological module; l-equivalence; G-equivalence

Classification: 54H11, 54H13

All topological spaces in this paper are assumed to be Tychonoff. Throughout this paper, by dimension we mean Čech-Lebesgue (covering) dimension dim. By $\mathbb{N}$ we denote the set of all natural numbers, $\mathbb{Z}$ stands for the discrete additive group of integers and $\mathbb{R}$ is the additive group (ring, field) of reals with its usual topology.

1. Introduction

Let X and Y be topological spaces. We denote by $C(X,Y)$ the set of all continuous functions from X to Y. If G is a topological group, then $C_p(X,G)$ denotes the (topological) subgroup $C(X,G)$ of the topological group G^X taken with the subspace topology.

Following [12], we say that spaces X and Y are *G-equivalent* provided that the topological groups $C_p(X,G)$ and $C_p(Y,G)$ are topologically isomorphic.

Let us recall the classical notion of Arhangel'skiĭ. Spaces X and Y are said to be *l-equivalent* if the topological vector spaces $C_p(X,\mathbb{R})$ and $C_p(Y,\mathbb{R})$ (with

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the standard scalar multiplication by real numbers) are topologically isomorphic. Tkachuk noticed in [14] that l-equivalence coincides with $\mathbb{R}$-equivalence.

Based on this observation, the following notion was introduced in [12]. A topological property $\mathscr{T}$ is said to be preserved by G-equivalence within a given class $\mathscr{C}$ of topological spaces if for every pair X, Y of G-equivalent topological spaces such that $X, Y \in \mathscr{C}$, the space X has property $\mathscr{T}$ whenever Y has it. Extending the well-established line of research in the classical C_p-theory of Arhangel'skiĭ, in [12] the authors showed that many important topological properties and cardinal invariants are preserved by G-equivalence for certain classes of topological groups G (that include the real line $\mathbb{R}$).

Our main motivation for writing this manuscript was to present some classes of topological groups G for which the Čech-Lebesgue (covering) dimension dim is preserved by G-equivalence within the class of all Tychonoff spaces. Since this class contains also the additive group $\mathbb{R}$ of reals with its standard topology, our result covers also the famous theorem of Pestov who accomplished the effort of a great number of mathematicians (see [1], [5], [9], [15], [16]) by proving that l-equivalence (that is $\mathbb{R}$-equivalence in our notation) preserves the covering dimension. This result was later generalized by Gulko for u-equivalence (see [4] for details). Very recently M. Krupski generalized this result even for t-equivalence in [7].

The manuscript is structured as follows. In Section 2 a topological version of our main theorem is proved. We present this topological version in order to emphasize that in the background of our main theorem (Theorem 4.2) no algebra is needed. Section 3 collects some technical lemmas needed in the sequel. In Section 4 our main general result (Theorem 4.2) is stated an proved. Roughly speaking, this theorem says that if there exists a covariant functor F from some subcategory of the category of Tychonoff spaces into a certain category of "topologized" R-modules, and if certain technical conditions on F are satisfied, then spaces with the same image under F must have the same covering dimension. In order to prove this theorem, we further develop the technique of Pestov from [10].

The rest of the paper is devoted to applications of Theorem 4.2.

A typical example of a functor satisfying the conditions of Theorem 4.2 is a functor that assigns to each Tychonoff space its free topological module in certain class of topological modules. In Section 5 we recall the definition of the free topological module over a topological space in a given class $\mathscr{M}$ of topological R-modules that is closed under taking arbitrary products and R-submodules, and we prepare the ground for application of our main theorem in Section 6. There we prove that if the class $\mathscr{M}$ satisfies that the unit interval is $\mathscr{M}$-Hausdorff (see Definition 5.4(i)) and for every space of countable weight its free topological R-module in $\mathscr{M}$ has countable network, then any two Tychonoff spaces having topologically isomorphic their free R-modules in $\mathscr{M}$ must have the same dimension (Theorem 6.1). In particular, if R is a topological ring with countable network and $\mathscr{M}$ is a class of topological R-modules such that the closed unit interval is $\mathscr{M}$-Hausdorff, then every two Tychonoff spaces having topologically R-isomorphic their free topological modules in $\mathscr{M}$ must have the same dimension (Corollary 6.2). Replacing general ring R with the discrete ring $\mathbb{Z}$ we obtain Corollary 6.3 in which free topological modules in a given class of topological R-modules are replaced by free abelian topological groups in a given class $\mathscr{G}$ of abelian topological groups. In particular, we prove that if two Tychonoff spaces have topologically isomorphic their free precompact abelian groups, then their dimensions must coincide; Corollary 6.4.

For a space X and a topological group G we associate in Section 8 the group $C_p(X, G)$ with certain topological module and prove that under some simple assumption this module is a free topological module over X (Theorem 7.10). In Section 8 we then use this result to derive that if a nontrivial abelian separable metrizable pathwise connected group G is NSS or has self-slender completion, then G-equivalence preserves dimension; Theorem 8.5. In order to extend this result for a wider class of topological groups, we show in Section 9 that if the pathwise connected component $c_0(G)$ of identity of a topological group G is closed in G, then G-equivalence implies $c_0(G)$-equivalence; see Corollary 9.3. Finally, we use the latter fact to prove our most general result about preservation of covering dimension by G-equivalence. Namely we prove that if G is a topological group such that its pathwise connected component $c_0(G)$ is closed in G and $c_0(G) = H^\kappa$, where κ is an arbitrary nonzero cardinal, and H is a nontrivial abelian separable metrizable group that is either NSS, or has self-slender completion, then G-equivalence preserves the covering dimension.

2. Topological version of the main theorem

Definition 2.1. Let $(\mathbb{P}, \leq)$ be a partially ordered set (poset). For $P \subseteq \mathbb{P}$ an element $q \in \mathbb{P}$ is called the *supremum of* P provided that the following conditions hold:

(i) $p \leq q$ for every $p \in P$;
(ii) if $r \in \mathbb{P}$ is such that $p \leq r$ for every $p \in P$, then $q \leq r$.

If the supremum of P exists, then it is unique, and we denote it by $\sup P$.

Definition 2.2. A subset P of a poset $(\mathbb{P}, \leq)$ is:

(i) *closed* in $(\mathbb{P}, \leq)$ provided that $\sup\{p_n : n \in \mathbb{N}\} \in P$ for every sequence $\{p_n : n \in \mathbb{N}\} \subseteq P$ such that $p_0 \leq p_1 \leq \dots \leq p_n \leq p_{n+1} \leq \dots$,
(ii) *unbounded* in $(\mathbb{P}, \leq)$ provided that for every $q \in \mathbb{P}$ there exists $p \in P$ such that $q \leq p$,
(iii) a *club* in $(\mathbb{P}, \leq)$ if P is both closed and unbounded in $(\mathbb{P}, \leq)$.

Our next lemma is a part of folklore. We include its proof for the reader's convenience.

Lemma 2.3. *Let* $\{P_n : n \in \mathbb{N}\}$ *be a sequence of clubs in a poset* $(\mathbb{P}, \leq)$*. Then* $P = \bigcap\{P_n : n \in \mathbb{N}\}$ *is a club in* $(\mathbb{P}, \leq)$*.*

Proof: Obviously, P is closed. To prove that it is unbounded fix an arbitrary $q \in \mathbb{P}$. Since each P_n is unbounded, using the standard diagonal argument we can find a sequence $\{p_i : i \in \mathbb{N}\} \subseteq \mathbb{P}$ such that $q \leq p_0 \leq p_1 \leq \dots \leq p_i \leq p_{i+1} \leq \dots$ and the set $I_n = \{i \in \mathbb{N} : p_i \in P_n\}$ is infinite for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Let $I_n = \{i(j,n) : j \in \mathbb{N}\}$ be the order preserving one-to-one enumeration of the infinite set I_n. Then $p_{i(0,n)} \leq p_{i(1,n)} \leq \dots \leq p_{i(j,n)} \leq p_{i(j+1,n)} \leq \dots$ is the sequence of elements of P_n. Since P_n is closed in $(\mathbb{P}, \leq)$, we have $r_n = \sup\{p_i : i \in I_n\} = \sup\{p_{i(j,n)} : j \in \mathbb{N}\} \in P_n$.

Clearly, $q \leq p_{i(0,0)} \leq r_0$. It remains only to check that $r_n = r_0$ for every $n \in \mathbb{N} \setminus \{0\}$, as this would yield $r_0 \in \bigcap\{P_n : n \in \mathbb{N}\}$.

Fix $n \in \mathbb{N} \setminus \{0\}$. Let $j \in \mathbb{N}$. Since I_0 is infinite, there exists $i_0 \in I_0$ such that $p_{i(j,n)} \leq p_{i_0}$. From this and $r_0 = \sup\{p_i : i \in I_0\}$, we conclude that $p_{i(j,n)} \leq r_0$. Since this inequality holds for every $j \in \mathbb{N}$, it follows that $r_n = \sup\{p_{i(j,n)} : j \in \mathbb{N}\} \leq r_0$. The reverse inequality is proved similarly, using the fact that the set I_n is infinite. $\square$

Definition 2.4. Let X be a space.

(i) We denote by $\mathbb{Q}(X)$ the class of all continuous functions from X onto spaces with countable weight.

(ii) For $f, f' \in \mathbb{Q}(X)$ we write $f \preceq f'$ provided that there exists a continuous map $h : f'(X) \to f(X)$ such that $f = h \circ f'$.

(iii) For $f, f' \in \mathbb{Q}(X)$ we write $f \approx f'$ provided that both $f \preceq f'$ and $f' \preceq f$ hold.

(iv) For $f \in \mathbb{Q}(X)$ we denote by $[f]$ the equivalence class of f with respect to the equivalence relation $\approx$ on $\mathbb{Q}(X)$.

(v) Define $\mathbb{P}_X = \{[f] : f \in \mathbb{Q}(X)\}$. For $[f], [f'] \in \mathbb{P}_X$ we define $[f] \leq [f']$ by $f \preceq f'$.

Clearly, $\mathbb{P}_X$ is a poset. With a certain abuse of notation, in the sequel, we will not distinguish between $f \in \mathbb{Q}(X)$ and its equivalence class $[f]$.

Our next lemma shows among others that the poset $\mathbb{P}_X$ is closed under taking suprema of countable subsets. Its straightforward proof is left to the reader.

Lemma 2.5. *Let X be a space and S a countable subset of $\mathbb{P}_X$. Let $f = \triangle S : X \to f(X) \subseteq \prod_{g \in S} g(X)$ be the diagonal product of the set S. Then $f \in \mathbb{P}_X$ and $f = \sup S$.*

Definition 2.6. Let X be a space. For $f \in \mathbb{P}_X$ we define $|f|$ to be the function from the set X to the set $f(X)$. (In other words, $|f|$ is an image of f under the forgetful functor from the category of topological spaces to the category of sets.) We put $|\mathbb{P}_X| = \{|f| : f \in \mathbb{P}_X\}$. If $f, g \in |\mathbb{P}_X|$ we write $f \preceq g$ provided that there exists a function $h : g(X) \to f(X)$ such that $f = h \circ g$.

Lemma 2.7. *Let X be a space and $f, g' \in \mathbb{P}_X$. Assume that $|f| \preceq |g'|$. Then there exists $g \in \mathbb{P}_X$ such that $f \leq g$, $g' \leq g$ and $|g'| = |g|$ (and consequently $|g|(X) = |g'|(X)$).*

Proof: Let $\mathcal{B}_{g'}, \mathcal{B}_f$ be countable bases of the topology of $g'(X)$ and $f(X)$ respectively. By our assumptions there exists function $h : |g'|(X) \to |f|(X)$, such that $|f| = h \circ |g'|$. Put $\mathcal{B} = \mathcal{B}_{g'} \cup \{h^{-1}(U) : U \in \mathcal{B}_f\}$. Then $\mathcal{B}$ is a countable subbase of some topology $\mathcal{T}$ on $|g'|(X)$. We claim that the function $g : X \to (|g'|(X), \mathcal{T})$ is continuous. Indeed, for $V \in \mathcal{B}$ there are two cases. If $V \in \mathcal{B}_{g'}$, then $g^{-1}(V)$ is open since g' is continuous. If $V = h^{-1}(U)$ for some $U \in \mathcal{B}_f$, then $g^{-1}(V) = f^{-1}(U)$ is open due to the continuity of f. Thus $g \in \mathbb{P}_X$. By the construction of g, we get $|g'| = |g|$. Since $\mathcal{T}$ is stronger then the topology of $g'(X)$, we get $g' \leq g$. By the construction of $\mathcal{T}$, the function $h : g(X) \to f(X)$ is continuous. This gives us $f \leq g$. □

The following simple lemma can be derived from Lemmas 2.7 and 2.5. It verifies that the condition (iii) of Theorem 2.11 makes sense.

Lemma 2.8. *Let X be a space. If $f, g \in |\mathbb{P}_X|$ are such that $f \preceq g$ and $g \preceq f$, then $f = g$. In particular, $|\mathbb{P}_X|$ is a poset. Furthermore, if $S \subseteq \mathbb{P}_X$ is countable, then* $\sup |S| = |\sup S|$.

The next two lemmas form an essence of the proof of Theorem 2.11.

Lemma 2.9. *Let X be a Tychonoff space, $n \in \mathbb{N}$ and $S_n(X) = \{f \in \mathbb{P}_X : \dim f(X) \leq n\}$ then the following conditions are equivalent:*

(i) $\dim X \leq n$;
(ii) $S_n(X)$ *is unbounded in* $\mathbb{P}_X$;
(iii) $S_n(X)$ *is a club in* $\mathbb{P}_X$.

Proof: The equivalence of (i) and (ii) is proved in [10, Theorem 2]. The implication (ii)→(iii) can be found for example in [11, Lemma 11]. The implication (iii)→(ii) is trivial. □

Let X, Y be sets and 2^Y be a set of all nonempty subsets of Y. We call a mapping $F : X \to 2^Y$ a *set-valued mapping from X to Y* and denote this fact by the symbol $F : X \Rightarrow Y$. A mapping $F : X \Rightarrow Y$ is called *finite-valued* provided that $F(x)$ is finite for every $x \in X$. When X and Y are spaces, a set-valued mapping $F : X \Rightarrow Y$ is called *lower semi-continuous* (abbreviated by *lsc*) provided that the set

$$F^{-1}(U) = \{x \in X : F(x) \cap U \neq \emptyset\}$$

is open in X for every open set $U \subseteq Y$.

The next lemma is a particular case of [13, Corollary 5.4].

Lemma 2.10. *For $i \in \{0,1\}$ let X_i be a separable metric space, and $\varphi_i : X_i \Rightarrow X_{1-i}$ a finite-valued lsc mapping such that for every $x_1 \in X_1$ there exists $x_0 \in \varphi_1(x_1)$ with $x_1 \in \varphi_0(x_0)$. Then $\dim X_1 \leq \dim X_0$.*

Theorem 2.11. *For $i \in \{0,1\}$ let X_i be a Tychonoff space and $\mathfrak{f}_i : \mathbb{P}_{X_i} \to \mathbb{P}_{X_{1-i}}$ a map satisfying the following conditions:*

(i) $|\mathfrak{f}_i(f)| \preceq |\mathfrak{f}_i(g)|$ *for all $f, g \in \mathbb{P}_{X_i}$ such that* $|f| \preceq |g|$;

(ii) *for* $\mathfrak{h}_i = \mathfrak{f}_{1-i} \circ \mathfrak{f}_i$ *we have* $|f| \preceq |\mathfrak{h}_i(f)|$ *for every* $f \in \mathbb{P}_{X_i}$*;*
(iii) $\sup\{|\mathfrak{f}_i(f)| : f \in s\} = |\mathfrak{f}_i(\sup s)|$ *for every increasing sequence* $s \subseteq \mathbb{P}_{X_i}$*;*
(iv) *for each* $f \in \mathbb{P}_{X_i}$ *there exists finite-valued lsc mapping* $\varphi_{|f|,i} : \mathfrak{f}_i(f)(X_{1-i}) \Rightarrow f(X_i)$ *(that depends only on* $|f|$*) such that if* $f \in \mathbb{P}_{X_0}$*, then for every* $x_1 \in X_1$ *there exists* $x_0 \in \varphi_{|f|,0}(x_1)$ *satisfying* $x_1 \in \varphi_{|\mathfrak{f}_0(f)|,1}(x_0)$ *whenever* $|\mathfrak{h}_0(f)| = |f|$*;*
(v) *for every increasing sequence* $s_i = (f_0, f_1, \dots)$ *in* $\mathbb{P}_{X_i}$ *and every increasing sequence* $s_{1-i} = (g_0, g_1, \dots)$ *in* $\mathbb{P}_{X_{1-i}}$ *such that* $|\mathfrak{f}_i(f_n)| = |g_n|$ *and* $\varphi_{|f_n|,i} : g_n(X_{1-i}) \Rightarrow f_n(X_i)$ *is lsc for all* $n \in \mathbb{N}$*, the mapping* $\varphi_{|f|,i} : \mathfrak{f}_i(f)(X_{1-i}) \Rightarrow f(X_i)$ *is lsc, where* $f = \sup s_i$*.*

Then $\dim X_1 \leq \dim X_0$.

PROOF: If $\dim X_0$ is infinite, the desired inequality follows. Otherwise $\dim X_0 = n$ for some $n \in \mathbb{N}$. Let P be a subset of $\mathbb{P}_{X_0} \times \mathbb{P}_{X_1}$ consisting of all pairs (f_0, f_1) such that $|f_{1-i}| = |\mathfrak{f}_i(f_i)|$ holds for some $i \in \{0, 1\}$. We take P with the product order inherited from $\mathbb{P}_{X_0} \times \mathbb{P}_{X_1}$.

Claim 1. Let $i \in \{0, 1\}$, and assume that K is a club in $\mathbb{P}_{X_i}$. Then the set $A_K = \{(f_0, f_1) : f_i \in K, |f_{1-i}| = |\mathfrak{f}_i(f_i)|\}$, is a club in P.

PROOF: To show that A_K is unbounded fix $(f_0, f_1) \in P$ arbitrarily. By our assumption, there exists $g_i \in K$ such that $f_i \leq g_i$. Put $g'_{1-i} = \mathfrak{f}_i(g_i)$. Then $|f_{1-i}| \preceq |g'_{1-i}|$. Indeed, if $|f_{1-i}| = |\mathfrak{f}_i(f_i)|$, then the inequality follows directly from (i) and if $|f_i| = |\mathfrak{f}_{1-i}(f_{1-i})|$, then $|f_{1-i}| \preceq |\mathfrak{h}_{1-i}(f_{1-i})| \preceq |g'_{1-i}|$ by (ii). Thus, by Lemma 2.7, there exists $g_{1-i} \in \mathbb{P}_{X_{1-i}}$ such that $f_{1-i} \leq g_{1-i}$ and $|g'_{1-i}| = |g_{1-i}|$. Clearly, $(g_0, g_1) \in A_K$ and $(f_0, f_1) \leq (g_0, g_1)$.

The fact that A_K is closed follows immediately from the assumption that K is closed in $\mathbb{P}_{X_i}$ and from (iii). □

Claim 2. The set $B \subseteq P$ consisting of all pairs $(f_0, f_1) \in P$ such that $\dim f_0(X_0) \leq n$ and $|f_1| = |\mathfrak{f}_0(f_0)|$ is a club in P.

PROOF: By Lemma 2.9, $S_n(X_0)$ is a club in $\mathbb{P}_{X_0}$. Hence the conclusion follows from Claim 1. □

Claim 3. For both $i \in \{0, 1\}$ the set $C_i \subseteq P$ consisting of all pairs (f_0, f_1) such that $|f_{1-i}| = |\mathfrak{f}_i(f_i)|$ and the function $\varphi_{|f_i|,i} : f_{1-i}(X_{1-i}) \Rightarrow f_i(X_i)$ is lsc is a club in P.

PROOF: By Claim 1, the set $A_i = \{(f_0, f_1) \in P : |f_{1-i}| = |\mathfrak{f}_i(f_i)|\}$ is a club in P. Since $C_i \subseteq A_i$, it suffices to prove that C_i is a club in A_i. To show that C_i is unbounded, pick (f_0, f_1) arbitrarily. Put $g_i = f_i$ and $g'_{1-i} = \mathfrak{f}_i(f_i)$. By (iv), the finite valued mapping $\varphi_{|f_i|,i} : g'_{1-i}(X_{1-i}) \Rightarrow f_i(X_i)$ is lsc. Since $|f_{1-i}| = |\mathfrak{f}_i(f_i)| = |g'_{1-i}|$, Lemma 2.7 implies existence of $g_{1-i} \in \mathbb{P}_{X_{1-i}}$ such that $f_{1-i} \leq g_{1-i}$, $|g'_{1-i}| = |g_{1-i}|$ and $g'_{1-i} \leq g_{1-i}$. In particular, the topology of $g_{1-i}(X_{1-i})$ is stronger then that of $g'_{1-i}(X_{1-i})$. Hence $\varphi_{|f_i|,i} : g_{1-i}(X_{1-i}) \Rightarrow f_i(X_i)$ is lsc, and $(f_0, f_1) \leq (g_0, g_1) \in C_i$.

The fact that C_i is closed follows directly from (iii) and (v). □

By Lemma 2.9, to show that $\dim X_1 \leq n$ we have to prove that $S_n(X_1)$ is unbounded in $\mathbb{P}_{X_1}$. To do so, pick $f_1 \in \mathbb{P}_{X_1}$ arbitrarily and put $f_0 = \mathfrak{f}_1(f_1)$. Then $(f_0, f_1) \in P$. Put $D = B \cap C_0 \cap C_1$. Then D is a club in P by Lemma 2.3. Therefore, there exists $(g_0, g_1) \in D$ such that $(f_0, f_1) \leq (g_0, g_1)$. By the definition of D it follows that $\dim g_0(X_0) \leq n$, $\varphi_{|g_i|,i} : g_{1-i}(X_{1-i}) \Rightarrow g_i(X_i)$ is lsc for both $i \in \{0,1\}$, and $\mathfrak{h}_0(g_0) = g_0$. Now, from (iii) we get that for every $x_1 \in X_1$ there exists $x_0 \in \varphi_{|g_0|,0}(x_1)$ satisfying $x_1 \in \varphi_{|g_1|,1}(x_0)$. Applying Lemma 2.10, we conclude that $\dim g_1(X_1) \leq \dim g_0(X_0) \leq n$. Thus, $g_1 \in S_n(X_1)$ satisfies $f_1 \leq g_1$. This finishes the proof. $\square$

3. Technical lemmas

Lemma 3.1. *Let X be a separable metric space and Y a space with a countable network. Assume that $\varphi : Y \Rightarrow X$ is an lsc mapping. Then there exists weaker separable metrizable topology $\mathcal{T}$ on the underlying set $|Y|$ of Y such that $\varphi : (|Y|, \mathcal{T}) \Rightarrow X$ is lsc.*

Proof: Since $(Y, \mathcal{T}_Y)$ is a space with a countable network, there is a topology $\mathcal{T}'$ on Y with a countable base $\mathcal{B}$ such that $\mathcal{T}' \subseteq \mathcal{T}_Y$. (A proof of this simple folklore fact can be found, for example, in the appendix of [6].)

Let $\mathcal{C}$ be a countable base of the topology of X. Then $\{\varphi^{-1}(U) : U \in \mathcal{C}\} \cup \mathcal{B}$ is a countable subbase of topology $\mathcal{T}$ on $|Y|$. Since φ is lsc, it follows that $\mathcal{T} \subseteq \mathcal{T}_Y$. By the definition of $\mathcal{T}$, $\varphi : (|Y|, \mathcal{T}) \Rightarrow X$ is lsc. $\square$

An *R-module* is a left module over a ring R. The additive identities of a ring R and a module M will be denoted by 0_R and 0_M respectively. Having R-modules M, M' and a mapping $f : M \to M'$, it may happen that f is a homomorphism with respect to the (abelian) group structure of M and M' but it is not a homomorphism with respect to their module structure. To avoid confusion, we will call a homomorphism with respect to the R-module structure an *R-homomorphism*, while the term homomorphism will be reserved for a group homomorphism. In the same spirit we shall use the terms like *R-isomorphism* or *R-isomorphic*.

Definition 3.2. For an R-module M, a set $X \subseteq M$ is called:

(i) *generating* if every element of M is a finite sum of elements of X multiplied by coefficients in R, or equivalently, if M is the smallest R-submodule of M containing X;

(ii) *free* if $\sum_{i=1}^{k} r_i x_i = 0_M$ implies $r_1 = r_2 = \ldots = r_k = 0_R$ whenever $k \in \mathbb{N}$, $r_1, \ldots, r_k \in R$ and the elements $x_1, \ldots, x_k \in X$ are pairwise distinct;

(iii) a *basis of M* if X is both generating and free.

A *free R-module* is an R-module that has a free basis.

Definition 3.3. Let M be a free R-module with a free basis X. It follows from items (i) and (ii) of Definition 3.2 that for every $a \in M$ there exist the unique finite set $K_a \subseteq X$ and the unique family $\{r_x : x \in K_a\} \subseteq R \setminus \{0_R\}$ of scalars such that

$$a = \sum_{x \in K_a} r_x x. \tag{1}$$

We shall call (1) the *canonical representation of a with respect to X* and we define the *support function* $\varphi_X : M \Rightarrow X$ by $\varphi_X(a) = K_a$ for all $a \in M$.

The straightforward proof of the following lemma is left to the reader.

Lemma 3.4. *Let M, N be R-modules and let X be a free basis of M. Then:*

(i) *for every function $f : X \to N$ there exists the unique R-homomorphism $\widehat{f} : M \to N$ extending f;*
(ii) *if Y is a free basis of N and $f : M \to N$ is an R-homomorphism such that $f(X) = Y$, then $\varphi_Y(f(a)) \subseteq f(\varphi_X(a))$ for every $a \in M$;*
(iii) *if Z is a free basis of M, then for every $x \in X$ there exists $z \in \varphi_Z(x)$ such that $x \in \varphi_X(z)$.*

Lemma 3.5. *Let M be an R-module with a free basis X. For every $i \in \mathbb{N}$ let M_i be an R-module, $f_i : M \to M_i$ be an R-homomorphism and $p_i : M_{i+1} \to M_i$ be an R-homomorphism such that $f_i = p_i \circ f_{i+1}$ and $f_i(X)$ is the free basis of M_i. Let $f = \triangle\{f_i : i \in \mathbb{N}\}$ be the diagonal product of the family $\{f_i : i \in \mathbb{N}\}$. For every $i \in \mathbb{N}$ let $\pi_i : f(M) \to M_i$ be the unique R-homomorphism satisfying $f_i = \pi_i \circ f$. Then:*

(i) *$f(X)$ is the free basis of $f(M)$;*
(ii) *for every $z \in f(M)$ there exists $n \in \mathbb{N}$ such that $\pi_m(\varphi_{f(X)}(z)) = \varphi_{f_m(X)}(\pi_m(z))$ for all integers $m \geq n$.*

Furthermore, let $\mathcal{T}$ be a topology on M and for every $i \in \mathbb{N}$ let $\mathcal{T}_i$ be a topology on M_i such that the maps $f_i : (M, \mathcal{T}) \to (M_i, \mathcal{T}_i)$ and $p_i : (M_{i+1}, \mathcal{T}_{i+1}) \to (M_i, \mathcal{T}_i)$ are continuous and the map $\varphi_i = \varphi_{f_i(X)} : M_i \Rightarrow f_i(X)$ is lsc with respect to $\mathcal{T}_i$. Then:

(iii) *the support function $\varphi_{f(X)} : f(M) \Rightarrow f(X)$ is lsc with respect to the subspace topology inherited by $f(M)$ from the Tychonoff product $\prod_{i \in \mathbb{N}}(M_i, \mathcal{T}_i)$.*

Proof: For every finite set $Y \subseteq f(M)$ one can easily choose $n \in \mathbb{N}$ such that

$$\pi_m(y) \neq \pi_m(y') \text{ whenever } m \in \mathbb{N}, m \geq n, y, y' \in Y \text{ and } y \neq y'. \tag{2}$$

This easily implies (i). To check (ii), pick $z \in f(M)$ arbitrarily. Apply the above observation to $Y = \varphi_{f(X)}(z) \subseteq f(X)$ to fix $n \in \mathbb{N}$ satisfying (2). Fix $m \in \mathbb{N}$ with $m \geq n$. Let $z = \sum_{y \in Y} r_y y$ be the canonical representation of z with respect to $f(X)$. Since $f_m(X)$ is the free basis of $M_m = \pi_m(M)$, $\{\pi_m(y) : y \in Y\} \subseteq \pi_m(f(X)) = f_m(X)$ and $\pi_m(z) = \pi_m(\sum_{y \in Y} r_y y) = \sum_{y \in Y} r_y \pi_m(y)$, it follows that $\varphi_{f_m(X)}(\pi_m(z)) = \{\pi_m(y) : y \in Y\} = \pi_m(Y) = \pi_m(\varphi_{f(X)}(z))$.

To prove (iii), fix an open subset U of $f(X)$. We have to show that the set $\varphi^{-1}_{f(X)}(U)$ is open in $f(M)$. Without loss of generality, we may assume that U is a basic open set; that is, $U = \pi_k^{-1}(U_k)$ for some $k \in \mathbb{N}$ and an open subset U_k of $f_k(X)$. For every integer $m \geq k$ the set $U_m = (p_k \circ p_{k+1} \circ \ldots \circ p_{m-1})^{-1}(U_k)$ is open in $f_m(X)$ and $\pi_m^{-1}(U_m) = U$.

Let $z \in \varphi_{f(X)}^{-1}(U)$. Then $\varphi_{f(X)}(z) \cap U \neq \emptyset$. By (ii), there exists an integer $m \geq k$ such that $\pi_m(\varphi_{f(X)}(z)) = \varphi_{f_m(X)}(\pi_m(z)) = \varphi_m(\pi_m(z))$. From $\varphi_{f(X)}(z) \cap U \neq \emptyset$ we get

$$\pi_m(\varphi_{f(X)}(z)) \cap \pi_m(U) = \pi_m(\varphi_{f(X)}(z)) \cap U_m = \varphi_m(\pi_m(z)) \cap U_m \neq \emptyset.$$

Consequently, $z \in \pi_m^{-1}(\varphi_m^{-1}(U_m))$. Since φ_m is lsc and U_m is open in X_m, the set $\varphi_m^{-1}(U_m)$ is open in M_m. Since π_m is continuous, $V = \pi_m^{-1}(\varphi_m^{-1}(U_m))$ is an open subset of $f(M)$. To finish the proof, it remains to show that $V \subseteq \varphi_{f(X)}^{-1}(U)$. Let $x \in V$. Then $\pi_m(x) \in \varphi_m^{-1}(U_m)$, and so $\varphi_m(\pi_m(x)) \cap U_m \neq \emptyset$. Since $\varphi_m(\pi_m(x)) \subseteq \pi_m(\varphi_{f(X)}(x))$ by Lemma 3.4(ii), this yields $\pi_m(\varphi_{f(X)}(x)) \cap U_m \neq \emptyset$. Therefore, $\varphi_{f(X)}(x) \cap \pi_m^{-1}(U_m) = \varphi_{f(X)}(x) \cap U \neq \emptyset$, and so $x \in \varphi_{f(X)}^{-1}(U)$. □

4. Main theorem

Let us first recall several basic notions from the category theory.

Let $\mathbf{A}, \mathbf{B}$ be categories. We use $Ob(\mathbf{A})$ to denote the class of all objects of $\mathbf{A}$, and $Mor(\mathbf{A})$ to denote the class of all morphisms of $\mathbf{A}$. If $X, Y \in Ob(\mathbf{A})$, then $hom(X, Y)$ stays for the set of all $\mathbf{A}$-morphisms from X to Y and $id_X \in hom(X, X)$ denotes the identity morphism on X. If $f \in hom(X, Y)$ then we call X a *domain* and Y a *codomain* of f. We say that a functor $\mathscr{U} : \mathbf{A} \to \mathbf{B}$ is *faithful* provided that it is injective on $hom(A, B)$ for all $A, B \in Ob(\mathbf{A})$. By $Id_{\mathbf{A}}$ we denote the identity functor from $\mathbf{A}$ to $\mathbf{A}$. If $F, G : \mathbf{A} \to \mathbf{B}$ are functors, then a *natural transformation* η from F to G is a map that assigns to each $A \in Ob(\mathbf{A})$ a morphism $\eta_A \in hom(F(A), G(A))$ in such a way that for every morphism $f \in hom(A, A') \subseteq Mor(\mathbf{A})$ it holds

$$\eta_{A'} \circ F(f) = G(f) \circ \eta_A.$$

As usual **Top** stays for the category of all topological spaces and their continuous mappings and **Tych** for its full subcategory of all Tychonoff spaces.

Here comes the definition of an embedding functor in terms of natural transformation.

Definition 4.1. Let $\mathbf{T}$ be a subcategory of the category **Top**. We say that a functor $F : \mathbf{T} \to \mathbf{Top}$ is an *embedding* functor provided that there exists a natural transformation η between $Id_{\mathbf{T}}$ and F such that $\eta_X \in hom(X, F(X))$ is a homeomorphic embedding for each $X \in Ob(\mathbf{T})$.

Now we can state the main theorem in a categorial fashion:

Theorem 4.2. *Let* $\mathbf{T}$ *be some full subcategory of* **Tych** *containing all separable metrizable spaces. Let* $\mathbf{M}$ *be some subcategory of* **Tych** *consisting of* R*-modules* *and R-homomorphisms. Let $F : \mathbf{T} \to \mathbf{M}$ be an embedding functor with natural transformation η such that for all $X \in Ob(\mathbf{T})$*

(i) *$\eta_X(X)$ is a free basis of $F(X)$ and*

(ii) *if X has a countable weight, then $F(X)$ has countable network and the support function $\varphi_{\eta_X(X)} : F(X) \Rightarrow \eta_X(X)$ is lsc.*

Let $X_0, X_1 \in Ob(\mathbf{T})$, and assume that $F(X_0)$ and $F(X_1)$ are isomorphic in $\mathbf{M}$. Then $\dim X_0 = \dim X_1$.

Proof: Without loss of generality, we may identify X with $\eta_X(X)$ for every $X \in \mathbf{T}$, and we may assume that $F(X_0) = F(X_1) = M$. Thus both, X_0 and X_1, form a free basis of M and every (continuous) function $f : X_i \to f(X_i)$, where $f(X_i) \in Ob(\mathbf{T})$, can be uniquely extended to a (continuous) R-homomorphism $F(f) : M \to F(M)$ for both $i \in \{0, 1\}$.

It suffices to prove that $\dim X_1 \leq \dim X_0$ since the other inequality follows then from symmetry.

For both $i \in \{0, 1\}$ and every $f \in \mathbb{P}_{X_i}$ consider the space $Y_{1-i} = F(f)(X_{1-i})$. By (ii) it has a countable network and the finite-valued function $\varphi_{|f|,i} = \varphi_{f(X_i)} \restriction Y_{1-i}$ is lsc. By Lemma 3.1 there exists a weaker separable metric topology $\mathcal{T}$ on the underlining set $|Y_{1-i}|$ of Y_{1-i} such that $\varphi_{|f|,i} : (|Y_{1-i}|, \mathcal{T}) \Rightarrow f(X_i)$ is lsc. Let $\iota_{f,i} : Y_{1-i} \to (|Y_{1-i}|, \mathcal{T})$ be the (continuous) identity mapping. Define $\mathfrak{f}_i(f) = \iota_{f,i} \circ F(f) \restriction_{X_{1-i}}$. Clearly, $\mathfrak{f}_i(f) \in \mathbb{P}_{X_{1-i}}$. We claim that X_i, the map $\mathfrak{f}_i : \mathbb{P}_{X_i} \to \mathbb{P}_{X_{1-i}}$ and the finite-valued mapping $\varphi_{|f|,i} : \mathfrak{f}_i(f)(X_{1-i}) \Rightarrow f(X_i)$ satisfy conditions (i)–(v) of Theorem 2.11 for every $i \in \{0, 1\}$ and every $f \in \mathbb{P}_{X_i}$.

Conditions (i) and (ii) follow immediately from Lemma 3.4(i). Condition (iii) follows from Lemmas 3.5(i) and 2.5. The first part of condition (iv) is satisfied by the definition of $\varphi_{|f|,i}$ and the second part follows from Lemma 3.4(iii). Finally, condition (v) is satisfied by Lemmas 2.5 and 3.5(iii). Applying Theorem 2.11 gives us $\dim X_1 \leq \dim X_0$. $\square$

5. Background on free topological modules

Recall that a *topological ring* is a ring R which is also a topological space such that both the addition and the multiplication are continuous as maps from $R \times R$ to R. By a *topological R-module* we mean a module M over a topological ring R which carries at the same time a topology that makes M with the group operations a topological group and the scalar multiplication $(r, x) \mapsto rx$ continuous as a map from $R \times M$ to M.

Definition 5.1. Let $\mathscr{M}$ be a class of topological R-modules. By $\mathscr{M}_R$ we denote the smallest class of topological R-modules containing $\mathscr{M}$ that is closed under taking arbitrary products and (topological) R-submodules.

If $\mathscr{M} = \mathscr{M}_R$, then we will say that the class $\mathscr{M}$ is *R-closed*.

Definition 5.2. Let $\mathscr{M}$ be an R-closed class of topological modules. We say that $F \in \mathscr{M}$ is a *free topological module over a space X in $\mathscr{M}$* provided that there exists $\eta \in C(X, F)$ called a *unit of F* such that, for every $M \in \mathscr{M}$ and each

$f \in C(X, M)$, one can find a unique continuous R-homomorphism $\widehat{f} : F \to M$ satisfying $f = \widehat{f} \circ \eta$.

The next proposition about existence and uniqueness of free topological modules is a well-known particular case of the Freyd Adjoint Functor Theorem [3]. For the construction of free topological modules we refer the reader to the proof of Proposition 5.7.

Proposition 5.3. *Let $\mathscr{M}$ be an R-closed class of topological modules and X a topological space. Then there exists a free topological module F over X in $\mathscr{M}$ with a unit η. Furthermore, if F' is a free topological module over X in $\mathscr{M}$ with a unit η', then there exists a topological R-isomorphism $\phi : F \to F'$ such that $\eta' = \phi \circ \eta$.*

Definition 5.4. Let $\mathscr{M}$ be some class of topological R-modules. We say that a topological space X is:

(i) *$\mathscr{M}$-Hausdorff* provided that for every $y \in X$, each $x \in X \setminus \{y\}$ and all $r \in R \setminus \{0_R\}$, there exist $M \in \mathscr{M}$ and $f \in C(X, M)$ such that $f(y) = 0_M$ and $rf(x) \neq 0_M$;

(ii) *$\mathscr{M}$-regular* if for every closed $A \subseteq X$, every $x \in X \setminus A$ and each $r \in R \setminus \{0_R\}$, there exist $M \in \mathscr{M}$ and $f \in C(X, M)$ such that $f(A) \subseteq \{0_M\}$ and $rf(x) \neq 0_M$.

Lemma 5.5. *Let $\mathscr{M}$ be some class of topological R-modules. Suppose that the closed unit interval $[0, 1]$ is $\mathscr{M}$-Hausdorff. Then every Tychonoff space is $\mathscr{M}$-regular.*

PROOF: Let X be Tychonoff, $A \subseteq X$ closed, $x \in X \setminus A$ and $r \in R \setminus \{0_R\}$. Then there exists $f \in C(X, [0, 1])$ such that $f(A) \subseteq \{0\}$ and $f(x) = 1$. Since $[0, 1]$ is $\mathscr{M}$-Hausdorff, there exist $M \in \mathscr{M}$ and $g \in C([0, 1], M)$ such that $g(0) = 0_M$ and $rg(1) \neq 0_M$. Then the function $g \circ f \in C(X, M)$ witnesses that X is $\mathscr{M}$-regular. □

Proposition 5.6. *Let M be an R-module with a free basis X and $\mathcal{T}$ some topology on M. Assume that for every $\mathcal{T}\restriction_X$-closed subset A of X, all $x \in X \setminus A$ and each scalar $r \in R \setminus \{0_R\}$, there exist some R-module M' with a topology $\mathcal{T}'$ and a continuous R-homomorphism $f : (M, \mathcal{T}) \to (M', \mathcal{T}')$ such that $f(A) \subseteq \{0_{M'}\}$ and $rf(x) \neq 0_{M'}$. Then $\varphi_X : (M, \mathcal{T}) \Rightarrow (X, \mathcal{T}\restriction_X)$ is lsc.*

PROOF: Let U be an arbitrary open subset of X (with respect to $\mathcal{T}\restriction_X$), and assume that $z \in \varphi_X^{-1}(U)$. We have to show that there exists an open set V in $(M, \mathcal{T})$ such that $z \in V \subseteq \varphi_X^{-1}(U)$. Let $z = \sum_{i=1}^{n} r_i x_i$, where $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ are pairwise distinct and $r_1, \dots, r_n \in R \setminus \{0_R\}$. Since $\varphi_X(z)$ is finite, there is some open set $U' \subseteq U$ such that $U' \cap \varphi_X(z) = \{x_k\}$ for some $k \in \{1, \dots, n\}$. Put $A = X \setminus U'$. By our assumption, there exist an R-module M' with a topology $\mathcal{T}'$ and a continuous R-homomorphism $f : (M, \mathcal{T}) \to (M', \mathcal{T}')$ such that

$f(A) \subseteq \{0_{M'}\}$ and $r_k f(x_k) \neq 0_{M'}$. Thus we get

$$f(z) = f\left(\sum_{i=1}^{n} r_i x_i\right) = \sum_{i=1}^{n} r_i f(x_i) = r_k f(x_k) \neq 0_{M'}.$$

It follows that $z \in V = f^{-1}(M' \setminus \{0_{M'}\})$, where V is open by the continuity of f. Pick $z' \in V$ arbitrarily. Observe that $\varphi_X(z') \subseteq A$ would imply $f(z') = 0_{M'}$, because $f(A) \subseteq \{0_{M'}\}$. Therefore, since $f(z') \neq 0_{M'}$, we get $\varphi_X(z') \cap U' \neq \emptyset$. Consequently, $V \subseteq \varphi_X^{-1}(U)$, and φ_X is lsc. $\square$

Proposition 5.7. *Let $\mathscr{M}$ be an R-closed class of topological modules, X an $\mathscr{M}$-regular topological space and F the free topological module over X in $\mathscr{M}$ with a unit η. Then:*

- (i) *η is a homeomorphic embedding,*
- (ii) *$\eta(X)$ is a free basis of F, and*
- (iii) *the support function $\varphi_{\eta(X)} : F \Rightarrow \eta(X)$ is lsc.*

Proof: There exists an indexed set $\{(M_s, f_s) : s \in S\}$ such that:

- (a) for each $s \in S$, $M_s \in \mathscr{M}$, $f_s : X \to M_s$ is continuous and $f_s(X)$ generates M_s;
- (b) if $M \in \mathscr{M}$ and $f : X \to M$ is a continuous function, then there exist $t \in S$, a submodule M'_t of M and a topological R-isomorphism $i_t : M_t \to M'_t$ such that $f = i_t \circ f_t$.

The diagonal product $\eta' = \triangle_{s\in S} f_s : X \to \prod_{s\in S} M_s$ of the family $\{f_s : s \in S\}$ is a continuous function. Let F' be the R-submodule generated by $\eta'(X)$ in the topological R-module $\prod_{s\in S} M_s$. Since $\mathscr{M}$ is R-closed, we have $F' \in \mathscr{M}$. Let $M \in \mathscr{M}$ and $f \in C(X, M)$. Let $t \in S$ and i_t be as in the conclusion of item (b), and let $\pi_t : \prod_{s\in S} M_s \to M_t$ be the projection on t's coordinate. Then $g = i_t \circ \pi_t \restriction_{F'} : F' \to M'_t$ is a continuous R-homomorphism such that $g \circ \eta = i_t \circ \pi_t \restriction_{F'} \circ \eta' = i_t \circ f_t = f$. Since $\eta'(X)$ generates F', it follows that g is unique. We have verified that F' is a free topological module over X in $\mathscr{M}$ with a unit η'.

By Proposition 5.3, the free topological module F over X in $\mathscr{M}$ and its unit η are unique up to a topological R-isomorphism. Thus, we may assume, without loss of generality, that $F = F'$ and $\eta = \eta'$.

(i) Since X is $\mathscr{M}$-regular, continuous functions from X to elements of $\mathscr{M}$ separate points and closed sets. Consequently, the unit $\eta : X \to F$, being the diagonal mapping, is a homeomorphic embedding.

(ii) The set $\eta(X)$ is generating for F by the construction of η and F. To show that it is a free set, take pairwise distinct points $x_1, \dots, x_n \in \eta(X)$ and scalars $r_1, \dots, r_n \in R$ such that $\sum_{i=1}^{n} r_i x_i = 0_F$. Suppose, for contradiction, that $r_k \neq 0_R$ for some integer k satisfying $1 \leq k \leq n$. Since X is $\mathscr{M}$-regular, so is $\eta(X)$ by (i). Thus, there exist $M \in \mathscr{M}$ and $f \in C(\eta(X), M)$ such that $r_k f(x_k) \neq 0_M$ and $f(x_j) = 0_M$ for all $j \neq k$.

Since F is the free module over X, there exists a unique continuous R-homomorphism $\widehat{f} : F \to M$ that extends $f \circ \eta$. Now

$$0_M = \widehat{f}(0_F) = \widehat{f}\left(\sum_{j=1}^{n} r_j x_j\right) = \sum_{j=1}^{n} r_j f(x_j) = r_k f(x_k) \neq 0_M$$

gives a contradiction.

(iii) Since $\eta(X)$ is a free basis of F, the support function $\varphi_{\eta(X)}$ is well defined. Finally, observe that since $\eta(X)$ is $\mathscr{M}$-regular and every $f \in C(X, M)$ can be extended to a continuous R-homomorphism for all $M \in \mathscr{M}$, conditions of Proposition 5.6 are satisfied (with F at the place of M and $\eta(X)$ at the place of X). Item (iii) follows. □

6. Applications to free topological modules

Theorem 6.1. *Let $\mathscr{M}$ be an R-closed class of topological modules. Assume that the closed unit interval $[0,1]$ is $\mathscr{M}$-Hausdorff and the free topological module $F(Z)$ over Z in $\mathscr{M}$ has countable network whenever Z has countable weight. If X, Y are Tychonoff spaces and $F(X)$, $F(Y)$ are topologically R-isomorphic free topological modules over X and Y in $\mathscr{M}$ respectively, then $\dim X = \dim Y$.*

Proof: Let $\mathbf{T}$ be the category of all Tychonoff spaces and for every $X \in Ob(\mathbf{T})$ let $F(X)$ denote the free topological module over X in $\mathscr{M}$.

Let $\mathbf{M}$ be the category having as objects the class $\mathscr{M}$ and as morphisms all continuous R-homomorphisms between objects of $\mathbf{M}$. Since the closed unit interval is $\mathscr{M}$-Hausdorff, every $X \in Ob(\mathbf{T})$ is $\mathscr{M}$-regular by Lemma 5.5. Therefore, Proposition 5.7 verifies that the functor $F : \mathbf{T} \to \mathbf{M}$ that assigns to each $X \in Ob(\mathbf{T})$ the free topological module $F(X)$ and to each $f \in Mor(\mathbf{T})$ the unique extension $F(f) \in Mor(\mathbf{M})$ of f satisfies the conditions of Theorem 4.2. Consequently, if $F(X)$ and $F(Y)$ are topologically R-isomorphic Tychonoff spaces, then $\dim X = \dim Y$. □

Corollary 6.2. *Let R be a topological ring with a countable network, $\mathscr{M}$ be an R-closed class of topological modules, and assume that the closed unit interval $[0,1]$ is $\mathscr{M}$-Hausdorff. If X, Y are Tychonoff spaces and $F(X)$, $F(Y)$ are topologically R-isomorphic free topological modules over X and Y in $\mathscr{M}$ respectively, then $\dim X = \dim Y$.*

Proof: Let Z be a space with a countable weight, $F(Z)$ the free topological module over Z in $\mathscr{M}$, and η_Z a unit of $F(Z)$. Since network weight is preserved by continuous mappings, $\eta_Z(Z)$ has a countable network. Since $F(Z)$ is generated by $\eta_Z(Z)$ (see the proof of Proposition 5.7), and R has a countable network, $F(Z)$ has a countable network. It remains to apply Theorem 6.1. □

Since every abelian topological group can be viewed as a topological $\mathbb{Z}$-module, where $\mathbb{Z}$ is taken with the discrete topology, we immediately obtain the following result (for a definition of the free abelian topological group over a space X in a given class of topological groups see [12]).

Corollary 6.3. *Let $\mathscr{G}$ be a $\mathbb{Z}$-closed class of abelian topological groups, and assume that the closed unit interval $[0,1]$ is $\mathscr{G}$-Hausdorff. If X, Y are Tychonoff spaces and $F(X)$, $F(Y)$ are topologically isomorphic free abelian topological groups over X and Y in $\mathscr{G}$ respectively, then* $\dim X = \dim Y$.

Recall that a topological group is called *precompact* if it is a subgroup of some compact topological group. The free topological group over a space X in the class of all (abelian) precompact groups is then called the free precompact (abelian) group over X.

Corollary 6.4. *If Tychonoff spaces X and Y have topologically isomorphic their free precompact abelian groups, then* $\dim X = \dim Y$.

Proof: Obviously, the class $\mathscr{P}$ of all precompact abelian groups is closed under taking arbitrary products and (topological) subgroups. In order to apply Corollary 6.3, we have to show that the closed unit interval $[0,1]$ is $\mathscr{P}$-Hausdorff. The group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is (pre)compact, abelian and pathwise connected. Let t be an arbitrary non-torsion element of $\mathbb{T}$, and x and y be two distinct points of $[0,1]$. Since $\mathbb{T}$ is pathwise connected, there exists a continuous function $f : [0,1] \to \mathbb{T}$ such that $f(x) = t$ and $f(y) = 0_{\mathbb{T}}$. Since t is non-torsion, we have $rf(x) \neq 0_{\mathbb{T}}$ for every $r \in \mathbb{Z} \setminus \{0\}$. Thus, the closed unit interval $[0,1]$ is $\mathscr{P}$-Hausdorff. □

Corollary 6.3 can be further generalized for some non-abelian free topological groups. It was proved in [12, Theorem 9.9] that if $\mathscr{G}' \subseteq \mathscr{G}$ are two classes of topological groups, and X and Y topological spaces such that the free topological groups over X and Y in $\mathscr{G}$ are topologically isomorphic, then the free topological groups over X and Y in $\mathscr{G}'$ are topologically isomorphic as well. This together with Corollary 6.3 gives the following result.

Corollary 6.5. *Let $\mathscr{G}'$ be a $\mathbb{Z}$-closed class of abelian topological groups, and assume that the closed unit interval $[0,1]$ is $\mathscr{G}'$-Hausdorff. Suppose that $\mathscr{G}$ is some class of topological groups containing $\mathscr{G}'$.*

If X, Y are Tychonoff spaces and $F(X)$, $F(Y)$ are topologically isomorphic free topological groups over X and Y in $\mathscr{G}$ respectively, then $\dim X = \dim Y$.

Remark 6.6. Let us note that Theorem 4.2 can be stated in a more general way, where the R-module structure can be replaced by a more general algebraic structure that includes free groups as well. Stronger version of Corollary 6.5 would then follow from such a generalization. However, this topic exceeds the scope of this paper, since we are mostly interested in applications to G-equivalence.

7. $\mathrm{Hom}_p(C_p(X,G),G)$ as a free module

For a topological group H we denote by $\mathrm{Hom}(G,H)$ the subset of $C(G,H)$ consisting of all continuous group homomorphisms. It is well known and easy to observe that if H is abelian, then $\mathrm{Hom}(G,H)$ is a subgroup of $C(G,H)$.

Let G be an abelian topological group. For $f, g \in C(G,G)$ define the multiplication $fg \in C(G,G)$ by $fg(x) = f \circ g(x)$ for all $x \in G$. The group $\mathrm{Hom}(G,G)$ together with such multiplication forms a ring that is called the *endomorphism ring* of G and denoted by $\mathrm{End}(G)$. If not stated otherwise, we will consider $\mathrm{End}(G)$ with the discrete topology.

For topological groups G and H, we denote by $\mathrm{Hom}_p(G,H)$ the set $\mathrm{Hom}(G,H)$ equipped with the topology of pointwise convergence (so that $\mathrm{Hom}_p(G,H)$ becomes the topological subgroup of $C_p(G,H)$).

Recall that every abelian topological group G can be naturally considered as a topological $\mathrm{End}(G)$-module. Indeed, for $r \in \mathrm{End}(G)$ and $g \in G$ the scalar multiplication of r and g is defined by $rg = r(g)$.

Similarly, if X is a set, then G^X, being the product of topological $\mathrm{End}(G)$-modules, is itself a topological $\mathrm{End}(G)$-module. Here the scalar multiplication for $r \in \mathrm{End}(G)$ and $f \in G^X$ is defined pointwisely; that is, $rf(x) = r(f(x))$ for all $x \in X$.

Lemma 7.1. *Let G be an abelian topological group and X a topological space. Then $C_p(X,G)$ is an* $\mathrm{End}(G)$*-submodule of G^X, and* $\mathrm{Hom}_p(C_p(X,G),G)$ *is an* $\mathrm{End}(G)$*-submodule of $G^{C_p(X,G)}$.*

Definition 7.2. Let G be an abelian topological group, X a set, and H a subspace of G^X.

(i) By $\pi_B : G^X \to G^B$, where $B \subseteq X$, we denote the projection.
(ii) We define $\psi_H : X \to C(H,G)$ by $\psi_H(x) = \pi_{\{x\}} \restriction_H$ for $x \in X$.

If M and N are topological R-modules, we denote by $\mathrm{Hom}_R(M,N)$ the subgroup of $\mathrm{Hom}(M,N)$ consisting of all continuous R-homomorphisms from M to N.

Lemma 7.3. *Let G be an abelian topological group, X a set and H an* $\mathrm{End}(G)$*-submodule of G^X. Then π_B is a continuous* $\mathrm{End}(G)$*-homomorphism for every $B \subseteq X$. Consequently, $\pi_B \restriction_H$ is a continuous* $\mathrm{End}(G)$*-homomorphism for all $B \subseteq X$. In particular, $\psi_H(X) \subseteq \mathrm{Hom}_{\mathrm{End}(G)}(H,G)$.*

Given an R-module M and its subset X, we denote by $\langle X \rangle_R$ the R-submodule of M generated by X.

Proposition 7.4. *Let G be an abelian topological group and X a space. Put $\eta = \psi_{C_p(X,G)}$, and consider the* $\mathrm{End}(G)$*-submodule $M = \langle \eta(X) \rangle_{\mathrm{End}(G)}$ of the* $\mathrm{End}(G)$*-module* $\mathrm{Hom}_p(C_p(X,G),G)$*. Then M is the free topological module over X in the class $\{G\}_{\mathrm{End}(G)}$ of topological* $\mathrm{End}(G)$*-modules, and η is its unit.*

Proof: It suffices to prove that

(i) $M \in \{G\}_{\mathrm{End}(G)}$, and
(ii) for every $H \in \{G\}_{\mathrm{End}(G)}$ and each $f \in C(X,H)$ there exists a unique continuous $\mathrm{End}(G)$-homomorphism $\bar{f} : M \to H$ such that $f = \bar{f} \circ \eta$.

Since M is an $\mathrm{End}(G)$-submodule of $\mathrm{Hom}_p(C_p(X,G),G)$ and the latter module is an $\mathrm{End}(G)$-submodule of $G^{C(X,G)}$ by Lemma 7.1, item (i) follows from the fact that $\{G\}_{\mathrm{End}(G)}$ is $\mathrm{End}(G)$-closed.

To prove (ii), fix $H \in \{G\}_{\mathrm{End}(G)}$ and $f \in C(X,H)$. Since $H \in \{G\}_{\mathrm{End}(G)}$, there exists a set A such that H is an $\mathrm{End}(G)$-submodule of G^A. For every $a \in A$ put $f_a = \pi_a \circ f$, where $\pi_a : G^A \to G$ is the projection on ath coordinate. Clearly, $f_a \in C_p(X,G)$ for all $a \in A$, and $f = \triangle\{f_a : a \in A\}$ is the diagonal product. For each $a \in A$ define $\bar{f}_a : M \to G$ by $\bar{f}_a(\mu) = \mu(f_a)$ for every $\mu \in M$. That is, $\bar{f}_a$ is the projection at f_a. Thus, by Lemma 7.3 (with M instead of H and $C(X,G)$ instead of X), $\bar{f}_a \in \mathrm{Hom}_{\mathrm{End}(G)}(M,G)$.

By the definition of η we have

$$f_a(x) = \pi_{\{x\}}(f_a) = \bar{f}_a(\pi_{\{x\}}) = \bar{f}_a(\eta(x)) = \bar{f}_a \circ \eta(x)$$

for every $x \in X$. Consequently, $\bar{f} = \triangle\{\bar{f}_a : a \in A\}$ is a continuous $\mathrm{End}(G)$-homomorphism from M to G^A such that $f = \bar{f} \circ \eta$. Since H is an $\mathrm{End}(G)$-module, $M = \langle \eta(X) \rangle_{\mathrm{End}(G)}$ and $\bar{f}(\eta(X)) \subseteq H$, it follows that $\bar{f}$ is the unique continuous $\mathrm{End}(G)$-homomorphism satisfying $f = \bar{f} \circ \eta$ and $\bar{f}(M) \subseteq H$. □

Lemma 7.5. *Let G be an abelian topological group, B a finite set and $\nu : G^B \to G$ a continuous homomorphism. Then for every $b \in B$ there exists a unique $r_b \in \mathrm{End}(G)$ such that $\nu = \sum_{b \in B} r_b \circ \pi_b$, where $\pi_b : G^B \to G$ is the projection defined by $\pi_b(g) = g(b)$ for $b \in B$.*

Proof: For $a \in B$ let $\theta_a : G \to G^B$ be the unique homomorphism satisfying $\pi_a \circ \theta_a(g) = g$ and $\pi_b \circ \theta_a(g) = 0$ for every $g \in G$ and $b \in B \setminus \{a\}$. Then the endomorphisms $r_b = \nu \circ \theta_b \in \mathrm{End}(G)$ $(b \in B)$ are as required. □

Definition 7.6. Let X be a set, G a topological group, H a subgroup of G^X and $\mu \in \mathrm{Hom}(H,G)$.

(i) We say that $B \subseteq X$ is a *supporting set for* μ, or B *supports* μ, provided that $f(B) \subseteq \{e\}$ implies $\mu(f) = e$ for every $f \in H$.
(ii) By $S(\mu)$ we denote the set $\{B \subseteq X : B \text{ supports } \mu\}$.
(iii) We say that μ is *finitely supported* provided there exists some finite $K \in S(\mu)$.

Lemma 7.7. *Let X be a set, G a topological group, H a subgroup of G^X, $\mu \in \mathrm{Hom}(H,G)$ and $B \in S(\mu)$. If $f, g \in H$ and $f \restriction_B = g \restriction_B$, then $\mu(f) = \mu(g)$.*

Proof: Since $(fg^{-1})(B) \subseteq \{e\}$, it follows that $\mu(fg^{-1}) = e$. Consequently, $\mu(f) = \mu(fg^{-1}g) = \mu(fg^{-1})\mu(g) = \mu(g)$. □

Proposition 7.8. *Let G be an abelian topological group, X a set and H a subgroup of G^X such that $\pi_K \restriction_H$ is surjective for every finite set $K \subseteq X$. Then $\langle \psi_H(X) \rangle_{\mathrm{End}(G)} = \mathrm{Hom}(H,G)$ if and only if every $\mu \in \mathrm{Hom}(H,G)$ is finitely supported.*

Proof: We start with the "only if" part. Take $\mu \in \langle \psi_H(X) \rangle_{\mathrm{End}(G)} = \mathrm{Hom}(H,G)$. Then $\mu = \sum_{i=1}^n r_i \psi_H(x_i)$ for some $n \in \mathbb{N}$, $x_1, \dots, x_n \in X$ and $r_1, \dots, r_n \in \mathrm{End}(G)$. Clearly, $\{x_1, \dots, x_n\} \in S(\mu)$, and thus μ is finitely supported.

To prove the "if" part, take $\mu \in \mathrm{Hom}(H, G)$ and a finite set $B \in S(\mu)$. By our assumption on H, $\chi = \pi_B \restriction_H$ is surjective, so for every $g \in G^B$ we can chose $f_g \in \chi^{-1}(g)$. Define $\nu : G^B \to G$ by $\nu(g) = \mu(f_g)$ for $g \in G^B$. By Lemma 7.7, $\mu(f) = \mu(f_g)$ whenever $f \in H$ and $\chi(f) = \chi(f_g)$, so ν is well-defined. A straightforward verification shows that $\nu : G^B \to G$ is a continuous homomorphism. Clearly, $\mu = \nu \circ \chi$. For $b \in B$ let $r_b \in \mathrm{End}(G)$ be as in the conclusion of Lemma 7.5. Then $\mu = \sum_{b \in B} r_b \circ \pi_b \circ \chi$. Since $\pi_b \circ \chi = \psi_H(b)$, we get $\mu = \sum_{b \in B} r_b \circ \psi_H(b)$. Thus $\mu \in \langle \psi_H(X) \rangle_{\mathrm{End}(G)}$. □

We will need the notion of $G^{\star\star}$-regularity from [12]. Given a topological group G, a topological space X is called $G^{\star\star}$*-regular* provided that, whenever $g \in G$, $x \in X$ and F is a closed subset of X satisfying $x \notin F$, there exists $f \in C_p(X, G)$ such that $f(x) = g$ and $f(F) \subseteq \{e\}$.

Lemma 7.9. *Let G be a topological group and X a $G^{\star\star}$-regular space. Then $\pi_K \restriction_{C_p(X,G)}$ is surjective for every finite set $K \subseteq X$. In particular, $C_p(X, G)$ is dense in G^X.*

Theorem 7.10. *Let G be an abelian topological group, and X a $G^{\star\star}$-regular space such that every $\mu \in \mathrm{Hom}_p(C_p(X, G), G)$ is finitely supported. Then $\mathrm{Hom}_p(C_p(X, G), G)$ is the free topological module over X in the class $\{G\}_{\mathrm{End}(G)}$.*

Proof: Since X is $G^{\star\star}$-regular, the conclusion of Lemma 7.9 shows that the assumptions of Proposition 7.8 are satisfied. Therefore, $\mathrm{Hom}_p(C_p(X, G), G) = \langle \psi_{C_p(X,G)}(X) \rangle_{\mathrm{End}(G)}$. Now the conclusion of our theorem follows from Proposition 7.4. □

8. Application to G-equivalence

Theorem 8.1. *Let G be a nontrivial abelian separable metrizable pathwise connected group. Assume that every $\mu \in \mathrm{Hom}_p(C_p(X, G), G)$ is finitely supported for every Tychonoff space X. Then G-equivalence preserves the covering dimension.*

Proof: Let X be a Tychonoff space. Since G is pathwise connected and nontrivial, X is $G^{\star\star}$-regular [12, Proposition 2.3(i)]. By Theorem 7.10, $\mathrm{Hom}_p(C_p(X, G), G)$ is the free topological module over X in the class $\mathscr{M} = \{G\}_{\mathrm{End}(G)}$ of topological $\mathrm{End}(G)$-modules.

Clearly, $\mathscr{M}$ is $\mathrm{End}(G)$-closed. Since G is pathwise connected, the closed unit interval is $\mathscr{M}$-Hausdorff.

Suppose that Z is a space with a countable base. It is well known that $nw(C_p(X, Y)) \leq nw(X)$ for every space Y with a countable base. Applying this fact twice, we conclude that both $C_p(Z, G)$ and $C_p(C_p(Z, G), G)$ have a countable network. Since $\mathrm{Hom}_p(C_p(Z, G), G)$ is a subspace of $C_p(C_p(Z, G), G)$, it follows that $\mathrm{Hom}_p(C_p(Z, G), G)$ has a countable network.

From Theorem 6.1 we conclude that $\dim X = \dim Y$ whenever X and Y are Tychonoff spaces such that $\mathrm{Hom}_p(C_p(X, G), G)$ is topologically $\mathrm{End}(G)$-isomorphic to $\mathrm{Hom}_p(C_p(Y,G),G)$. Finally, it remains to observe that if X and Y are G-equivalent, that is, $C_p(X,G)$ and $C_p(Y,G)$ are topologically isomorphic, then $\mathrm{Hom}_p(C_p(X,G),G)$ and $\mathrm{Hom}_p(C_p(Y,G),G)$ are topologically $\mathrm{End}(G)$-isomorphic. □

Next we will present some classes of topological groups G for which the condition that every $\mu \in \mathrm{Hom}_p(C_p(X,G),G)$ is finitely supported for every Tychonoff space X is satisfied. The following definition comes from [8].

Definition 8.2. We say that a topological group G is *self-slender* provided that for every set X and each continuous homomorphism $\mu : G^X \to G$ there exist a finite set $K \subseteq X$ and a continuous homomorphism $\varphi : G^K \to G$ such that $\mu = \varphi \circ \pi_K$, where $\pi_K : G^X \to G^K$ is the projection.

Lemma 8.3. *A topological group G is self-slender if and only if each continuous homomorphism $\phi : G^X \to G$ is finitely supported for every set X.*

Proof: Fix a set X and a continuous homomorphism $\phi : G^X \to G$. If G is self-slender, then there exist some finite $K \subseteq X$ and a continuous homomorphism $\varphi : G^K \to G$ such that $\phi = \varphi \circ \pi_K$. Obviously, $K \in S(\phi)$ and thus ϕ is finitely supported. On the other hand, if ϕ is finitely supported, then there is some finite $K \in S(\phi)$. For every $f \in G^K$ pick $f' \in G^X$ such that $\pi_K(f') = f$, and define $\varphi(f) = \phi(f')$. Since $K \in S(\phi)$, it follows from Lemma 7.7 that $\varphi : G^K \to G$ is a well-defined homomorphism. Obviously, $\phi = \varphi \circ \pi_K$. It remains to observe that φ is continuous. For every $V \subseteq G$ we have $\varphi^{-1}(V) = \pi_K(\phi^{-1}(V))$. Consequently, the continuity of φ follows from the continuity of ϕ and from the fact that π_K is an open mapping. □

Recall that a topological group is called NSS provided that there exists a neighborhood of the identity containing no nontrivial subgroup.

Lemma 8.4. *Let G be a topological group, X be a set and H be a subgroup of G^X such that one of the following conditions hold:*

(i) *G is NSS,*

(ii) *the Raikov completion $\widehat{G}$ of G is self-slender and H is dense in G^X.*

Then every $\mu \in \mathrm{Hom}(H,G)$ is finitely supported.

Proof: Assume that (i) holds.

Fix $\mu \in \mathrm{Hom}(H,G)$ and a neighborhood U of the identity e in G containing no nontrivial subgroup. Since μ is continuous, there exist a finite set $K \subseteq X$ and an open neighborhood $V \subseteq G$ of e such that for $A = \{f \in H : f(K) \subseteq V\}$ we have $\mu(A) \subseteq U$. We claim that $K \in S(\mu)$. Indeed, if $g \in H$ satisfies $g(K) \subseteq \{e\}$, then $g^z \in A$, and consequently, $\mu(g)^z \subseteq U$ for every $z \in \mathbb{Z}$. In other words, U contains the subgroup generated by $\mu(g)$. Therefore $\mu(g) = e$. It follows that $K \in S(\mu)$.

Suppose now that (ii) holds. Since H is dense in G^X, it follows that $\widehat{H} = \widehat{G}^X$. Pick $\mu \in \mathrm{Hom}(H,G)$ and take the unique continuous homomorphism $\widehat{\mu} : \widehat{G}^X \to \widehat{G}$ extending μ. Then $\widehat{\mu}$ is finitely supported by Lemma 8.3. Obviously, $\mu = \widehat{\mu} \restriction_H$ is finitely supported as well. □

Theorem 8.5. *Let G be a nontrivial abelian separable metrizable pathwise connected group. Assume that G is either an NSS group or the completion $\widehat{G}$ of G is self-slender. Then G-equivalence preserves the covering dimension.*

Proof: If G is NSS, then every $\mu \in \mathrm{Hom}_p(C_p(X,G),G)$ is finitely supported for every Tychonoff space X by Lemma 8.4. If the completion of G is self-slender, then $C_p(X,G)$ is dense in G^X for every Tychonoff space X by Lemma 7.9. Here we are using the fact that every Tychonoff space is $G^{\star\star}$-regular because G is pathwise connected. Therefore, by Lemma 8.4 also in this case every $\mu \in \mathrm{Hom}_p(C_p(X,G),G)$ is finitely supported for every Tychonoff space X. Thus, the conclusion follows from Theorem 8.1. □

Since $\mathbb{R}$ is an NSS, pathwise connected, separable metrizable, abelian group, Theorem 8.5 implies the following result of Pestov [10].

Corollary 8.6 (Pestov)**.** *The covering dimension is preserved by l-equivalence.*

9. Further generalizations

If H is a subgroup of a topological group G, then $C_p(X,H)$ is a subgroup of $C_p(X,G)$ for every space X. If we succeed to prove that each topological isomorphism between $C_p(X,G)$ and $C_p(Y,G)$ must map $C_p(X,H)$ onto $C_p(Y,H)$, then this would mean that G-equivalence implies H-equivalence. This usually happens when H is a "significant" subgroup of G, for example, its center or (arcwise) connected component.

Recall that the set $Z(G) = \{g \in G : gh = hg \text{ for every } h \in G\}$ is called the *center* of a group G. Obviously, $Z(G)$ is an abelian subgroup of G.

Proposition 9.1. *Let G be a topological group. Then G-equivalence implies $Z(G)$-equivalence.*

Proof: Assume that spaces X and Y are G-equivalent, and let $\varphi : C_p(X,G) \to C_p(Y,G)$ be a topological isomorphism. One can easily check that $Z(C_p(X,G)) = C_p(X,Z(G))$ and $Z(C_p(Y,G)) = C_p(Y,Z(G))$. Since an isomorphism between topological groups maps the center onto the center, we must have $\varphi(C_p(X,Z(G))) = C_p(Y,Z(G))$. Thus, X and Y are $Z(G)$-equivalent. □

Given a topological group G, we denote by $c(G)$ the connected component of the identity of G and by $c_0(G)$ the pathwise connected component of the identity of G. Recall that both, $c(G)$ and $c_0(G)$ are topological subgroups of G.

If X is a topological space and $A \subseteq X$, then by $\mathrm{Cl}_X(A)$ we will denote the closure of A in X.

Proposition 9.2. *Let X be a space and G a topological group. Then $\mathrm{Cl}_{C_p(X,G)}(c_0(C_p(X,G))) = C_p(X,\mathrm{Cl}_G(c_0(G)))$.*

Proof: First let us prove the inclusion "$\subseteq$". Pick $f \in \mathrm{Cl}_{C_p(X,G)}(c_0(C_p(X,G)))$ and $x \in X$. Let $\pi : C_p(X,G) \to G$ be the projection at x. Since a continuous image of a pathwise connected space is pathwise connected and π is continuous, it follows that $\pi(c_0(C_p(X,G))) \subseteq c_0(G)$ and consequently, $\pi(\mathrm{Cl}_{C_p(X,G)}(c_0(C_p(X,G)))) \subseteq \mathrm{Cl}_G(c_0(G))$. Thus, $f \in C_p(X, \mathrm{Cl}_G(c_0(G)))$.

To show the reverse inclusion "$\supseteq$", fix $f \in C_p(X, \mathrm{Cl}_G(c_0(G)))$, and let O be an open neighborhood of f in $C_p(X, \mathrm{Cl}_G(c_0(G)))$. We must show that $O \cap c_0(C_p(X,G)) \neq \emptyset$. There exist $n \in \mathbb{N}$, pairwise distinct elements $x_1, \ldots, x_n \in X$ and a sequence $U_1, \ldots, U_n$ of open subsets of $\mathrm{Cl}_G(c_0(G))$ with

$$f \in \bigcap_{i=1}^{n} W(x_i, U_i) \subseteq O, \tag{3}$$

where $W(x_i, U_i) = \{g \in C(X, \mathrm{Cl}_G(c_0(G))) : g(x_i) \in U_i\}$. Fix an integer i satisfying $1 \le i \le n$. From (3) it follows that $U_i \subseteq \mathrm{Cl}_G(c_0(G))$ is nonempty. Thus we can choose $g_i \in U_i \cap c_0(G)$. Therefore, there exists a continuous map $\varphi_i : [0,1] \to G$ such that

$$\varphi_i(0) = e \text{ and } \varphi_i(1) = g_i. \tag{4}$$

Let $\psi_i : X \to [0,1]$ be a continuous function such that

$$\psi_i(x_i) = 1 \text{ and } \psi_i(x_j) = 0 \text{ for every } j \in \{1, \ldots, n\} \text{ with } j \neq i. \tag{5}$$

Let $\varphi : [0,1] \to C_p(X,G)$ be the map which assigns to every $t \in [0,1]$ the function $\varphi(t) \in C_p(X,G)$ defined by

$$\varphi(t)(x) = \prod_{i=1}^{n} \varphi_i(t\psi_i(x)) \text{ for } x \in X. \tag{6}$$

From (4) and (6) we conclude that $\varphi(0)$ is the identity element of $C_p(X,G)$. One can easily check that φ is continuous, so $\varphi([0,1])$ is a path between $\varphi(0) = e$ and $h = \varphi(1) \in C_p(X,G)$. Therefore, $h \in c_0(C_p(X,G))$. Finally, from (4), (5) and (6) we conclude that $h(x_i) = g_i \in U_i$ for every integer i with $1 \le i \le n$. That is, $h \in \bigcap_{i=1}^{n} W(x_i, U_i)$. Combining this with (3), we conclude that $h \in O$. Hence $h \in O \cap c_0(C_p(X,G)) \neq \emptyset$. $\square$

Since a closure of a subgroup of a topological group G is again a subgroup of G, it follows that $\mathrm{Cl}_G(c_0(G))$ is a subgroup of G.

Corollary 9.3. *G-equivalence implies $\mathrm{Cl}_G(c_0(G))$-equivalence for every topological group G.*

Proof: Assume that spaces X and Y are G-equivalent, and let $\varphi : C_p(X,G) \to C_p(Y,G)$ be a topological isomorphism. Since a topological isomorphism between topological groups maps the pathwise connected component onto the pathwise connected component, we must have

$$\varphi(\mathrm{Cl}_{C_p(X,G)}(c_0(C_p(X,G)))) = \mathrm{Cl}_{C_p(Y,G)}(c_0(C_p(Y,G))).$$

From this and Proposition 9.2 we obtain that

$$\begin{aligned} C_p(X, \mathrm{Cl}_G(c_0(G))) &= \mathrm{Cl}_{C_p(X,G)}(c_0(C_p(X,G))) \\ &\cong \mathrm{Cl}_{C_p(Y,G)}(c_0(C_p(Y,G))) = C_p(Y, \mathrm{Cl}_G(c_0(G))). \end{aligned}$$

Thus, X and Y are $\mathrm{Cl}_G(c_0(G))$-equivalent. □

Corollary 9.4. *Let G be a topological group such that $c_0(G)$ is dense in $c(G)$. Then G-equivalence implies $c(G)$-equivalence.*

Proof: Since $c(G)$ is closed in G, from our assumption we get $c(G) = \mathrm{Cl}_G(c_0(G))$. It remains to apply Corollary 9.3. □

Theorem 9.5. *Let G be a topological group. Assume that $c_0(G)$ is closed in G and $c_0(G) = H^\kappa$, where κ is an arbitrary nonzero cardinal, and H is a nontrivial abelian separable metrizable group that is either NSS, or has self-slender completion. Then G-equivalence preserves the covering dimension.*

Proof: It follows from Corollary 9.3 that G-equivalence implies $c_0(G)$-equivalence (that is, H^κ-equivalence). It follows from [12, Corollary 2.14] that H^κ-equivalence preserves the covering dimension if and only if H-equivalence does. Since H^κ is pathwise connected, so is H, and thus H-equivalence preserves dimension by Theorem 8.5. □

Remark 9.6. Examples of compact abelian self-slender groups that are not NSS can be found in [2]. Therefore, conditions from items (i) and (ii) of Lemma 8.4, as well as the corresponding assumptions in Corollary 8.5 and Theorem 9.5, cannot be "merged" into a single general statement.

Remark 9.7. In [12] we presented some classes $\mathscr{G}, \mathscr{H}$ of topological groups for which G-equivalence implies H-equivalence whenever $G \in \mathscr{G}$ and $H \in \mathscr{H}$. Using these results as well as Proposition 9.1 together with Theorem 9.5 can result into extending the class of topological groups G for which it is known that G-equivalence preserves the covering dimension.

10. Final remarks

Given a space X (no separation axioms are assumed) and a topological group G, we say that X is *G-Tychonoff* provided that it can be embedded in some power of G. Consider the image of X under the diagonal product r_G of all maps $f \in C(X,G)$. A straightforward check shows that the map $\Psi : C_p(r_G(X), G) \to C_p(X,G)$ defined by $\Psi(f) = f \circ r_G$ is a topological isomorphism. In particular, X and $r_G(X)$ are G-equivalent. Obviously, $r_G(X)$ is always G-Tychonoff while X is G-Tychonoff if and only if it is homeomorphic to $r_G(X)$. A reader familiar with the theory of categories can readily recognize that r_G is a reflection of the category of all topological spaces in the category of all G-Tychonoff spaces.

Let us give a simple example showing that the requirement on H to be nontrivial in Theorem 9.5 cannot be omitted.

Example 10.1. Let G be the discrete two-point group $\mathbb{Z}(2)$. Then G is separable metric and NSS but G-equivalence does not preserve dimension within the class of compact metric spaces.

Indeed, for the closed unit interval I its reflection $r_G(I)$ is a singleton, because every continuous $\mathbb{Z}(2)$-valued function on I is constant. Since $r_G(I)$ is G-equivalent to I the conclusion follows.

The previous extreme example shows that in order to obtain meaningful theorems about preservation of dimension by G-equivalence we have to restrict ourselves only to the class of G-Tychonoff spaces. Therefore, when investigating the preservation of dimension by G-equivalence within the class of all Tychonoff spaces it is reasonable to require all Tychonoff spaces to be G-Tychonoff which happens if and only if I is G-Tychonoff if and only if $c_0(G)$ is nontrivial.

Question 10.2. (i) Can "separable metrizable" be omitted in Theorem 8.5 and Theorem 9.5?
(ii) Does G-equivalence preserve the covering dimension for every NSS abelian group G within the class of G-Tychonoff spaces?
(iii) Does G-equivalence preserve the covering dimension for every compact self-slender abelian group G within the class of G-Tychonoff spaces?
(iv) Can the assumption that $c_0(G)$ is closed be omitted in Theorem 9.5?

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