On computation of minimal free resolutions over solvable polynomial algebras

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Abstract. Let $A = K[a_1, \dots, a_n]$ be a (noncommutative) solvable polynomial algebra over a field K in the sense of A. Kandri-Rody and V. Weispfenning [*Non-commutative Gröbner bases in algebras of solvable type*, J. Symbolic Comput. **9** (1990), 1–26]. This paper presents a comprehensive study on the computation of minimal free resolutions of modules over A in the following two cases: (1) $A = \bigoplus_{p\in\mathbb{N}} A_p$ is an $\mathbb{N}$-graded algebra with the degree-0 homogeneous part $A_0 = K$; (2) A is an $\mathbb{N}$-filtered algebra with the filtration $\{F_pA\}_{p\in\mathbb{N}}$ determined by a positive-degree function on A.

Keywords: solvable polynomial algebra; Gröbner basis; minimal free resolution

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1. Introduction and preliminaries

1.1 Introduction. Since the late 1980s, the Gröbner basis theory for commutative polynomial algebras and their modules (cf. [Bu1], [Bu 2], [Sch], [BW], [AL2], [Fröb], [KR1], [KR2]) has been successfully generalized to (noncommutative) solvable polynomial algebras and their modules (cf. [AL1], [Gal], [K-RW], [Kr2], [LW], [Li1], [Lev]). It is now well known that the class of solvable polynomial algebras covers numerous significant algebras such as enveloping algebras of Lie algebras, Weyl algebras (including algebras of partial differential operators with polynomial coefficients over a field of characteristic 0), more generally a large number of operator algebras, iterated Ore extensions, and many quantum (quantized) algebras. In particular, after [K-RW] successfully established a noncommutative version of the Buchberger's criterion and a noncommutative version of Buchberger algorithm for computing (one-sided, two-sided) Gröbner bases of (one-sided, two-sided) ideals in general solvable polynomial algebras (see the module versions presented as Theorem 1.3.2 and **Algorithm 1** in the current paper), the noncommutative version of Buchberger algorithm has been implemented in some well-developed computer algebra systems, such as Modula-2 [KP] and Singular [DGPS]. Based on such an *effective Gröbner basis theory* and the fact that *every solvable polynomial algebra is a* (*left and right*) *Noetherian domain of finite global homological dimension* (see Theorem 1.2.3 and Theorem 2.3.3 in the current paper), in this paper we present a comprehensive study on the computation of minimal graded free resolutions and minimal filtered free resolutions over $\mathbb{N}$-graded solvable polynomial algebras and $\mathbb{N}$-filtered solvable polynomial algebras respectively. More precisely, after this preliminary Section 1, there are two sections devoted to the goal of this paper. In Section 2, we show that the methods and algorithms, developed in [CDNR], [KR2] for computing minimal homogeneous generating sets of graded submodules and graded quotient modules of free modules over a commutative polynomial algebra, can be adapted for computing minimal homogeneous generating sets of graded submodules and graded quotient modules of free modules over a noncommutative $\mathbb{N}$-graded solvable polynomial K-algebra A with the degree-0 homogeneous part $A_0 = K$ (where K is a field), and consequently algorithmic procedures for computing minimal finite graded free resolutions of finitely generated modules over A can be achieved. This work, in turn, makes a solid foundation for our next step of computing minimal finite filtered free resolutions over $\mathbb{N}$-filtered solvable polynomial algebras. In Section 3, we first specify the $\mathbb{N}$-filtered solvable polynomial algebras that have an $\mathbb{N}$-filtration determined by a positive-degree function $d(\ )$, and over such filtered solvable polynomial algebras we introduce filtered free modules that enable us to establish the filtered-graded transfer principle of Gröbner bases for submodules of free modules, which had not been systematically done before (e.g., comparing with the filtered-graded transfer of Gröbner bases for left ideals and applications presented in [Li1]). Secondly, over an $\mathbb{N}$-filtered solvable polynomial algebra A with respect to a positive-degree function $d(\ )$, we introduce minimal filtered free resolutions for finitely generated A-modules by introducing minimal F-bases and minimal standard bases respectively for A-modules and their submodules with respect to good filtration (note that good filtration always exists for finitely generated modules over a filtered ring, cf. [LVO]), where, as we remarked in Subsection 3.3, the standard bases are just noncommutative analogues of the classical Macaulay bases in the commutative case. We verify the appropriateness of the definitions for minimal F-bases, minimal standard bases, and minimal filtered free resolutions by showing that any two minimal F-bases, respectively any two minimal standard bases, have the same number of elements and the same number of elements of the same filtered degree, and that minimal filtered free resolutions are unique up to strict filtered isomorphism of chain complexes in the category of filtered A-modules. Comparing with classical minimal graded free resolutions (e.g. [Eis, Chapter 19], [Kr1, Chapter 3], [Li3]), this therefore means that the definition of minimal filtered free resolutions we introduced for finitely generated modules over the $\mathbb{N}$-filtered solvable polynomial algebra A is a right definition of "minimal free resolutions". Furthermore, combining the results of Section 2, we show that minimal finite filtered free resolutions can be algorithmically computed in case A has a graded monomial ordering $\prec_{gr}$. At this stage, it is worth noting that once a minimal finite filtered free resolution $\mathcal{L}_{\bullet}$ is constructed for a finitely generated A-module M, then, as a by-product, $\mathcal{L}_{\bullet}$ gives rise to a minimal graded free resolution for the associated graded module $G(M)$ and the Rees module $\widetilde{M}$, respectively (see Theorem 3.5.3).

Throughout this paper, K denotes a field, $K^* = K - \{0\}$; $\mathbb{N}$ denotes the additive monoid of all nonnegative integers, and $\mathbb{Z}$ denotes the additive group of all integers; all algebras are associative K-algebras with the multiplicative identity 1, and modules over an algebra are meant left unitary modules.

1.2 Solvable polynomial algebras. In this subsection we recall briefly some basics on solvable polynomial algebras. The main references are [AL1], [Gal], [K-RW], [Kr2], [LW], [Li1], [Li2], [Li4].

Let K be a field and let $A = K[a_1, \ldots, a_n]$ be a finitely generated K-algebra with the *minimal set of generators* $\{a_1, \ldots, a_n\}$. If, for some permutation $\tau = i_1 i_2 \cdots i_n$ of $1, 2, \ldots, n$, the set $\mathscr{B} = \{a^\alpha = a_{i_1}^{\alpha_1} \cdots a_{i_n}^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$, forms a K-basis of A, then $\mathscr{B}$ is referred to as a *PBW K-basis* of A (where the phrase "PBW K-basis" is abbreviated from the well-known *Poincaré-Birkhoff-Witt Theorem* concerning the standard K-basis of the enveloping algebra of a Lie algebra, e.g., see [Hu, p. 92]). It is clear that if A has a PBW K-basis, then we can always assume that $i_1 = 1, \ldots, i_n = n$. Thus, we make the following convention once for all.

Convention. From now on in this paper, if we say that an algebra A has the PBW K-basis $\mathscr{B}$, then it means that

$$\mathscr{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}.$$

Moreover, adopting the commonly used terminology in computational algebra, elements of $\mathscr{B}$ are referred to as *monomials* of A.

Suppose that A has the PBW K-basis $\mathscr{B}$ as presented above and that $\prec$ is a total ordering on $\mathscr{B}$. Then every nonzero element $f \in A$ has a unique expression

$$f = \sum_{j=1}^{m} \lambda_j a^{\alpha(j)}, \quad \text{where } \lambda_j \in K^*,\ a^{\alpha(j)} = a_1^{\alpha_{1j}} a_2^{\alpha_{2j}} \cdots a_n^{\alpha_{nj}} \in \mathscr{B},\ 1 \le j \le m.$$

If $a^{\alpha(1)} \prec a^{\alpha(2)} \prec \cdots \prec a^{\alpha(m)}$ in the above representation, then the *leading monomial of* f is defined as $\mathbf{LM}(f) = a^{\alpha(m)}$, the *leading coefficient of* f is defined as $\mathbf{LC}(f) = \lambda_m$, and the *leading term of* f is defined as $\mathbf{LT}(f) = \lambda_m a^{\alpha(m)}$.

1.2.1 Definition. Suppose that the K-algebra $A = K[a_1, \dots, a_n]$ has the PBW K-basis $\mathscr{B}$. Let $\prec$ be a total ordering on $\mathscr{B}$ that satisfies the following three conditions:

(1) $\prec$ is a well-ordering;

(2) for $a^\gamma, a^\alpha, a^\beta, a^\eta \in \mathscr{B}$, if $a^\alpha \prec a^\beta$ and $\mathbf{LM}(a^\gamma a^\alpha a^\eta)$, $\mathbf{LM}(a^\gamma a^\beta a^\eta) \notin K$, then $\mathbf{LM}(a^\gamma a^\alpha a^\eta) \prec \mathbf{LM}(a^\gamma a^\beta a^\eta)$;

(3) for $a^\gamma, a^\alpha, a^\beta, a^\eta \in \mathscr{B}$, if $a^\beta \ne a^\gamma$, and $a^\gamma = \mathbf{LM}(a^\alpha a^\beta a^\eta)$, then $a^\beta \prec a^\gamma$ (thereby $1 \prec a^\gamma$ for all $a^\gamma \ne 1$).

Then $\prec$ is called a *monomial ordering* on $\mathscr{B}$ (or a monomial ordering on A).

If $\prec$ is a monomial ordering on $\mathscr{B}$, then we call $(\mathscr{B}, \prec)$ an *admissible system* of A.

Remark. (i) Definition 1.2.1 is indeed borrowed from the theory of Gröbner bases for general finitely generated K-algebras, in which the algebras considered may be noncommutative, may have divisors of zero, and the K-bases used may not be a PBW basis, but with a (one-sided, two-sided) monomial ordering such algebras may theoretically have a (one-sided, two-sided) Gröbner basis theory. For more details on this topic, one may referrer to [Li2, Section 3.1 of Chapter 3 and Section 8.3 of Chapter 8]. Also, to see the essential difference between Definition 1.2.1 and the classical definition of a monomial ordering in the commutative case, one may refer to Definition 1.4.1 and the proof of Theorem 1.4.6 given in [AL2].

(ii) Note that the conditions (2) and (3) in Definition 1.2.1 mean that $\prec$ is *two-sided compatible with the multiplication operation of the algebra* A. Originally in [K-RW], the use of a (two-sided) monomial ordering $\prec$ on a solvable polynomial algebra A first guarantees that A *is a domain*, and furthermore guarantees an effective (left, right, two-sided) finite Gröbner basis theory for A (Theorem 1.2.3 below).

Note that if a K-algebra $A = K[a_1, \dots, a_n]$ has the PBW K-basis $\mathscr{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$, then for any given n-tuple $(m_1, \dots, m_n) \in \mathbb{N}^n$, a *weighted degree function* $d(\ )$ is well defined on nonzero elements of A, namely, for each $a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \in \mathscr{B}$, $d(a^\alpha) = m_1\alpha_1 + \cdots + m_n\alpha_n$, and for each nonzero $f = \sum_{i=1}^{s} \lambda_i a^{\alpha(i)} \in A$ with $\lambda_i \in K^*$ and $a^{\alpha(i)} \in \mathscr{B}$, $d(f) = \max\{d(a^{\alpha(i)}) \mid 1 \le i \le s\}$. If $d(a_i) = m_i > 0$ for $1 \le i \le n$, then $d(\ )$ is referred to as a *positive-degree function* on A.

Let $d(\ )$ be a positive-degree function on A. If $\prec$ is a monomial ordering on $\mathscr{B}$ such that for all $a^\alpha, a^\beta \in \mathscr{B}$,

$$(*) \qquad a^\alpha \prec a^\beta \ \text{ implies } \ d(a^\alpha) \le d(a^\beta),$$

then we call $\prec$ a *graded monomial ordering* with respect to $d(\ )$, and unless otherwise stated, from now on we always use $\prec_{gr}$ to denote a graded monomial ordering.

As one may see from the literature that in both the commutative and noncommutative computational algebra, the most popularly used graded monomial orderings on an algebra $A = K[a_1, \dots, a_n]$ with the PBW K-basis $\mathscr{B}$ are those graded (reverse) lexicographic orderings with respect to the degree function $d(\ )$ such that $d(a_i) = 1$, $1 \le i \le n$.

Originally, a noncommutative solvable polynomial algebra (or an algebra of solvable type) R' was defined in [K-RW] by first fixing a monomial ordering $\prec$ on the standard K-basis $\mathscr{B} = \{X_1^{\alpha_1} \cdots X_n^{\alpha_n} \mid \alpha_i \in \mathbb{N}\}$ of the commutative polynomial algebra $R = K[X_1, \dots, X_n]$ in n variables $X_1, \dots, X_n$ over a field K, and then introducing a new multiplication $*$ on R, such that certain axioms [K-RW, Axioms 1.2] are satisfied. In [LW] the definition of a solvable polynomial algebra was modified, in the formal language of associative K-algebras, as follows.

1.2.2 Definition. Suppose that the K-algebra $A = K[a_1, \dots, a_n]$ has the PBW K-basis $\mathscr{B}$ and an admissible system $(\mathscr{B}, \prec)$. If for all $a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n}$, $a^\beta = a_1^{\beta_1} \cdots a_n^{\beta_n} \in \mathscr{B}$, the following condition is satisfied:

$$\begin{aligned} a^\alpha a^\beta \ &= \ \lambda_{\alpha,\beta} a^{\alpha+\beta} + f_{\alpha,\beta}, \\ &\quad \text{where } \ \lambda_{\alpha,\beta} \in K^*, \ a^{\alpha+\beta} = a_1^{\alpha_1+\beta_1} \cdots a_n^{\alpha_n+\beta_n}, \ \text{ and} \\ &\quad f_{\alpha,\beta} \in K\text{-span } \mathscr{B} \ \text{ with } \ \mathbf{LM}(f_{\alpha,\beta}) \prec a^{\alpha+\beta} \ \text{ whenever } \ f_{\alpha,\beta} \ne 0, \end{aligned}$$

then A is said to be a *solvable polynomial algebra.*

Remark. Let $A = K[a_1, \dots, a_n]$ be a finitely generated K-algebra and $K\langle X\rangle = K\langle X_1, \dots, X_n\rangle$ the free K-algebra on $\{X_1, \dots, X_n\}$. Then it follows from [Li4] that A is a solvable polynomial algebra if and only if

(1) $A \cong K\langle X\rangle/\langle G\rangle$ in which $\langle G\rangle$ is the two-sided ideal of $K\langle X\rangle$ generated by a finite subset $G = \{g_1, \dots, g_m\}$ and, with respect to some monomial ordering $\prec_{X}$ on $K\langle X\rangle$, G is a Gröbner basis of $\langle G\rangle$ such that the set of normal monomials (mod G) gives rise to a PBW K-basis $\mathscr{B}$ for A, and
(2) there is a monomial ordering $\prec$ on $\mathscr{B}$ such that the condition on monomials given in Definition 1.2.2 is satisfied.

Thus, solvable polynomial algebras are completely determinable and constructible in a computational way.

By Definition 1.2.2 it is straightforward that if A is a solvable polynomial algebra and $f, g \in A$ with $\mathbf{LM}(f) = a^\alpha$, $\mathbf{LM}(g) = a^\beta$, then

$$(\mathbb{P}1) \qquad \mathbf{LM}(fg) = \mathbf{LM}(\mathbf{LM}(f)\mathbf{LM}(g)) = \mathbf{LM}(a^\alpha a^\beta) = a^{\alpha+\beta}.$$

We shall freely use this property in the rest of this paper without additional indication.

The results mentioned in the theorem below are summarized from [K-RW, Sections 2–5].

1.2.3 Theorem. *Let $A = K[a_1, \ldots, a_n]$ be a solvable polynomial algebra with admissible system $(\mathscr{B}, \prec)$. The following statements hold.*

(i) *A is a* (*left and right*) *Noetherian domain.*

(ii) *With respect to the given monomial ordering $\prec$ on $\mathscr{B}$, every nonzero left ideal I of A has a finite left Gröbner basis $\mathcal{G} = \{g_1, \ldots, g_t\} \subset I$ in the sense that*

- *if $f \in I$ and $f \neq 0$, then there is a $g_i \in \mathcal{G}$ such that $\mathbf{LM}(g_i)|\mathbf{LM}(f)$, i.e., there is some $a^\gamma \in \mathscr{B}$ such that $\mathbf{LM}(f) = \mathbf{LM}(a^\gamma \mathbf{LM}(g_i))$, or equivalently, with $\gamma(i_j) = (\gamma_{i_{1j}}, \gamma_{i_{2j}}, \ldots, \gamma_{i_{nj}}) \in \mathbb{N}^n$, f has a left Gröbner representation:*

$$f = \sum_{i,j} \lambda_{ij} a^{\gamma(i_j)} g_j,$$

where $\lambda_{ij} \in K^*$, $a^{\gamma(i_j)} \in \mathscr{B}$, $g_j \in \mathcal{G}$, satisfying $\mathbf{LM}(a^{\gamma(i_j)} g_j) \preceq \mathbf{LM}(f)$ for all (i, j).

(iii) *The Buchberger's criterion and Buchberger algorithm* [Bu1], [Bu2], *that is for computing a finite Gröbner basis of a finitely generated commutative polynomial ideal, has a complete noncommutative version for computing a finite left Gröbner basis of a finitely generated left ideal $I = \sum_{i=1}^m Af_i$ of A* (*see Theorem* 1.3.2 *and* **Algorithm 1** *given in the end of the next subsection, which, of course, are versions for modules*)*.*

(iv) *Similar results of* (ii) *and* (iii) *hold for right ideals and two-sided ideals of A.*

1.3 Gröbner bases for submodules of free modules. In this subsection we recall briefly some basics on left Gröbner bases for submodules of free left modules over solvable polynomial algebras. The main references are [AL2], [Eis], [KR1], [KR2], [K-RW], and [Lev].

Let $A = K[a_1, \ldots, a_n]$ be a solvable polynomial algebra with admissible system $(\mathscr{B}, \prec)$, and let $L = \bigoplus_{i=1}^s Ae_i$ be a free A-module with the A-basis $\{e_1, \ldots, e_s\}$. Then L is a Noetherian module with the K-basis

$$\mathscr{B}(e) = \{a^\alpha e_i \mid a^\alpha \in \mathscr{B},\ 1 \le i \le s\}.$$

We also call elements of $\mathscr{B}(e)$ *monomials* in L. If $\prec_e$ is a total ordering on $\mathscr{B}(e)$, and if $\xi = \sum_{j=1}^m \lambda_j a^{\alpha(j)} e_{i_j} \in L$, where $\lambda_j \in K^*$ and $\alpha(j) = (\alpha_{j_1}, \ldots, \alpha_{j_n}) \in \mathbb{N}^n$, such that $a^{\alpha(1)}e_{i_1} \prec_e a^{\alpha(2)}e_{i_2} \prec_e \cdots \prec_e a^{\alpha(m)}e_{i_m}$, then by $\mathbf{LM}(\xi)$ we denote the *leading monomial* $a^{\alpha(m)}e_{i_m}$ of ξ, by $\mathbf{LC}(\xi)$ we denote the *leading coefficient* λ_m of ξ, and by $\mathbf{LT}(\xi)$ we denote the *leading term* $\lambda_m a^{\alpha(m)}e_{i_m}$ of f.

With respect to the given monomial ordering $\prec$ on $\mathscr{B}$, a total ordering $\prec_e$ on $\mathscr{B}(e)$ is called a *left monomial ordering* if the following two conditions are satisfied:

(1) $a^\alpha e_i \prec_e a^\beta e_j$ implies $\mathbf{LM}(a^\gamma a^\alpha e_i) \prec_e \mathbf{LM}(a^\gamma a^\beta e_j)$ for all $a^\alpha e_i$, $a^\beta e_j \in \mathscr{B}(e)$, $a^\gamma \in \mathscr{B}$;

(2) $a^\beta \prec a^\beta$ implies $a^\alpha e_i \prec_e a^\beta e_i$ for all $a^\alpha, a^\beta \in \mathscr{B}$ and $1 \le i \le s$.

From the definition it is straightforward to check that every left monomial ordering $\prec_e$ on $\mathscr{B}(e)$ is a well-ordering. Moreover, if $f \in A$ with $\mathbf{LM}(f) = a^\gamma$ and $\xi \in L$ with $\mathbf{LM}(\xi) = a^\alpha e_i$, then, by referring to the property ($\mathbb{P}1$) given in Subsection 1.2, we have

$$(\mathbb{P}2) \qquad \mathbf{LM}(f\xi) = \mathbf{LM}(\mathbf{LM}(f)\mathbf{LM}(\xi)) = \mathbf{LM}(a^\gamma a^\alpha e_i) = a^{\gamma+\alpha}e_i.$$

We shall also freely use this property in the rest of this paper without additional indication.

Actually as in the commutative case (e.g. [AL2], [Eis], [KR1], [KR2]), any monomial ordering $\prec$ on $\mathscr{B}$ may induce two left monomial orderings on $\mathscr{B}(e)$:

(**TOP** ordering) $\quad a^\alpha e_i \prec_e a^\beta e_j \Leftrightarrow a^\alpha \prec a^\beta$, or $a^\alpha = a^\beta$ and $i < j$;
(**POT** ordering) $\quad a^\alpha e_i \prec_e a^\beta e_j \Leftrightarrow i < j$, or $i = j$ and $a^\alpha \prec a^\beta$.

Let $\prec_e$ be a left monomial ordering on the K-basis $\mathscr{B}(e)$ of L, and $a^\alpha e_i$, $a^\beta e_j \in \mathscr{B}(e)$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$. We say that $a^\alpha e_i$ *divides* $a^\beta e_j$, denoted $a^\alpha e_i | a^\beta e_j$, if $i = j$ and $a^\beta e_i = \mathbf{LM}(a^\gamma a^\alpha e_i)$ for some $a^\gamma \in \mathscr{B}$. It follows from the foregoing property ($\mathbb{P}2$) that

$$a^\alpha e_i | a^\beta e_j \text{ if and only if } i = j \text{ and } \beta_i \ge \alpha_i,\ 1 \le i \le n.$$

This division of monomials can be extended to a division algorithm of dividing an element ξ by a finite subset of nonzero elements $U = \{\xi_1, \ldots, \xi_m\}$ in L. That is, if there is some $\xi_{i_1} \in U$ such that $\mathbf{LM}(\xi_{i_1}) | \mathbf{LM}(\xi)$, i.e., there is a monomial $a^{\alpha(i_1)} \in \mathscr{B}$ such that $\mathbf{LM}(\xi) = \mathbf{LM}(a^{\alpha(i_1)}\xi_{i_1})$, then $\xi' := \xi - \frac{\mathbf{LC}(\xi)}{\mathbf{LC}(a^{\alpha(i_1)}\xi_{i_1})} a^{\alpha(i_1)}\xi_{i_1}$; otherwise, $\xi' := \xi - \mathbf{LT}(\xi)$. Executing this procedure for ξ' and so on, it follows from the well-ordering property of $\prec_e$ that after finitely many repetitions, ξ has an expression

$$\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)}\xi_j + \eta,$$

where $\lambda_{ij} \in K$, $a^{\alpha(i_j)} \in \mathscr{B}$, $\xi_j \in U$, $\eta = 0$ or $\eta = \sum_k \lambda_k a^{\gamma(k)}e_k$ with $\lambda_k \in K$, $a^{\gamma(k)}e_k \in \mathscr{B}(e)$, satisfying $\mathbf{LM}(a^{\alpha(i_j)}\xi_j) \preceq_e \mathbf{LM}(\xi)$ for all $\lambda_{ij} \neq 0$, and if $\eta \neq 0$, then $a^{\gamma(k)}e_k \preceq_e \mathbf{LM}(\xi)$, $\mathbf{LM}(\xi_i) \nmid a^{\gamma(k)}e_k$ for all $\xi_i \in U$ and all $\lambda_k \neq 0$.

The element η appeared in the above expression is called a *remainder* of ξ on division by U, and is usually denoted by $\overline{\xi}^U$, i.e., $\overline{\xi}^U = \eta$. If $\overline{\xi}^U = 0$, then we say that ξ is *reduced to zero* on division by U. A nonzero element $\xi \in L$ is said to be *normal* (mod U) if $\xi = \overline{\xi}^U$.

Based on the division algorithm, the notion of a *left Gröbner basis* for a submodule N of the free module $L = \bigoplus_{i=1}^{s} Ae_i$ comes into play. Since A is a Noetherian domain, it follows that L is a Noetherian A-module and the following proposition holds.

1.3.1 Theorem. *With respect to the given $\prec_e$ on $\mathscr{B}(e)$, every nonzero submodule N of L has a finite left Gröbner basis $\mathcal{G} = \{g_1, \ldots, g_m\} \subset N$ in the sense that*

> *if $\xi \in N$ and $\xi \neq 0$, then $\mathbf{LM}(g_i)|\mathbf{LM}(\xi)$ for some $g_i \in \mathcal{G}$, i.e., there is a monomial $a^\gamma \in \mathscr{B}$ such that $\mathbf{LM}(\xi) = \mathbf{LM}(a^\gamma \mathbf{LM}(g_i))$, or equivalently, ξ has a* left Gröbner representation *$\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)} g_j$, where $\lambda_{ij} \in K^*$, $a^{\alpha(i_j)} \in \mathscr{B}$ with $\alpha(i_j) = (\alpha_{i_{j1}}, \ldots, \alpha_{i_{jn}}) \in \mathbb{N}^n$, $g_j \in \mathcal{G}$, satisfying $\mathbf{LM}(a^{\alpha(i_j)} g_j) \preceq_e \mathbf{LM}(\xi)$.*

Moreover, starting with any finite generating set of N, such a left Gröbner basis $\mathcal{G}$ can be computed by running a noncommutative version of the Buchberger algorithm for modules over solvable polynomial algebras (see **Algorithm 1** *presented below).* □

The noncommutative version of Buchberger algorithm is based on the noncommutative version of Buchberger's criterion that makes the strategy for computing left Gröbner bases of modules over solvable polynomial algebras. For the reader's convenience and the use in subsequent sections, we recall both of them as follows.

Let $N = \sum_{i=1}^{m} A\xi_i$ with $U = \{\xi_1, \ldots, \xi_m\} \subset L$. For $\xi_i, \xi_j \in U$ with $1 \leq i < j \leq m$, $\mathbf{LM}(\xi_i) = a^{\alpha(i)} e_p$, $\mathbf{LM}(\xi_j) = a^{\alpha(j)} e_q$, where $\alpha(i) = (\alpha_{i_1}, \ldots, \alpha_{i_n})$, $\alpha(j) = (\alpha_{j_1}, \ldots, \alpha_{j_n})$, put $\gamma = (\gamma_1, \ldots, \gamma_n)$ with $\gamma_k = \max\{\alpha_{i_k}, \alpha_{j_k}\}$. The *left S-polynomial* of ξ_i and ξ_j is defined as

$$S_\ell(\xi_i, \xi_j) = \begin{cases} \frac{1}{\mathbf{LC}(a^{\gamma-\alpha(i)}\xi_i)} a^{\gamma-\alpha(i)}\xi_i - \frac{1}{\mathbf{LC}(a^{\gamma-\alpha(j)}\xi_j)} a^{\gamma-\alpha(j)}\xi_j, & \text{if } p = q \\ 0, & \text{if } p \neq q. \end{cases}$$

1.3.2 Theorem (Noncommutative version of Buchberger's criterion)**.** *With notation as above, U is a left Gröbner basis of the submodule N if and only if every $S_\ell(\xi_i, \xi_j)$ is reduced to 0 on division by U, i.e., $\overline{S_\ell(\xi_i, \xi_j)}^U = 0$.*

Algorithm 1 (Noncommutative version of Buchberger algorithm)

INPUT : $U = \{\xi_1, \ldots, \xi_m\}$
OUTPUT : $\mathcal{G} = \{g_1, \ldots, g_t\}$, a left Gröbner basis for $N = \sum_{i=1}^{m} A\xi_i$,
INITIALIZATION : $m' := m$, $\mathcal{G} := \{g_1 = \xi_1, \ldots, g_{m'} = \xi_m\}$,

$$\mathcal{S} := \left\{ S_\ell(g_i, g_j) \;\middle|\; \begin{array}{l} g_i, g_j \in \mathcal{G},\ i < j, \text{ and for some } e_t, \\ \mathbf{LM}(g_i) = a^\alpha e_t,\ \mathbf{LM}(g_j) = a^\beta e_t \end{array} \right\}$$

BEGIN
  WHILE $\mathcal{S} \neq \emptyset$ DO
    Choose any $S_\ell(g_i, g_j) \in \mathcal{S}$
    $\mathcal{S} := \mathcal{S} - \{S_\ell(g_i, g_j)\}$
    $\overline{S_\ell(g_i, g_j)}^{\mathcal{G}} = \eta$
      IF $\eta \neq 0$ with $\mathbf{LM}(\eta) = a^\rho e_k$ THEN
        $m' := m' + 1,\ g_{m'} := \eta$
        $\mathcal{S} := \mathcal{S} \cup \{S_\ell(g_j, g_{m'}) \mid g_j \in \mathcal{G},\ \mathbf{LM}(g_j) = a^\nu e_k\}$
        $\mathcal{G} := \mathcal{G} \cup \{g_{m'}\}$,
      END
  END
END

One is referred to the up-to-date computer algebra systems Singular [DGPS] for the implementation of **Algorithm 1**. Also, nowadays there have been optimized algorithms, such as the signature-based algorithm for computing Gröbner bases in solvable polynomial algebras [SWMZ], which is based on the celebrated F5 algorithm [Fau], and may be used to speed-up the computation of left Gröbner bases for modules.

2. Computation of minimal graded free resolutions

The work of this section is completely a noncommutative analogue of the well-known commutative case, that is, we show that the methods and algorithms, developed in [CDNR], [KR2] for computing minimal homogeneous generating sets of graded submodules and graded quotient modules of free modules over commutative polynomial algebras, can be adapted for computing minimal homogeneous generating sets of graded submodules and graded quotient modules of free modules over a noncommutative $\mathbb{N}$-graded solvable polynomial K-algebras A with the degree-0 homogeneous part $A_0 = K$. Consequently the algorithmic procedures for computing minimal graded free resolutions of finitely generated modules over A can be achieved.

In the literature, a finitely generated $\mathbb{N}$-graded K-algebra $A = \bigoplus_{p \in \mathbb{N}} A_p$ with the degree-0 homogeneous part $A_0 = K$ is referred to as a *connected* $\mathbb{N}$-*graded* K-*algebra*. Concerning introductions to minimal resolutions of graded modules over a (commutative or noncommutative) connected $\mathbb{N}$-graded K-algebra (or more generally an $\mathbb{N}$-graded local K-algebra) and relevant results, one may refer to [Eis, Chapter 19], [Kr1, Chapter 3], and [Li3].

Noticing that commutative polynomial algebras are certainly the type of $\mathbb{N}$-graded solvable polynomial algebras we specified, and that the noncommutative version of Buchberger's criterion as well as Buchberger algorithm for modules over solvable polynomial algebras (Theorem 1.3.2 and **Algorithm 1** presented in the end of Section 1) looks as if working in the same way as in the commutative case by reducing the S-polynomials, one might think that the extension of methods and algorithms provided by [CDNR], [KR2] to modules over noncommutative $\mathbb{N}$-graded solvable polynomial algebras could be naturally holding true as a folklore. However, from the literature (e.g. [AL2], [BW], [Eis], [Fröb], [KR1], [KR2]) we learnt that in developing the Gröbner basis (including the n-truncated Gröbner basis) theory for a commutative polynomial K-algebra $R = K[x_1, \dots, x_n]$, two featured algebraic structures play the key role in both the theoretical proofs and technical calculations, namely

- the multiplicative monoid $\mathscr{B}$, where $\mathscr{B}$ is the PBW basis of R, which is furthermore turned into an ordered multiplicative monoid with respect to any monomial ordering $\prec$;
- monomial ideals, i.e., ideals generated by monomials from $\mathscr{B}$.

For instance, a version of Dickson's lemma for monomial ideals holds true, thereby a Gröbner basis of an ideal I in R is usually defined (or characterized) in terms of generators of the monomial ideal generated by leading monomials of I. In the proof of Buchberger's criterion, reduction of a monomial does not cause any trouble (e.g. see [AL2, p. 41, l -5]); especially, the already known Noetherianess of R (or Dickson's lemma for monomial ideals) guarantees the termination of Buchberger algorithm (e.g. see [AL2, p. 43]), and this algorithm, in turn, gives rise to more relevant algorithms not only for ideals but also for submodules of free R-modules (e.g. [KR2, Proposition 4.5.10, Theorem 4.6.3]). While due to the *noncommutativity* of a solvable polynomial algebra $A = K[a_1, \dots, a_n]$, *the PBW basis* $\mathscr{B}$ *of* A *is no longer a multiplicative monoid*. Thereby, in developing a (one-sided, two-sided) Gröbner basis theory of A, all steps using reduction by monomials from $\mathscr{B}$ cannot be simply replicated from the commutative case. Moreover, (one-sided, two-sided) ideals generated by monomials from $\mathscr{B}$ can no longer play the role as in the commutative case, and the Noetherianess of A is not known until the existence of finite Gröbner bases for (one-sided) ideals is algorithmically established (note that in general A is not necessarily an iterated Ore extension of the base ring K or some Noetherian ring). Since **Algorithm 2** and **Algorithm 3** below in this section essentially depend on **Algorithm 1** presented in the last section, at this point, one is referred to [K-RW] for the nontrivial and detailed argumentation on how the barrier of noncommutativity is broken down, in order to reach the main results we recalled in Section 1 (Theorem 1.2.3, Theorem 1.3.1, Theorem 1.3.2, **Algorithm 1** though this is for modules). Moreover, so far in the literature there had been no clear and systematical presentation showing that the commutative n-truncated Gröbner basis theory and the algorithmic principle for $\mathbb{N}$-graded modules presented in [CDNR], [KR2] can be adapted for $\mathbb{N}$-graded modules over general noncommutative $\mathbb{N}$-graded solvable polynomial K-algebras with the degree-0 homogeneous part equal to K. So, from a mathematical viewpoint, we are naturally concerned about how to trust that this is a true story and then, how to give a precise quotation source when the relevant results are applied to noncommutative setting (for instance, in our Section 3). Following the rule "*to see is to believe*", which we understand as understanding more than merely observing, all what we pointed out above motivates us to provide a detailed argumentation and demonstration on the topic of this section, which one may also compare with the corresponding argumentations given in [KR2, Chapter 4].

All notions, notations and conventions introduced before are maintained.

2.1 $\mathbb{N}$-graded solvable polynomial algebras and graded free modules. In this subsection we first specify, by means of positive-degree functions (as defined in Subsection 1.2), the structure of $\mathbb{N}$-graded solvable polynomial K-algebras with the degree-0 homogeneous part equal to K, and then we demonstrate how to construct graded free modules over such $\mathbb{N}$-graded algebras.

Let $A = K[a_1, \ldots, a_n]$ be a solvable polynomial K-algebra with admissible system $(\mathscr{B}, \prec)$, where $\mathscr{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$ is the PBW K-basis of A and $\prec$ is a monomial ordering on $\mathscr{B}$. Suppose that A is an $\mathbb{N}$-graded algebra with the degree-0 homogeneous part equal to K, namely $A = \bigoplus_{p\in\mathbb{N}} A_p$, where the degree-$p$ homogeneous part A_p is a K-subspace of A, $A_0 = K$, and $A_{p_1}A_{p_2} \subseteq A_{p_1+p_2}$ for all $p_1, p_2 \in \mathbb{N}$. Then, writing $d_{\rm gr}(f) = p$ for the *gr-degree of a nonzero homogeneous element* $f \in A_p$, we have

$$d_{\rm gr}(a_i) = m_i > 0, \quad 1 \le i \le n,$$

for, conventionally, any generator a_i of A is not contained in the ground field K. It turns out that if $a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \in \mathscr{B}$, then $d_{\rm gr}(a^\alpha) = \sum_{i=1}^n \alpha_i m_i$. This shows that A is a *weighted* $\mathbb{N}$*-graded algebra*, and that $d_{\rm gr}(\ )$ gives rise to positive-degree function on A as defined in Subsection 1.2, such that

(1) $A_p = K\text{-span}\{a^\alpha \in \mathscr{B} \mid d_{\rm gr}(a^\alpha) = p\}$, $p \in \mathbb{N}$;
(2) for $1 \le i < j \le n$, all the relations $a_j a_i = \lambda_{ji} a_i a_j + f_{ji}$ with $f_{ji} = \sum \mu_k a^{\alpha(k)}$ derived from Definition 1.2.2 satisfy $d_{\rm gr}(a^{\alpha(k)}) = d_{\rm gr}(a_i a_j)$ whenever $\mu_k \ne 0$.

Conversely, given a positive-degree function $d(\ )$ on A (as defined in Subsection 1.2) such that $d(a_i) = m_i > 0$, $1 \le i \le n$, we know that A has an $\mathbb{N}$-graded K-module structure, i.e., $A = \bigoplus_{p\in\mathbb{N}} A_p$ with $A_p = K\text{-span}\{a^\alpha \in \mathscr{B} \mid d(a^\alpha) = p\}$, in particular, $A_0 = K$. It is straightforward to verify that if furthermore for $1 \le i < j \le n$, all the relations $a_j a_i = \lambda_{ji} a_i a_j + f_{ji}$ with $f_{ji} = \sum \mu_k a^{\alpha(k)}$ derived from Definition 1.2.2 satisfy $d(a^{\alpha(k)}) = d(a_i a_j)$ whenever $\mu_k \ne 0$, then $A_{p_1}A_{p_2} \subseteq A_{p_1+p_2}$ holds for all $p_1, p_2 \in \mathbb{N}$, i.e., A is turned into an $\mathbb{N}$-graded solvable polynomial algebra with the degree-0 homogeneous part $A_0 = K$.

Summing up, we have reached the following observation.

Observation. A solvable polynomial algebra $A = K[a_1, \ldots, a_n]$ is an $\mathbb{N}$-graded algebra with the degree-0 homogeneous part $A_0 = K$ if and only if there is a positive-degree function $d(\ )$ on A (as defined in Subsection 1.2) such that for $1 \le i < j \le n$, all the relations $a_j a_i = \lambda_{ji} a_i a_j + f_{ji}$ with $f_{ji} = \sum \mu_k a^{\alpha(k)}$ derived from Definition 1.2.2 satisfy $d(a^{\alpha(k)}) = d(a_i a_j)$ whenever $\mu_k \ne 0$.

To make the compatibility with the structure of $\mathbb{N}$-filtered solvable polynomial algebras specified in Section 3, it is necessary to emphasize the role played by a positive-degree function in the structure of $\mathbb{N}$-graded solvable polynomial algebras we specified in this subsection, that is, from now on in the rest of this paper we keep using the following convention.

Convention. An $\mathbb{N}$-graded solvable polynomial K-algebra $A = \bigoplus_{p\in\mathbb{N}} A_p$ with $A_0 = K$ is always referred to as an $\mathbb{N}$-*graded solvable polynomial algebra with respect to a positive-degree function* $d(\ )$.

Remark. Let $A = K[a_1, \dots, a_n] = \bigoplus_{p\in\mathbb{N}} A_p$ be an $\mathbb{N}$-graded solvable polynomial algebra with respect to a positive-degree function $d(\ )$.

(i) We emphasize that *every* $a^\alpha \in \mathscr{B}$ *is a homogeneous element of* A *and* $d(a^\alpha) = d_{\rm gr}(a^\alpha)$, where $d_{\rm gr}(\ )$, as we defined above, is the gr-degree function on nonzero homogeneous elements of A.

(ii) Since A is a domain (Theorem 1.2.3), the gr-degree function $d_{\rm gr}(\ )$ has the property: for all nonzero homogeneous elements $h_1, h_2 \in A$,

$$(\mathbb{P}3) \qquad d_{\rm gr}(h_1h_2) = d_{\rm gr}(h_1) + d_{\rm gr}(h_2).$$

From now on we shall freely use this property without additional indication.

In view of the above observation, if $\prec_{gr}$ is a graded monomial ordering on $\mathscr{B}$ with respect to some given positive-degree function $d(\ )$ on A (see Subsection 1.2 for the definition), then it is easy to check whether A is an $\mathbb{N}$-graded algebra with respect to $d(\ )$ or not.

Typical $\mathbb{N}$-graded solvable polynomial algebras with respect to given positive-degree functions are those iterated skew polynomial K-algebras $A = K[a_1, \dots, a_n]$ subject to the relations

$$a_ja_i = \lambda_{ji}a_ia_j, \qquad \lambda_{ji} \in K^*,\ 1 \le i < j \le n,$$

where the positive-degree function on A can be defined by setting $d(a_i) = m_i$ for any fixed tuple $(m_1, \dots, m_n)$ of positive integers. Such algebras include the well-known coordinate rings of quantum affine n-spaces. Another well-known $\mathbb{N}$-graded solvable polynomial algebra is the coordinate ring $M_q(2) = K[a, b, c, d]$ of the manifold of quantum 2×2 matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ introduced in [Man], which has the defining relations

$$\begin{array}{ll} ab = q^{-1}ba, & db = bd - (q - q^{-1})ac, \\ cb = qbc, & da = qad, \\ ca = ac, & dc = qcd, \end{array}$$

where each generator is assigned the degree 1. More generally, let $\Lambda = (\lambda_{ij})$ be a multiplicatively antisymmetric $n \times n$ matrix over K, and let $\lambda \in K^*$ with $\lambda \ne -1$. Considering the multiparameter coordinate ring of quantum $n \times n$ matrices over K

(see [Good]), namely the K-algebra $\mathcal{O}_{\lambda,\Lambda}(M_n(K))$ generated by n^2 elements a_{ij} $(1 \le i, j \le n)$ subject to the relations

$$a_{\ell m}a_{ij} = \begin{cases} \lambda_{\ell i}\lambda_{jm}a_{ij}a_{\ell m} + (\lambda - 1)\lambda_{\ell i}a_{im}a_{\ell j} & (\ell > i,\ m > j) \\ \lambda\lambda_{\ell i}\lambda_{jm}a_{ij}a_{\ell m} & (\ell > i,\ m \le j) \\ \lambda_{jm}a_{ij}a_{\ell m} & (\ell = i,\ m > j) \end{cases}$$

Then $\mathcal{O}_{\lambda,\Lambda}(M_n(K))$ is an $\mathbb{N}$-graded solvable polynomial algebra, where each generator has degree 1.

Moreover, by [LW], [Li1], or the later Section 3, the associated graded algebra and the Rees algebra of every $\mathbb{N}$-filtered solvable polynomial algebra with a graded monomial ordering $\prec_{gr}$ (as defined in Subsection 1.2) are $\mathbb{N}$-graded solvable polynomial algebras of the type we specified in this subsection.

Let $A = K[a_1, \ldots, a_n]$ be an $\mathbb{N}$-graded solvable polynomial algebra with respect to a positive-degree function $d(\ )$, and let $(\mathscr{B}, \prec)$ be an admissible system of A. We next turn to construct $\mathbb{N}$-graded free A-modules.

If $L = \bigoplus_{i=1}^{s} Ae_i$ is a free left A-module with the A-basis $\{e_1, \ldots, e_s\}$, then L has the K-basis $\mathscr{B}(e) = \{a^\alpha e_i \mid a^\alpha \in \mathscr{B},\ 1 \le i \le n\}$, and for an *arbitrarily* fixed $\{b_1, \ldots, b_s\} \subset \mathbb{N}$, L can be turned into an $\mathbb{N}$-*graded free* A-*module* in the sense that $L = \bigoplus_{q\in\mathbb{N}} L_q$ in which

$$L_q = \{0\} \text{ if } q < \min\{b_1, \ldots, b_s\};\ \text{otherwise } L_q = \sum_{p_i + b_i = q} A_{p_i}e_i, \quad q \in \mathbb{N},$$

or alternatively, for $q \ge \min\{b_1, \ldots, b_s\}$,

$$L_q = K\text{-span}\{a^\alpha e_i \in \mathscr{B}(e) \mid d(a^\alpha) + b_i = q\},\ q \in \mathbb{N},$$

such that $A_pL_q \subseteq A_{p+q}$ for all $p, q \in \mathbb{N}$. For each $q \in \mathbb{N}$, elements in L_q are called *homogeneous elements of degree* q, and accordingly L_q is called the *degree-q homogeneous part* of L. If $\xi \in L_q$, then we write $d_{\rm gr}(\xi)$ for the *gr-degree* of ξ as a homogeneous element of L, i.e., $d_{\rm gr}(\xi) = q$. In particular, $d_{\rm gr}(e_i) = b_i$, $1 \le i \le s$. As with the gr-degree of homogeneous elements in A, noticing that $d_{\rm gr}(a^\alpha e_i) = d(a^\alpha) + b_i$ for all $a^\alpha e_i \in \mathscr{B}(e)$ and that A is a domain, from now on we shall freely use the following property without additional indication: for all nonzero homogeneous elements $h \in A$ and all nonzero homogeneous elements $\xi \in L$,

$$(\mathbb{P}4) \qquad d_{\rm gr}(h\xi) = d_{\rm gr}(h) + d_{\rm gr}(\xi).$$

Remark. Although we have remarked that $d(a^\alpha) = d_{\rm gr}(a^\alpha)$ for all $a^\alpha \in \mathscr{B}$, $d(a^\alpha)$ is used in constructing L_q just for highlighting the role of $d(\ )$.

Convention. Unless otherwise stated, from now on throughout the subsequent sections if we say that L is a graded free module over an $\mathbb{N}$-graded solvable polynomial algebra A with respect to a positive-degree function $d(\ )$, then it always means that L has an $\mathbb{N}$-gradation as constructed above.

Let $L = \bigoplus_{q\in\mathbb{N}} L_q$ be a graded free A-module. If a submodule N is generated by homogeneous elements, then N is called a *graded submodule* of L. A graded submodule N has the $\mathbb{N}$-graded structure $N = \bigoplus_{q\in\mathbb{N}} N_q$ with $N_q = N \cap L_q$, such that $A_pN_q \subseteq N_{p+q}$ for all $p, q \in \mathbb{N}$. Note that monomials in $\mathscr{B}$ are homogeneous elements of A, thereby left S-polynomials of homogeneous elements are homogeneous elements, and remainders of homogeneous elements on division by homogeneous remain homogeneous elements. It follows that if a graded submodule $N = \sum_{i=1}^{m} A\xi_i$ of L is generated by the set of homogeneous elements $\{\xi_1, \ldots, \xi_m\}$, then, running the noncommutative version of Buchberger's algorithm for modules over solvable polynomial algebras (**Algorithm 1** in Subsection 1.3) with respect to a fixed monomial ordering $\prec_e$ on $\mathscr{B}(e)$, it produces a finite *homogeneous left Gröbner basis* $\mathcal{G}$ for N, that is, $\mathcal{G}$ consists of homogeneous elements.

2.2 Computation of minimal homogeneous generating sets. In this subsection, $A = K[a_1, \ldots, a_n]$ denotes an $\mathbb{N}$-graded solvable polynomial algebra with respect to a positive-degree function $d(\ )$, $(\mathscr{B}, \prec)$ denotes a fixed admissible system of A, $L = \bigoplus_{i=1}^{s} Ae_i$ denotes a graded free A-module such that $d_{\mathrm{gr}}(e_i) = b_i$, $1 \le i \le s$, and $\prec_e$ denotes a fixed left monomial ordering on the K-basis $\mathscr{B}(e)$ of L. Moreover, as in Section 1 we write $S_\ell(\xi_i, \xi_j)$ for the left S-polynomial of two elements ξ_i, $\xi_j \in L$.

Remark. To reach our goal of this subsection, let us first point out that although monomials from the PBW K-basis $\mathscr{B}$ of A *can no longer behave as well as monomials in a commutative polynomial algebra* (namely the product of two monomials is not necessarily a monomial), every monomial from $\mathscr{B}$ is a homogeneous element in the $\mathbb{N}$-graded structure of A (as we remarked before), thereby the product of two monomials gives rise to a homogeneous element. Bearing in mind this fact, the rule of division, and the properties ($\mathbb{P}$1)–($\mathbb{P}$3) mentioned in Section 1 and the foregoing ($\mathbb{P}$4), the argument below will go through without trouble.

We start by a discussion on computing n-truncated left Gröbner bases for graded submodules of L, which is similar to the commutative case (see [KR2, Section 4.5]).

2.2.1 Definition. Let $G = \{g_1, \ldots, g_t\}$ be a subset of homogeneous elements of L, $N = \sum_{i=1}^{t} Ag_i$ the graded submodule generated by G, and let $n \in \mathbb{N}$, $G_{\le n} = \{g_j \in G \mid d_{\mathrm{gr}}(g_j) \le n\}$. If, for each nonzero homogeneous element $\xi \in N$ with $d_{\mathrm{gr}}(\xi) \le n$, there is some $g_i \in G_{\le n}$ such that $\mathbf{LM}(g_i)|\mathbf{LM}(\xi)$ with respect to $\prec_e$, then we call $G_{\le n}$ an *n-truncated left Gröbner basis* of N with respect to $(\mathscr{B}(e), \prec_e)$.

Verification of the lemma below is straightforward.

2.2.2 Lemma. *Let $\mathcal{G} = \{g_1, \ldots, g_t\}$ be a homogeneous left Gröbner basis for the graded submodule $N = \sum_{i=1}^{t} Ag_i$ of L with respect to $(\mathscr{B}(e), \prec_e)$. For each $n \in \mathbb{N}$, put $\mathcal{G}_{\leq n} = \{g_j \in \mathcal{G} \mid d_{\rm gr}(g_j) \leq n\}$, $N_{\leq n} = \bigcup_{q=0}^{n} N_q$ where each N_q is the degree-q homogeneous part of N, and let $N(n) = \sum_{\xi \in N_{\leq n}} A\xi$ be the graded submodule generated by $N_{\leq n}$. The following statements hold.*

(i) *$\mathcal{G}_{\leq n}$ is an n-truncated left Gröbner basis of N. Thus, if $\xi \in L$ is a homogeneous element with $d_{\rm gr}(\xi) \leq n$, then $\xi \in N$ if and only if $\overline{\xi}^{\mathcal{G}_{\leq n}} = 0$, i.e., ξ is reduced to zero on division by $\mathcal{G}_{\leq n}$.*

(ii) *$N(n) = \sum_{g_j \in \mathcal{G}_{\leq n}} Ag_j$, and $\mathcal{G}_{\leq n}$ is an n-truncated left Gröbner basis of $N(n)$.*

In light of **Algorithm 1** (presented in Subsection 1.3), an n-truncated left Gröbner basis is characterized as follows.

2.2.3 Proposition. *Let $N = \sum_{i=0}^{s} Ag_i$ be the graded submodule of L generated by a set of homogeneous elements $G = \{g_1, \ldots, g_m\}$. For each $n \in \mathbb{N}$, put $G_{\leq n} = \{g_j \in G \mid d_{\rm gr}(g_j) \leq n\}$. The following statements are equivalent with respect to the given $(\mathscr{B}(e), \prec_e)$.*

(i) *$G_{\leq n}$ is an n-truncated left Gröbner basis of N.*

(ii) *Every nonzero left S-polynomial $S_\ell(g_i, g_j)$ of $d_{\rm gr}(S_\ell(g_i, g_j)) \leq n$ is reduced to zero on division by $G_{\leq n}$, i.e, $\overline{S_\ell(g_i, g_j)}^{G_{\leq n}} = 0$.*

Proof: Recall that if $g_i, g_j \in G$, $\mathbf{LT}(g_i) = \lambda_i a^\alpha e_t$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\mathbf{LT}(g_j) = \lambda_j a^\beta e_t$ with $\beta = (\beta_1, \ldots, \beta_n)$, and $\gamma = (\gamma_1, \ldots \gamma_n)$ with $\gamma_i = \max\{\alpha_i, \beta_i\}$, $1 \leq i \leq n$, then

$$S_\ell(g_i, g_j) = \frac{1}{\mathbf{LC}(a^{\gamma-\alpha} g_i)} a^{\gamma-\alpha} g_i - \frac{1}{\mathbf{LC}(a^{\gamma-\beta} g_j)} a^{\gamma-\beta} g_j$$

is a homogeneous element in N with $d_{\rm gr}(S_\ell(g_i, g_j)) = d(a^\gamma) + b_t$ by the foregoing property ($\mathbb{P}$4). If $d_{\rm gr}(S_\ell(g_i, g_j)) \leq n$, then it follows from (i) that (ii) holds.

Conversely, suppose that (ii) holds. To see that $G_{\leq n}$ is an n-truncated left Gröbner basis of N, let us run **Algorithm 1** (presented in Subsection 1.3) with the initial input data G. Without optimizing **Algorithm 1** we may certainly assume that $G \subseteq \mathcal{G}$, thereby $G_{\leq n} \subseteq \mathcal{G}_{\leq n}$ where $\mathcal{G}$ is the new input set returned by each pass through the WHILE loop. On the other hand, by the construction of $S_\ell(g_i, g_j)$ and the foregoing property ($\mathbb{P}$4) we know that if $d_{\rm gr}(S_\ell(g_i, g_j)) \leq n$, then $d_{\rm gr}(g_i) \leq n$, $d_{\rm gr}(g_j) \leq n$. Hence, the assumption (ii) implies that **Algorithm 1** does not append any new element of degree not greater than n to $\mathcal{G}$. Therefore, $G_{\leq n} = \mathcal{G}_{\leq n}$. By Lemma 2.2.2 we conclude that $G_{\leq n}$ is an n-truncated left Gröbner basis of N. □

2.2.4 Corollary. *Let $N = \sum_{i=1}^{m} Ag_i$ be the graded submodule of L generated by a set of homogeneous elements $G = \{g_1, \ldots, g_m\}$. Suppose that $G_{\leq n} = \{g_j \in G \mid d_{\rm gr}(g_j) \leq n\}$ is an n-truncated left Gröbner basis of N with respect to $(\mathscr{B}(e), \prec_e)$.*

(i) *If $\xi \in L$ is a nonzero homogeneous element of $d_{\rm gr}(\xi) = n$ such that $\mathbf{LM}(g_i) \not| \mathbf{LM}(\xi)$ for all $g_i \in G_{\le n}$, then $G' = G_{\le n} \cup \{\xi\}$ is an n-truncated left Gröbner basis for both the graded submodules $N' = N + A\xi$ and $N'' = \sum_{g_j \in G_{\le n}} Ag_j + A\xi$ of L.*

(ii) *If $n \le n_1$ and $\xi \in L$ is a nonzero homogeneous element of $d_{\rm gr}(\xi) = n_1$ such that $\mathbf{LM}(g_i) \not| \mathbf{LM}(\xi)$ for all $g_i \in G_{\le n}$, then $G' = G_{\le n} \cup \{\xi\}$ is an n_1-truncated left Gröbner basis for the graded submodule $N' = \sum_{g_j \in G_{\le n}} Ag_j + A\xi$ of L.*

Proof: If $\xi \in L$ is a nonzero homogeneous element of $d_{\rm gr}(\xi) = n_1 \ge n$ and $\mathbf{LM}(\xi_i) \not| \mathbf{LM}(\xi)$ for all $\xi_i \in G_{\le n}$, then noticing the property ($\mathbb{P}2$) mentioned in Section 1 and the foregoing ($\mathbb{P}4$), we see that every nonzero left S-polynomial $S_\ell(\xi, \xi_i)$ with $\xi_i \in G$ has $d_{\rm gr}(S_\ell(\xi, \xi_i)) > n$. Hence both (i) and (ii) hold by Proposition 2.2.3. □

2.2.5 Proposition (Compare with [KR2, Proposition 4.5.10]). *Given a finite set of nonzero homogeneous elements $U = \{\xi_1, \ldots, \xi_m\} \subset L$ with $d_{\rm gr}(\xi_1) \le d_{\rm gr}(\xi_2) \le \cdots \le d_{\rm gr}(\xi_m)$, and a positive integer $n_0 \ge d_{\rm gr}(\xi_1)$, the following algorithm computes an n_0-truncated left Gröbner basis $\mathcal{G} = \{g_1, \ldots, g_t\}$ for the graded submodule $N = \sum_{i=1}^{m} A\xi_i$ such that $d_{\rm gr}(g_1) \le d_{\rm gr}(g_2) \le \cdots \le d_{\rm gr}(g_t)$.*

Algorithm 2

INPUT : $U = \{\xi_1, \ldots, \xi_m\}$ with $d_{\rm gr}(\xi_1) \le d_{\rm gr}(\xi_2) \le \cdots \le d_{\rm gr}(\xi_m)$,
n_0, where $n_0 \ge d_{\rm gr}(\xi_1)$

OUTPUT : $\mathcal{G} = \{g_1, \ldots, g_t\}$ an n_0-truncated left Gröbner basis of N

INITIALIZATION : $\mathcal{S}_{\le n_0} := \emptyset,\ W := U,\ \mathcal{G} := \emptyset,\ t' := 0$

LOOP

$n := \min\{d_{\rm gr}(\xi_i),\ d_{\rm gr}(S_\ell(g_i, g_j)) \mid \xi_i \in W,\ S_\ell(g_i, g_j) \in \mathcal{S}_{\le n_0}\}$

$\mathcal{S}_n := \{S_\ell(g_i, g_j) \in \mathcal{S}_{\le n_0} \mid d_{\rm gr}(S_\ell(g_i, g_j)) = n\},\ W_n := \{\xi_j \in W \mid d_{\rm gr}(\xi_j) = n\}$

$\mathcal{S}_{\le n_0} := \mathcal{S}_{\le n_0} - \mathcal{S}_n,\ W := W - W_n$

WHILE $\mathcal{S}_n \ne \emptyset$ DO

Choose any $S_\ell(g_i, g_j) \in \mathcal{S}_n$

$\mathcal{S}_n := \mathcal{S}_n - \{S_\ell(g_i, g_j)\}$

$\overline{S_\ell(g_i, g_j)}^{\mathcal{G}} = \eta$

IF $\eta \ne 0$ with $\mathbf{LM}(\eta) = a^\rho e_k$ THEN

$t' := t' + 1,\ g_{t'} := \eta$

$\mathcal{S}_{\le n_0} := \mathcal{S}_{\le n_0} \cup \left\{ S_\ell(g_i, g_{t'}) \;\middle|\; \begin{array}{l} g_i \in \mathcal{G}, 1 \le i < t', \mathbf{LM}(g_i) = a^\tau e_k, \\ d_{\rm gr}(S_\ell(g_i, g_{t'})) \le n_0 \end{array} \right\}$

$\mathcal{G} := \mathcal{G} \cup \{g_{t'}\}$

END

END

    WHILE $W_n \neq \emptyset$ DO
        Choose any $\xi_j \in W_n$
        $W_n := W_n - \{\xi_j\}$
        $\overline{\xi_j}^{\mathcal{G}} = \eta$
        IF $\eta \neq 0$ with $\mathbf{LM}(\eta) = a^\rho e_k$ THEN
            $t' := t' + 1,\ g_{t'} := \eta$
            $\mathcal{S}_{\le n_0} := \mathcal{S}_{\le n_0} \cup \left\{ S_\ell(g_i, g_{t'}) \;\middle|\; \begin{array}{l} g_i \in \mathcal{G}, 1 \le i < t', \mathbf{LM}(g_i) = a^\tau e_k, \\ d_{\mathrm{gr}}(S_\ell(g_i, g_{t'})) \le n_0 \end{array} \right\}$
            $\mathcal{G} := \mathcal{G} \cup \{g_{t'}\}$
        END
    END
UNTIL $\mathcal{S}_{\le n_0} = \emptyset$
END

PROOF: First note that both the WHILE loops append new elements to $\mathcal{G}$ by taking the nonzero normal remainders on division by $\mathcal{G}$. Thus, with a fixed n, by the definition of a left S-polynomial and the normality of $g_{t'}$ (mod $\mathcal{G}$), it is straightforward to check that in both the WHILE loops every new appended $S_\ell(g_i, g_{t'})$ has $d_{\mathrm{gr}}(S_\ell(g_i, g_{t'})) > n$. To proceed, let us write $N(n)$ for the submodule generated by $\mathcal{G}$ which is obtained after W_n is exhausted in the second WHILE loop. If n_1 is the first number after n such that $\mathcal{S}_{n_1} \neq \emptyset$, and for some $S_\ell(g_i, g_j) \in \mathcal{S}_{n_1}$, $\eta = \overline{S_\ell(g_i, g_j)}^{\mathcal{G}} \neq 0$ in a certain pass through the first WHILE loop, then we note that this η is still contained in $N(n)$. Hence, after $\mathcal{S}_{n_1}$ is exhausted in the first WHILE loop, the obtained $\mathcal{G}$ generates $N(n)$ and $\mathcal{G}$ is an n_1-truncated left Gröbner basis of $N(n)$. Noticing that the algorithm starts with $\mathcal{S}_{\le n_0} = \emptyset$ and $\mathcal{G} = \emptyset$, inductively it follows from Proposition 2.2.3 and Corollary 2.2.4 that after W_{n_1} is exhausted in the second WHILE loop, the obtained $\mathcal{G}$ is an n_1-truncated left Gröbner basis of $N(n_1)$. Since n_0 is finite and all the generators of N with $d_{\mathrm{gr}}(\xi_j) \le n_0$ are processed through the second WHILE loop, the algorithm terminates and the eventually obtained $\mathcal{G}$ is an n_0-truncated left Gröbner basis of N. Finally, the fact that the degrees of elements in $\mathcal{G}$ are non-decreasingly ordered follows from the choice of the next n in the algorithm. □

Let the data $(A, \mathscr{B}, \prec)$ and $(L, \mathscr{B}(e), \prec_e)$ be fixed as before. Combining the foregoing results, we now proceed to show that the algorithm given in [KR2, Theorem 4.6.3] can be adapted for computing minimal homogeneous generating sets of graded submodules in free modules over A.

Let N be a graded submodule of the $\mathbb{N}$-graded free A-module L fixed above. We say that a homogeneous generating set U of N is a *minimal homogeneous generating set* if any proper subset of U cannot be a generating set of N. As preparatory result, we first show that a noncommutative analogue of [KR2, Proposition 4.6.1, Corollary 4.6.2] holds true for N.

2.2.6 Proposition. *Let $N = \sum_{i=1}^{m} A\xi_i$ be the graded submodule of L generated by a set of homogeneous elements $U = \{\xi_1, \ldots, \xi_m\}$, where $d_{\mathrm{gr}}(\xi_1) \le d_{\mathrm{gr}}(\xi_2) \le$*

$\cdots \leq d_{\rm gr}(\xi_m)$. *Put* $N_1 = \{0\}$, $N_i = \sum_{j=1}^{i-1} A\xi_j$, $2 \leq i \leq m$. *The following statements hold.*

(i) *U is a minimal homogeneous generating set of N if and only if $\xi_i \notin N_i$, $1 \leq i \leq m$.*

(ii) *The set $\overline{U} = \{\xi_k \mid \xi_k \in U,\ \xi_k \notin N_k\}$ is a minimal homogeneous generating set of N.*

Proof: (i) If U is a minimal homogeneous generating set of N, then clearly $\xi_i \notin N_i$, $1 \leq i \leq m$.

Conversely, suppose $\xi_i \notin N_i$, $1 \leq i \leq m$. If U is not a minimal homogeneous generating set of N, then, there is some i such that N is generated by $\{\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_m\}$, thereby $\xi_i = \sum_{j\neq i} h_j\xi_j$ for some nonzero homogeneous elements $h_j \in A$ such that $d_{\rm gr}(\xi_i) = d_{\rm gr}(h_jg_j) = d_{\rm gr}(h_j)+d_{\rm gr}(\xi_j)$, where the second equality follows from the foregoing property ($\mathbb{P}$4). Thus $d_{\rm gr}(\xi_j) \leq d_{\rm gr}(\xi_i)$ for all $j \neq i$. If $d_{\rm gr}(\xi_j) < d_{\rm gr}(\xi_i)$ for all $j \neq i$, then $\xi_i \in \sum_{j=1}^{i-1} A\xi_j$, which contradicts the assumption. If $d_{\rm gr}(\xi_i) = d_{\rm gr}(\xi_j)$ for some $j \neq i$, then since $h_j \neq 0$ we have $h_j \in A_0 - \{0\} = K^*$. Putting $i' = \max\{i,\ j \mid f_j \in K^*\}$, we then have $\xi_{i'} \in \sum_{j=1}^{i'-1} A\xi_j$, which again contradicts the assumption. Hence, under the assumption we conclude that U is a minimal homogeneous generating set of N.

(ii) In view of (i), it is sufficient to show that $\overline{U}$ is a homogeneous generating set of N. Indeed, if $\xi_i \in U - \overline{U}$, then $\xi_i \in \sum_{j=1}^{i-1} A\xi_j$. By checking ξ_{i-1} and so on, it follows that $\xi_i \in \sum_{\xi_k\in\overline{U}} A\xi_k$, as desired. □

2.2.7 Corollary. *Let $U = \{\xi_1, \ldots, \xi_m\}$ be a minimal homogeneous generating set of a graded submodule N of L, where $d_{\rm gr}(\xi_1) \leq d_{\rm gr}(\xi_2) \leq \cdots \leq d_{\rm gr}(\xi_m)$, and let $\xi \in L-N$ be a homogeneous element with $d_{\rm gr}(\xi_m) \leq d_{\rm gr}(\xi)$. Then $\widehat{U} = U\cup\{\xi\}$ is a minimal homogeneous generating set of the graded submodule $\widehat{N} = N + A\xi$.* □

2.2.8 Theorem (Compare with [KR2, Theorem 4.6.3])**.** *Let $U = \{\xi_1, \ldots, \xi_m\} \subset L$ be a finite set of nonzero homogeneous elements of L with $d_{\rm gr}(\xi_1) \leq d_{\rm gr}(\xi_2) \leq \cdots \leq d_{\rm gr}(\xi_m)$. Then the following algorithm returns a minimal homogeneous generating set $U_{\min} = \{\xi_{j_1}, \ldots, \xi_{j_r}\} \subset U$ for the graded submodule $N = \sum_{i=1}^{m} A\xi_i$; and meanwhile it returns a homogeneous left Gröbner basis $\mathcal{G} = \{g_1, \ldots, g_t\}$ for N such that $d_{\rm gr}(g_1) \leq d_{\rm gr}(g_2) \leq \cdots \leq d_{\rm gr}(g_t)$.*

Algorithm 3

INPUT : $U = \{\xi_1, \ldots, \xi_m\}$ with $d_{\rm gr}(\xi_1) \leq d_{\rm gr}(\xi_2) \leq \cdots \leq d_{\rm gr}(\xi_m)$
OUTPUT : $U_{\min} = \{\xi_{j_1}, \ldots, \xi_{j_r}\} \subset U$ a minimal homogeneous generating set of N;
$\mathcal{G} = \{g_1, \ldots, g_t\}$ a homogeneous left Gröbner basis of N
INITIALIZATION : $\mathcal{S} := \emptyset$, $W := U$, $\mathcal{G} := \emptyset$, $t' := 0$, $U_{\min} := \emptyset$

LOOP
$n := \min\{d_{\rm gr}(\xi_i),\ d_{\rm gr}(S_\ell(g_i,g_j)) \mid \xi_i \in W,\ S_\ell(g_i,g_j) \in \mathcal{S}\}$
$\mathcal{S}_n := \{S_\ell(g_i,g_j) \in \mathcal{S} \mid d_{\rm gr}(S_\ell(g_i,g_j)) = n\},\ W_n := \{\xi_j \in W \mid d_{\rm gr}(\xi_j) = n\}$
$\mathcal{S} := \mathcal{S} - \mathcal{S}_n,\ W := W - W_n$
  WHILE $\mathcal{S}_n \neq \emptyset$ DO
    Choose any $S_\ell(g_i,g_j) \in \mathcal{S}_n$
    $\mathcal{S}_n := \mathcal{S}_n - \{S_\ell(g_i,g_j)\}$
    $\overline{S_\ell(g_i,g_j)}^{\mathcal{G}} = \eta$
    IF $\eta \neq 0$ with $\mathbf{LM}(\eta) = a^\rho e_k$ THEN
      $t' := t'+1,\ g_{t'} := \eta$
      $\mathcal{S} := \mathcal{S} \cup \{S_\ell(g_i,g_{t'}) \mid g_i \in \mathcal{G},\ 1 \le i < t',\ \mathbf{LM}(g_i) = a^\tau e_k\}$
      $\mathcal{G} := \mathcal{G} \cup \{g_{t'}\}$
    END
  END
  WHILE $W_n \neq \emptyset$ DO
    Choose any $\xi_j \in W_n$
    $W_n := W_n - \{\xi_j\}$
    $\overline{\xi_j}^{\mathcal{G}} = \eta$
    IF $\eta \neq 0$ with $\mathbf{LM}(\eta) = a^\rho e_k$ THEN
      $U_{\min} := U_{\min} \cup \{\xi_j\}$
      $t' := t'+1,\ g_{t'} := \eta$
      $\mathcal{S} := \mathcal{S} \cup \{S_\ell(g_i,g_{t'}) \mid g_i \in \mathcal{G},\ 1 \le i < t',\ \mathbf{LM}(g_i) = a^\tau e_k\}$
      $\mathcal{G} := \mathcal{G} \cup \{g_{t'}\}$
    END
  END
UNTIL $\mathcal{S} = \emptyset$
END

PROOF: Since this algorithm is clearly a variant of **Algorithm 1** and **Algorithm 2** with a minimization procedure which works with the finite set U, it terminates after a certain integer n is executed, and the eventually obtained $\mathcal{G}$ is a homogeneous left Gröbner basis for N in which the degrees of elements are ordered non-decreasingly. It remains to prove that the eventually obtained $U_{\min}$ is a minimal homogeneous generating set of N.

As in the proof of Proposition 2.2.5, let us first bear in mind that for each n, in both the WHILE loops every new appended $S_\ell(g_i,g_{t'})$ has $d_{\rm gr}(S_\ell(g_i,g_{t'})) > n$. Moreover, for convenience, let us write $\mathcal{G}(n)$ for the $\mathcal{G}$ obtained after $\mathcal{S}_n$ is exhausted in the first WHILE loop, and write $U_{\min}[n]$, $\mathcal{G}[n]$ respectively for the $U_{\min}$, $\mathcal{G}$ obtained after W_n is exhausted in the second WHILE loop. Since the algorithm starts with $\mathcal{O} = \emptyset$ and $\mathcal{G} = \emptyset$, if, for a fixed n, we check carefully how the elements of $U_{\min}$ are chosen during executing the second WHILE loop, and how the new elements are appended to $\mathcal{G}$ after each pass through the first or the second WHILE loop, then it follows from Proposition 2.2.3 and Corollary 2.2.4

that after W_n is exhausted, the obtained $U_{\min}[n]$ and $\mathcal{G}[n]$ generate the same module, denoted $N(n)$, such that $\mathcal{G}[n]$ is an n-truncated left Gröbner basis of $N(n)$. We now use induction to show that the eventually obtained $U_{\min}$ is a minimal homogeneous generating set for N. If $U_{\min} = \emptyset$, then it is a minimal generating set of the zero module. To proceed, we assume that $U_{\min}[n]$ is a minimal homogeneous generating set for $N(n)$ after W_n is exhausted in the second WHILE loop. Suppose that n_1 is the first number after n such that $\mathcal{S}_{n_1} \neq \emptyset$. We complete the induction proof below by showing that $U_{\min}[n_1]$ is a minimal homogeneous generating set of $N(n_1)$.

If in a certain pass through the first WHILE loop, $\overline{S_\ell(g_i, g_j)}^{\mathcal{G}} = \eta \neq 0$ for some $S_\ell(g_i, g_j) \in \mathcal{S}_{n_1}$, then we note that $\eta \in N(n)$. It follows that after $\mathcal{S}_{n_1}$ is exhausted in the first WHILE loop, $\mathcal{G}(n_1)$ generates $N(n)$ and $\mathcal{G}(n_1)$ is an n_1-truncated left Gröbner basis of $N(n)$. Next, assume that $W_{n_1} = \{\xi_{j_1}, \ldots, \xi_{j_s}\} \neq \emptyset$ and that the elements of W_{n_1} are processed in the given order during executing the second WHILE loop. Since $\mathcal{G}(n_1)$ is an n_1-truncated left Gröbner basis of $N(n)$, if $\xi_{j_1} \in W_{n_1}$ is such that $\overline{\xi_{j_1}}^{\mathcal{G}(n_1)} = \eta_1 \neq 0$, then $\xi_{j_1}, \eta_1 \in L - N(n)$. By Corollary 2.2.4, we conclude that $\mathcal{G}(n_1) \cup \{\eta_1\}$ is an n_1-truncated Gröbner basis for the module $N(n) + A\eta_1$; and by Corollary 2.2.7, we conclude that $U_{\min}[n] \cup \{\xi_{j_1}\}$ is a minimal homogeneous generating set of $N(n) + A\eta_1$. Repeating this procedure, if $\xi_{j_2} \in W_{n_1}$ is such that $\overline{f_{j_2}}^{\mathcal{G}(n_1)\cup\{\eta_1\}} = \eta_2 \neq 0$, then $\xi_{j_2}, \eta_2 \in L - (N(n) + A\eta_1)$. By Corollary 2.2.4, we conclude that $\mathcal{G}(n_1) \cup \{\eta_1, \eta_2\}$ is an n_1-truncated left Gröbner basis for the module $N(n) + A\eta_1 + A\eta_2$; and by Corollary 2.2.7, we conclude that $U_{\min}[n] \cup \{\xi_{j_1}, \xi_{j_2}\}$ is a minimal homogeneous generating set of $N(n) + A\eta_1 + A\eta_2$. Continuing this procedure until W_{n_1} is exhausted, we assert that the returned $\mathcal{G}[n_1] = \mathcal{G}$ and $U_{\min}[n_1] = U_{\min}$ generate the same module $N(n_1)$ and $\mathcal{G}[n_1]$ is an n_1-truncated left Gröbner basis of $N(n_1)$ and $U_{\min}[n_1]$ is a minimal homogeneous generating set of $N(n_1)$, as desired. As all elements of U are eventually processed by the second WHILE loop, we conclude that the finally obtained $\mathcal{G}$ and $U_{\min}$ have the properties: $\mathcal{G}$ generates the module N, $\mathcal{G}$ is an n_0-truncated left Gröbner basis of N, and $U_{\min}$ is a minimal homogeneous generating set of N. □

Remark. If we are only interested in getting a minimal homogeneous generating set for the submodule N, then **Algorithm 3** can indeed be sped up. More precisely, with

$$d_{\rm gr}(\xi_1) \leq d_{\rm gr}(\xi_2) \leq \cdots \leq d_{\rm gr}(\xi_m) = n_0,$$

it follows from the proof above that if we stop executing the algorithm after S_{n_0} and W_{n_0} are exhausted, then the resulted $U_{\min}[n_0]$ is already the desired minimal homogeneous generating set for N, while $\mathcal{G}[n_0]$ is an n_0-truncated left Gröbner basis of N.

2.2.9 Corollary. *Let $U = \{\xi_1, \ldots, \xi_m\} \subset L$ be a finite set of nonzero homogeneous elements of L with $d_{\rm gr}(\xi_1) = d_{\rm gr}(\xi_2) = \cdots = d_{\rm gr}(\xi_m) = n_0$.*

(i) *If U satisfies $\mathbf{LM}(\xi_i) \neq \mathbf{LM}(\xi_j)$ for all $i \neq j$, then U is a minimal homogeneous generating set of the graded submodule $N = \sum_{i=1}^{m} A\xi_i$ of L, and meanwhile U is an n_0-truncated left Gröbner basis for N.*

(ii) *If U is a minimal left Gröbner basis of the graded submodule $N = \sum_{i=1}^{m} A\xi_i$ (i.e., U is a left Gröbner basis of N satisfying $\mathbf{LM}(\xi_i) \neq \mathbf{LM}(\xi_j)$ for all $i \neq j$), then U is a minimal homogeneous generating set of N.*

PROOF: By the assumption, it follows from the second WHILE loop of **Algorithm 3** that $U_{\min} = U$. □

Now, let us write $L_0 = \bigoplus_{i=1}^{s} Ae_i$ for the graded free A-module such that $d_{\mathrm{gr}}(e_i) = b_i$, $1 \leq i \leq s$, i.e., $L_0 = \bigoplus_{q\in\mathbb{N}} L_{0q}$ with $L_{0q} = K\text{-span}\{a^{\alpha}e_i \in \mathscr{B}(e) \mid d(a^{\alpha})+b_i = q\}$. Consider a graded submodule N of L_0 and the graded quotient module $M = L_0/N$. Our next goal is to compute a minimal homogeneous generating set for M.

Since A is Noetherian, N is a finitely generated graded submodule of L_0. Let $N = \sum_{j=1}^{m} A\xi_j$ be generated by the set of nonzero homogeneous elements $U = \{\xi_1, \ldots, \xi_m\}$, where $\xi_\ell = \sum_{k=1}^{s} f_{k\ell}e_k$ with $f_{k\ell} \in A$, $1 \leq \ell \leq m$. Then, every nonzero $f_{k\ell}$ is a homogeneous element of A such that $d_{\mathrm{gr}}(\xi_\ell) = d_{\mathrm{gr}}(f_{k\ell}e_k) = d_{\mathrm{gr}}(f_{k\ell}) + b_k$, where $b_k = d_{\mathrm{gr}}(e_k)$, $1 \leq k \leq s$, $1 \leq \ell \leq m$.

2.2.10 Lemma. *With every $\xi_\ell = \sum_{i=1}^{s} f_{i\ell}e_i$ fixed as above, $1 \leq \ell \leq m$, if the i-th coefficient f_{ij} of some ξ_j is a nonzero constant, say $f_{ij} = 1$ without loss of generality, then for each $\ell = 1, \ldots, j-1, j+1, \ldots, m$, the element $\xi'_\ell = \xi_\ell - f_{i\ell}\xi_j$ does not involve e_i. Putting $U' = \{\xi'_1, \ldots, \xi'_{j-1}, \xi'_{j+1}, \ldots, \xi'_m\}$, there is a graded A-module isomorphism $M' = L'_0/N' \cong L_0/N = M$, where $L'_0 = \bigoplus_{k\neq i} Ae_k$ and $N' = \sum_{\xi'_\ell\in U'} A\xi'_\ell$.*

PROOF: Since $f_{ij} = 1$ by the assumption, we see that every $\xi'_\ell = \sum_{k\neq i}(f_{k\ell} - f_{i\ell}f_{kj})e_k$ does not involve e_i. Let $U' = \{\xi'_1, \ldots, \xi'_{j-1}, \xi'_{j+1}, \ldots, \xi'_m\}$ and $N' = \sum_{\xi'_\ell\in U'} A\xi'_\ell$. Then $N' \subset L'_0 = \bigoplus_{k\neq i} Ae_k$. Again since $f_{ij} = 1$, we have $d_{\mathrm{gr}}(\xi_j) = d_{\mathrm{gr}}(e_i) = b_i$. It follows from the property ($\mathbb{P}4$) formulated in Subsection 2.1 that

$$\begin{aligned} d_{\mathrm{gr}}(f_{i\ell}f_{kj}e_k) &= d_{\mathrm{gr}}(f_{i\ell}) + d_{\mathrm{gr}}(f_{kj}e_k) \\ &= d_{\mathrm{gr}}(f_{i\ell}) + d_{\mathrm{gr}}(\xi_j) \\ &= d_{\mathrm{gr}}(f_{i\ell}) + b_i \\ &= d_{\mathrm{gr}}(f_{i\ell}e_i) \\ &= d_{\mathrm{gr}}(\xi_\ell) \\ &= d_{\mathrm{gr}}(f_{k\ell}e_k). \end{aligned}$$

Noticing that $d_{\mathrm{gr}}(f_{i\ell}\xi_j) = d_{\mathrm{gr}}(f_{i\ell}) + d_{\mathrm{gr}}(\xi_j)$, this shows that in the representation of ξ'_ℓ every nonzero term $(f_{k\ell} - f_{i\ell}f_{kj})e_k$ is a homogeneous element of degree $d_{\mathrm{gr}}(\xi_\ell) = d_{\mathrm{gr}}(f_{i\ell}\xi_j)$, thereby $M' = L'_0/N'$ is a graded A-module. Note that $N = N' + A\xi_j$ and that $\xi_j = e_i + \sum_{k\neq i} f_{kj}e_k$. Without making confusion, if we use the same notation $\overline{e}_k$ to denote the coset represented by e_k in M' and

M respectively, it is now clear that the desired graded A-module isomorphism $M' \xrightarrow{\varphi} M$ is naturally defined by $\varphi(\overline{e}_k) = \overline{e}_k$, $k = 1, \ldots, i-1, i+1, \ldots, s$. $\square$

Let $M = L_0/N$ be fixed as above with N generated by the set of nonzero homogeneous elements $U = \{\xi_1, \ldots, \xi_m\}$. Then since A is $\mathbb{N}$-graded with $A_0 = K$, it is well known that the homogeneous generating set $\overline{E} = \{\overline{e}_1, \ldots, \overline{e}_s\}$ of M is a minimal homogeneous generating set if and only if $\xi_\ell = \sum_{k=1}^{s} f_{k\ell} e_k$ implies $d_{\rm gr}(f_{k\ell}) > 0$ whenever $f_{k\ell} \neq 0$, $1 \leq \ell \leq m$.

2.2.11 Proposition (Compare with [KR2, Proposition 4.7.24]). *With notation fixed as above, the algorithm presented below returns a subset $\{e_{i_1}, \ldots, e_{i_{s'}}\} \subset \{e_1, \ldots, e_s\}$ and a subset $V = \{v_1, \ldots, v_t\} \subset N \cap L_0'$ such that $M \cong L_0'/N'$ as graded A-modules, where $L_0' = \bigoplus_{q=1}^{s'} Ae_{i_q}$ with $s' \leq s$ and $N' = \sum_{k=1}^{t} Av_k$, and such that $\{\overline{e}_{i_1}, \ldots, \overline{e}_{i_{s'}}\}$ is a minimal homogeneous generating set of M.*

Algorithm 4

INPUT : $E = \{e_1, \ldots, e_s\};\ U = \{\xi_1, \ldots, \xi_m\}$
  where $\xi_\ell = \sum_{k=1}^{s} f_{k\ell} e_k$ with homogeneous $f_{k\ell} \in A,\ 1 \leq \ell \leq m$
OUTPUT : $E' = \{e_{i_1}, \ldots, e_{i_{s'}}\};\ V = \{v_1, \ldots, v_t\} \subset N \cap L_0'$,
  such that $v_j = \sum_{q=1}^{s'} h_{qj} e_{i_q} \in L_0' = \bigoplus_{q=1}^{s'} Ae_{i_q}$ with $h_{qj} \notin K^*$
  whenever $h_{qj} \neq 0,\ 1 \leq j \leq t$
INITIALIZATION : $t := m;\ V := U;\ s' := s;\ E' := E;$
BEGIN
  WHILE there is a $v_j = \sum_{k=1}^{s'} f_{kj} e_k \in V$ satisfying
    $f_{kj} \notin K^*$ for $k < i$ and $f_{ij} \in K^*$ DO
    for $T = \{1, \ldots, j-1, j+1, \ldots, t\}$ compute
    $v_\ell' = v_\ell - \frac{1}{f_{ij}} f_{i\ell} v_j,\ \ell \in T,\ r = \#\{\ell \mid \ell \in T,\ v_\ell' = 0\}$
    $t := t - r - 1$
    $V := \{v_\ell = v_\ell' \mid \ell \in T,\ v_\ell' \neq 0\}$
    $= \{v_1, \ldots, v_t\}$ (after reordered)
    $s' := s' - 1$
    $E' := E' - \{e_i\} = \{e_1, \ldots, e_{s'}\}$ (after reordered)
  END
END

Proof: It is clear that the algorithm is finite. The correctness of the algorithm follows immediately from Lemma 2.2.10 and the remark we made before the proposition. $\square$

2.3 Computation of minimal finite graded free resolutions. Let $A = K[a_1, \ldots, a_n]$ be an $\mathbb{N}$-graded solvable polynomial algebra with respect to a positive-degree function $d(\ )$, and $(\mathscr{B}, \prec)$ a fixed admissible system of A. Then since A is Noetherian and $A_0 = K$, it is theoretically well known that up to a graded isomorphism of chain complexes in the category of graded A-modules, every finitely generated graded A-module M has a unique minimal graded free resolution (cf. [Eis, Chapter 19], [Kr1, Chapter 3], [Li3]). Combining the results of Subsection 2.1 and Subsection 2.2, in this section we work out the algorithmic procedures for constructing such a minimal graded free resolution over A. All notions, notations and conventions introduced before are maintained.

In what follows, $L = \bigoplus_{i=1}^{s} Ae_i$ denotes a graded free left A-module such that $d_{\rm gr}(e_i) = b_i$, $1 \le i \le s$, i.e., $L = \bigoplus_{q\in\mathbb{N}} L_q$ with $L_q = K\text{-span}\{a^\alpha e_i \in \mathscr{B}(e) \mid d(a^\alpha) + b_i = q\}$, and $\prec_e$ denotes a left monomial ordering on the K-basis $\mathscr{B}(e)$ of L. Moreover, as before we write $S_\ell(\xi_i, \xi_j)$ for the left S-polynomial of two elements ξ_i, $\xi_j \in L$.

Let $N = \sum_{i=1}^{m} A\xi_i$ be a graded submodule of L generated by the set of nonzero homogeneous elements $U = \{\xi_1, \dots, \xi_m\}$. We first demonstrate how to calculate a generating set of the syzygy module $\mathrm{Syz}(U)$ by means of a left Gröbner basis of N. To this end, let $\mathcal{G} = \{g_1, \dots, g_t\}$ be a left Gröbner basis of N with respect to $\prec_e$, then every nonzero left S-polynomial $S_\ell(g_i, g_j)$ has a left Gröbner representation $S_\ell(g_i, g_j) = \sum_{i=1}^{t} f_i g_i$ with $\mathbf{LM}(f_i g_i) \preceq_e \mathbf{LM}(S_\ell(g_i, g_j))$ whenever $f_i \neq 0$ (note that such a representation is obtained by using the division by $\mathcal{G}$ during executing the WHILE loop in **Algorithm 1** of Subsection 1.3 or **Algorithm 3** of Theorem 2.2.8). Consider the syzygy module $\mathrm{Szy}(\mathcal{G})$ of $\mathcal{G}$ in the free A-module $L_1 = \bigoplus_{i=1}^{t} A\varepsilon_i$, and put s_{ij} equal to

$$f_1\varepsilon_1 + \cdots + \left(f_i - \frac{a^{\gamma-\alpha(i)}}{\mathbf{LC}(a^{\gamma-\alpha(i)}\xi_i)}\right)\varepsilon_i + \cdots + \left(f_j + \frac{a^{\gamma-\alpha(j)}}{\mathbf{LC}(a^{\gamma-\alpha(j)}\xi_j)}\right)\varepsilon_j + \cdots + f_t\varepsilon_t,$$

$\mathcal{S} = \{s_{ij} \mid 1 \le i < j \le t\}$. Then it can be shown, actually as in the commutative case (cf. [AL, Theorem 3.7.3]), that $\mathcal{S}$ generates $\mathrm{Szy}(\mathcal{G})$. However, by employing an analogue of the Schreyer ordering $\prec_{s\text{-}\varepsilon}$ on the K-basis $\mathscr{B}(\varepsilon) = \{a^\alpha\varepsilon_i \mid a^\alpha \in \mathscr{B},\ 1 \le i \le m\}$ of L_1 induced by $\mathcal{G}$ with respect to $\prec_e$, which is defined subject to the rule: for $a^\alpha\varepsilon_i, a^\beta\varepsilon_j \in \mathscr{B}(\varepsilon)$,

$$a^\alpha\varepsilon_i \prec_{s\text{-}\varepsilon} a^\beta\varepsilon_j \Leftrightarrow \begin{cases} \mathbf{LM}(a^\alpha g_i) \prec_e \mathbf{LM}(a^\beta g_j), \\ \text{or} \\ \mathbf{LM}(a^\alpha g_i) = \mathbf{LM}(a^\beta g_j) \ \text{ and } \ i < j, \end{cases}$$

there is indeed a much stronger result, namely the noncommutative analogue of Schreyer's Theorem [Sch] (cf. Theorem 3.7.13 in [AL] for free modules over commutative polynomial algebras; Theorem 4.8 in [Lev] for free modules over solvable polynomial algebras).

2.3.1 Theorem. *With respect to the left monomial ordering $\prec_{s\text{-}\varepsilon}$ on $\mathscr{B}(\varepsilon)$ as defined above, the following statements hold.*

(i) *Let s_{ij} be determined by $S_\ell(g_i, g_j)$, where $i < j$, $\mathbf{LM}(g_i) = a^{\alpha(i)}e_s$ with $\alpha(i) = (\alpha_{i_1}, \dots, \alpha_{i_n})$, and $\mathbf{LM}(g_j) = a^{\alpha(j)}e_s$ with $\alpha(j) = (\alpha_{j_1}, \dots, \alpha_{j_n})$. Then $\mathbf{LM}(s_{ij}) = a^{\gamma-\alpha(j)}\varepsilon_j$, where $\gamma = (\gamma_1, \dots, \gamma_n)$ with each $\gamma_k = \max\{\alpha_{i_k}, \alpha_{j_k}\}$.*

(ii) *$\mathcal{S}$ is a left Gröbner basis of Syz($\mathcal{G}$), thereby $\mathcal{S}$ generates Syz($\mathcal{G}$).*

To go further, again let $\mathcal{G} = \{g_1, \ldots, g_t\}$ be the left Gröbner basis of N produced by running **Algorithm 1** presented in Subsection 1.3 (or **Algorithm 3** of Theorem 2.2.8) with the initial input data $U = \{\xi_1, \ldots, \xi_m\}$. Using the usual matrix notation for convenience, we have

$$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} = U_{m\times t}\begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix}, \quad \begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix} = V_{t\times m}\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix},$$

where the $m\times t$ matrix $U_{m\times t}$ (with entries in A) is obtained by the division by $\mathcal{G}$, and the $t\times m$ matrix $V_{t\times m}$ (with entries in A) is obtained by keeping track of the reductions during executing the WHILE loop of **Algorithm 1**. By Theorem 2.3.1, we may write $\mathrm{Syz}(\mathcal{G}) = \sum_{i=1}^{r} A\mathcal{S}_i$ with $\mathcal{S}_1, \ldots, \mathcal{S}_r \in L_1 = \bigoplus_{i=1}^{t} A\varepsilon_i$; and if $\mathcal{S}_i = \sum_{j=1}^{t} f_{ij}\varepsilon_j$, then we write $\mathcal{S}_i$ as a $1\times t$ row matrix, i.e., $\mathcal{S}_i = (f_{i_1} \ \ldots \ f_{it})$, whenever matrix notation is convenient in the according discussion. At this point, we note also that all the $\mathcal{S}_i$ may be written down one by one during executing the WHILE loop of **Algorithm 1** (or the first WHILE loop in **Algorithm 3**) successively. Furthermore, we write $D_{(1)}, \ldots, D_{(m)}$ for the rows of the matrix $D_{m\times m} = U_{m\times t}V_{t\times m} - E_{m\times m}$ where $E_{m\times m}$ is the $m\times m$ identity matrix. The following proposition is a noncommutative analogue of [AL, Theorem 3.7.6].

2.3.2 Proposition. *With notation fixed as above, the syzygy module Syz(U) of $U = \{\xi_1, \ldots, \xi_m\}$ is generated by*

$$\{\mathcal{S}_1V_{t\times m}, \ldots, \mathcal{S}_rV_{t\times m}, D_{(1)}, \ldots, D_{(m)}\},$$

where each $1\times m$ row matrix represents an element of the free A-module $\bigoplus_{i=1}^{m} A\omega_i$.

PROOF: Since

$$0 = \mathcal{S}_i\begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix} = (f_{i_1} \ \ldots \ f_{it})\begin{pmatrix} g_1 \\ \vdots \\ g_t \end{pmatrix} = (f_{i_1} \ \ldots \ f_{it})V_{t\times m}\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix},$$

we have $\mathcal{S}_iV_{t\times m}\in \mathrm{Syz}(U)$, $1\le i\le r$. Moreover, since

$$\begin{aligned} D_{m\times m}\begin{pmatrix}\xi_1\\ \vdots\\ \xi_m\end{pmatrix} &= (U_{m\times t}V_{t\times m}-E_{m\times m})\begin{pmatrix}\xi_1\\ \vdots\\ \xi_m\end{pmatrix}\\ &= U_{m\times t}V_{t\times m}\begin{pmatrix}\xi_1\\ \vdots\\ \xi_m\end{pmatrix}-\begin{pmatrix}\xi_1\\ \vdots\\ \xi_m\end{pmatrix}\\ &= U_{m\times t}\begin{pmatrix}g_1\\ \vdots\\ g_t\end{pmatrix}-\begin{pmatrix}\xi_1\\ \vdots\\ \xi_m\end{pmatrix}=\begin{pmatrix}\xi_1\\ \vdots\\ \xi_m\end{pmatrix}-\begin{pmatrix}\xi_1\\ \vdots\\ \xi_m\end{pmatrix}=0,\end{aligned}$$

we have $D_{(1)},\dots,D_{(r)}\in\mathrm{Syz}(U)$.

On the other hand, if $H=(h_1\ \dots\ h_m)$ represents the element $\sum_{i=1}^m h_i\omega_i\in\bigoplus_{i=1}^m A\omega_i$ such that $H\begin{pmatrix}\xi_1\\ \vdots\\ \xi_m\end{pmatrix}=0$, then $0=HU_{m\times t}\begin{pmatrix}g_1\\ \vdots\\ g_t\end{pmatrix}$. This means $HU_{m\times t}\in$ $\mathrm{Syz}(\mathcal{G})$. Hence, $HU_{m\times t}=\sum_{i=1}^r f_i\mathcal{S}_i$ with $f_i\in A$, and it follows that $HU_{m\times t}V_{t\times m}$ $=\sum_{i=1}^r f_i\mathcal{S}_iV_{t\times m}$. Therefore,

$$\begin{aligned} H &= H+HU_{m\times t}V_{t\times m}-HU_{m\times t}V_{t\times m}\\ &= H(E_m-U_{m\times t}V_{t\times m})+\textstyle\sum_{i=1}^r f_i\mathcal{S}_iV_{t\times m}\\ &= -HD_{m\times m}+\textstyle\sum_{i=1}^r f_i(\mathcal{S}_iV_{t\times m}).\end{aligned}$$

This shows that every element of $\mathrm{Syz}(U)$ is generated by $\{\mathcal{S}_1V_{t\times m},\dots,\mathcal{S}_rV_{t\times m},$ $D_{(1)},\dots,D_{(m)}\}$, as desired. □

Next, we recall the noncommutative version of Hilbert's syzygy theorem for solvable polynomial algebras. For a constructive proof of Hilbert's syzygy theorem by means of Gröbner bases respectively in the commutative case and the noncommutative case, we refer to Corollary 15.11 in [Eis] and Section 4.4 in [Lev].

2.3.3 Theorem. *Let $A=K[a_1,\dots,a_n]$ be a solvable polynomial algebra with admissible system $(\mathscr{B},\prec)$. Then every finitely generated left A-module M has a free resolution*

$$0\longrightarrow L_s\longrightarrow L_{s-1}\longrightarrow\cdots\longrightarrow L_0\longrightarrow M\longrightarrow 0$$

where each L_i is a free A-module of finite rank and $s\le n$. It follows that M has projective dimension $p.\dim_A M\le s$, and that A has global homological dimension $gl.\dim A\le n$. □

Now, we are able to give the main result of this subsection. Let $M=\sum_{i=1}^{s}Av_i$ be a finitely generated $\mathbb{N}$-graded left A-module with the set of homogeneous generators $\{v_1,\dots,v_s\}$ such that $d_{\rm gr}(v_i)=b_i$ for $1\le i\le s$, i.e., $M=\bigoplus_{b\in\mathbb{N}}M_b$ with each M_b the degree-b homogeneous part. Then M is isomorphic to a quotient module of the $\mathbb{N}$-graded free A-module $L_0=\bigoplus_{i=1}^{s}Ae_i$ which is equipped with the $\mathbb{N}$-gradation $L_0=\bigoplus_{q\in\mathbb{N}}L_{0q}$ such that $d_{\rm gr}(e_i)=b_i$, $1\le i\le s$, and $L_{0q}=K$-span$\{a^{\alpha}e_i\in\mathscr{B}(e)\mid d(a^{\alpha})+b_i=q\}$, where $d(\ )$ is the given positive-degree function on A. Thus we may write $M=L_0/N$, where N is a graded submodule of L_0.

Recall that a *minimal graded free resolution* of M is an exact sequence by free A-modules and A-module homomorphisms

$$\mathcal{L}_\bullet \qquad \cdots \xrightarrow{\varphi_{i+1}} L_i \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_2} L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

in which each L_i is an $\mathbb{N}$-graded free A-module with a finite homogeneous A-basis $E_i=\{e_{i_1},\dots,e_{i_{s_i}}\}$, and each φ_i is a graded A-module homomorphism of degree 0 (i.e., φ_i sends the degree-q homogeneous part of L_i into the degree-q homogeneous part of L_{i-1}), such that

(1) $\varphi_0(E_0)$ is a minimal homogeneous generating set of M, $\mathrm{Ker}\,\varphi_0=N$, and
(2) for $i\ge 1$, $\varphi_i(E_i)$ is a minimal homogeneous generating set of $\mathrm{Ker}\,\varphi_{i-1}$.

2.3.4 Theorem. *With notation fixed as above, suppose that $N=\sum_{i=1}^{m}A\xi_i$ with the set of nonzero homogeneous generators $U=\{\xi_1,\dots,\xi_m\}$. Then the graded A-module $M=L_0/N$ has a minimal graded free resolution of length $d\le n$:*

$$\mathcal{L}_\bullet \qquad 0\longrightarrow L_d \xrightarrow{\varphi_q} \cdots \xrightarrow{\varphi_2} L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

which can be constructed by implementing the following procedures:

Procedure 1. Run **Algorithm 4** of Proposition 2.2.11 with the initial input data $E=\{e_1,\dots,e_s\}$ and $U=\{\xi_1,\dots,\xi_m\}$ to compute a subset $E'=\{e_{i_1},\dots,e_{i_{s'}}\}\subset\{e_1,\dots,e_s\}$ and a subset $V=\{v_1,\dots,v_t\}\subset N\cap L_0'$ such that $M\cong L_0'/N'$ as graded A-modules, where $L_0'=\bigoplus_{q=1}^{s'}Ae_{i_q}$ with $s'\le s$ and $N'=\sum_{k=1}^{t}Av_k$, and such that $\{\overline{e}_{i_1},\dots,\overline{e}_{i_{s'}}\}$ is a minimal homogeneous generating set of M.

For convenience, after accomplishing Procedure 1 we may assume that $E=E'$, $U=V$ and $N=N'$. Accordingly we have the short exact sequence

$$0\longrightarrow N\longrightarrow L_0 \xrightarrow{\varphi_0} M\longrightarrow 0$$

such that $\varphi_0(E)=\{\overline{e}_1,\dots,\overline{e}_s\}$ is a minimal homogeneous generating set of M.

Procedure 2. Choose a left monomial ordering $\prec_e$ on the K-basis $\mathscr{B}(e)$ of L_0 and run **Algorithm 3** of Theorem 2.2.8 with the initial input data $U=\{\xi_1,\dots,\xi_m\}$ to compute a minimal homogeneous generating set $U_{\min}=\{\xi_{j_1},\dots,\xi_{j_{s_1}}\}$ and a left Gröbner basis $\mathcal{G}$ for N; at the same time, by keeping track of the reductions during executing the first WHILE loop and the second WHILE loop respectively, return the matrices $\mathcal{S}$ and V required by Proposition 2.3.2.

Procedure 3. By using the division by the left Gröbner basis $\mathcal{G}$ obtained in Procedure 2, compute the matrix U required by Proposition 2.3.2. Use the matrices $\mathcal{S}$, V obtained in Procedure 2, the matrix U and Proposition 2.3.2 to compute a homogeneous generating set of $N_1 = \mathrm{Syz}(U_{\min})$ in the $\mathbb{N}$-graded free A-module $L_1 = \bigoplus_{i=1}^{s_1} A\varepsilon_i$, where the gradation of L_1 is defined by setting $d_{\mathrm{gr}}(\varepsilon_k) = d_{\mathrm{gr}}(\xi_{j_k})$, $1 \le k \le s_1$.

Procedure 4. Construct the exact sequence

$$0 \longrightarrow N_1 \longrightarrow L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

where $\varphi_1(\varepsilon_k) = \xi_{j_k}$, $1 \le k \le s_1$. If $N_1 \ne 0$, then repeat Procedure 2–Procedure 4 for N_1 and so on.

Note that A is Noetherian (Theorem 1.2.3), A is $\mathbb{N}$-graded with the degree-0 homogeneous part $A_0 = K$, and that every finitely generated A-module M has finite projective dimension $\mathrm{p.dim}_A M \le n$ by the Hilbert's syzygy theorem for solvable polynomial algebras (Theorem 2.3.3), thereby A is an $\mathbb{N}$-graded local ring of finite global homological dimension. It follows from the literature ([Eis, Chapter 19], [Kr1, Chapter 3], [Li3]) that

$$\mathrm{p.dim}_A M = \text{ the length of a minimal graded free resolution of } M.$$

Hence, the desired minimal finite graded free resolution $\mathcal{L}_\bullet$ for M is then obtained after finite times of processing the above procedures.

3. Computation of minimal filtered free resolutions

Let $A = K[a_1, \ldots, a_n]$ be an $\mathbb{N}$-filtered solvable polynomial K-algebra with the filtration $\{F_pA\}_{p\in\mathbb{N}}$ determined by a positive-degree function $d(\ )$ on A (see Subsection 3.1 below). In this section we introduce minimal filtered free resolutions for finitely generated modules over A by introducing minimal F-bases and minimal standard bases for A-modules and their submodules with respect to good filtration. We show that any two minimal F-bases, respectively any two minimal standard bases, have the same number of elements and the same number of elements of the same filtered degree, minimal filtered free resolutions are unique up to strict filtered isomorphism of chain complexes in the category of filtered A-modules, and that minimal filtered free resolutions can be algorithmically computed in case A has a graded monomial ordering $\prec_{gr}$.

3.1 $\mathbb{N}$-filtered solvable polynomial algebras and filtered free modules. Recall that the $\mathbb{N}$-filtered solvable polynomial algebras with $\mathbb{N}$-filtration determined by the natural length of elements from the PBW K-basis (especially the quadric solvable polynomial algebras) were studied in [LW], [Li1]. In this subsection, we formulate more generally the structure of $\mathbb{N}$-filtered solvable polynomial algebras by means of positive-degree functions, and we construct filtered free modules over such algebras. Since the standard bases we are going to introduce in terms of good filtration are generalization of classical Macaulay bases (see a remark given in Subsection 3.3), while a classical Macaulay basis V is characterized in terms of both the leading homogeneous elements (degree forms) of V and the homogenized elements of V (cf. [KR2, p. 38, p. 55]), accordingly both the associated graded algebra (module) and the Rees algebra (module) of an $\mathbb{N}$-filtered solvable polynomial algebra (of a filtered module) are necessarily introduced in our text. All notions, notations and conventions used before are maintained.

Let $A = K[a_1, \dots a_n]$ be a solvable polynomial algebra with admissible system $(\mathscr{B}, \prec)$, where $\mathscr{B} = \{a^\alpha = a_1^{\alpha_1} \cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$ is the PBW K-basis of A and $\prec$ is a monomial ordering on $\mathscr{B}$, and let $d(\ )$ be a positive-degree function on A such that $d(a_i) = m_i > 0$, $1 \le i \le n$ (see Subsection 1.2). Put

$$F_pA = K\text{-span}\{a^\alpha \in \mathscr{B} \mid d(a^\alpha) \le p\}, \quad p \in \mathbb{N},$$

then it is clear that $F_pA \subseteq F_{p+1}A$ for all $p \in \mathbb{N}$, $A = \bigcup_{p\in\mathbb{N}} F_pA$, and $1 \in F_0A = K$.

3.1.1 Definition. With notation as above, if $F_pAF_qA \subseteq F_{p+q}A$ holds for all $p, q \in \mathbb{N}$, then we call A an $\mathbb{N}$-*filtered solvable polynomial algebra with respect to the positive-degree function* $d(\ )$, and accordingly we call $FA = \{F_pA\}_{p\in\mathbb{N}}$ the $\mathbb{N}$-*filtration of* A *determined by* $d(\ )$.

Note that the $\mathbb{N}$-filtration FA constructed above is clearly *separated* in the sense that if f is a nonzero element of A, then $f \in F_pA - F_{p-1}A$ for some p. Thus, if $f \in F_pA - F_{p-1}A$, then we say that f has *filtered degree* p and we use $d_{\rm fil}(f)$ to denote this degree, i.e.,

$$(\mathbb{P}5) \qquad d_{\rm fil}(f) = p \Longleftrightarrow f \in F_pA - F_{p-1}A.$$

Bearing in mind ($\mathbb{P}$5), the following featured property of FA will very much help us to deal with the associated graded structures of A and filtered A-modules.

3.1.2 Lemma. *If* $f = \sum_i \lambda_i a^{\alpha(i)}$ *with* $\lambda_i \in K^*$ *and* $a^{\alpha(i)} \in \mathscr{B}$, *then* $d_{\rm fil}(f) = p$ *if and only if* $d(a^{\alpha(i')}) = p$ *for some* i' *if and only if* $d(f) = p = d_{\rm fil}(f)$. □

Observation. Given a solvable polynomial algebra $A = K[a_1, \dots, a_n]$ and a positive-degree function $d(\ )$ on A, it follows from Definition 1.2.2, Definition 3.1.1 and Lemma 3.1.2 that

> A is an $\mathbb{N}$-filtered solvable polynomial algebra with respect to $d(\ )$ if and only if for $1 \le i < j \le n$, all the relations $a_ja_i = \lambda_{ji}a_ia_j + f_{ji}$ with $f_{ji} = \sum \mu_k a^{\alpha(k)}$ derived from Definition 1.2.2 satisfy $d(a^{\alpha(k)}) \le d(a_ia_j)$ whenever $\mu_k \ne 0$.

With the observation given above, the following examples may be better understood.

Example. (1) If $A = K[a_1, \ldots, a_n]$ is an $\mathbb{N}$-graded solvable polynomial algebra with respect to a positive-degree function $d(\ )$, i.e., $A = \bigoplus_{p\in\mathbb{N}} A_p$ with the degree-p homogeneous part $A_p = K\text{-span}\{a^\alpha \in \mathscr{B} \mid d(a^\alpha) = p\}$ (see Subsection 2.1), then, with respect to the same positive-degree function $d(\ )$ on A, A is turned into an $\mathbb{N}$-filtered solvable polynomial algebra with the $\mathbb{N}$-filtration $FA = \{F_pA\}_{p\in\mathbb{N}}$ where each $F_pA = \bigoplus_{q\le p} A_q$.

Example. (2) Let $A = K[a_1, \ldots a_n]$ be a solvable polynomial algebra with the admissible system $(\mathscr{B}, \prec_{gr})$, where $\prec_{gr}$ is a graded monomial ordering on $\mathscr{B}$ with respect to a given positive-degree function $d(\ )$ on A (see the definition of $\prec_{gr}$ given in Subsection 1.2). Then by referring to Definition 1.2.2 and the above observation, one easily sees that A is an $\mathbb{N}$-filtered solvable polynomial algebra with respect to the same $d(\ )$. In the case where $\prec_{gr}$ respects $d(a_i) = 1$ for $1 \le i \le n$, Definition 1.2.2 entails that the generators of A satisfy

$$\begin{array}{rcl} a_ja_i &=& \lambda_{ji}a_ia_j + \sum \lambda_{k\ell}^{ji}a_ka_\ell + \sum \lambda_t^{ji}a_t + \mu_{ji}, \\ && \text{where } 1 \le i < j \le n,\ \lambda_{ji} \in K^*,\ \lambda_{k\ell}^{ji}, \lambda_t^{ji}, \mu_{ji} \in K. \end{array}$$

In [Li1] such $\mathbb{N}$-filtered solvable polynomial algebras are referred to as *quadric solvable polynomial algebras* which include numerous significant algebras such as Weyl algebras and enveloping algebras of Lie algebras. One is referred to [Li1] for some detailed study on quadric solvable polynomial algebras.

The next example (quoted from [Li4]) provides $\mathbb{N}$-filtered solvable polynomial algebras in which some generators may have degree ≥ 2.

Example. (3) Considering the $\mathbb{N}$-graded structure of the free K-algebra $K\langle X\rangle = K\langle X_1, X_2, X_3\rangle$ by assigning X_1 the degree 2, X_2 the degree 1 and X_3 the degree 4, let I be the ideal of $K\langle X\rangle$ generated by the elements

$$\begin{aligned} &g_1 = X_1X_2 - X_2X_1, \\ &g_2 = X_3X_1 - \lambda X_1X_3 - \mu X_3X_2^2 - f(X_2), \\ &g_3 = X_3X_2 - X_2X_3, \end{aligned}$$

where $\lambda \in K^*$, $\mu \in K$, and $f(X_2) \in K\text{-span}\{1, X_2, X_2^2, \ldots, X_2^6\}$. If we use the graded lexicographic ordering $X_2 \prec_{grlex} X_1 \prec_{grlex} X_3$ on $K\langle X\rangle$, then it is straightforward to verify that $\mathcal{G} = \{g_1, g_2, g_3\}$ forms a Gröbner basis for I, and that $\mathscr{B} = \{a^\alpha = a_1^{\alpha_1}a_2^{\alpha_2}a_3^{\alpha_3} \mid \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3\}$ is a PBW basis for the K-algebra A, where $A = K[a_1, a_2, a_3] = K\langle X\rangle/I$ with a_1, a_2 and a_3 denoting the cosets $X_1 + I$, $X_2 + I$ and $X_3 + I$ in $K\langle X\rangle/I$ respectively. Since $a_3a_1 = \lambda a_1a_3 + \mu a_2^2a_3 + f(a_2)$, where $f(a_2) \in K\text{-span}\{1, a_2, a_2^2, \ldots, a_2^6\}$, we see that A has the monomial ordering $\prec_{lex}$ on $\mathscr{B}$ such that $a_3 \prec_{lex} a_2 \prec_{lex} a_1$ and $\mathbf{LM}(\mu a_2^2a_3 + f(a_2)) \prec_{lex} a_1a_3$. Thereby A is turned into an $\mathbb{N}$-filtered solvable polynomial algebra with respect to $\prec_{lex}$ and the degree function $d(\ )$ such that $d(a_1) = 2$, $d(a_2) = 1$, and $d(a_3) = 4$. Moreover, one may also check that with respect to the same degree function $d(\ )$, the graded lexicographic ordering $a_3 \prec_{grlex} a_2 \prec_{grlex} a_1$ is another choice to make A into an $\mathbb{N}$-filtered solvable polynomial algebra.

Let A be an $\mathbb{N}$-filtered solvable polynomial algebra with respect to a given positive-degree function $d(\ )$, and let $FA = \{F_pA\}_{p\in\mathbb{N}}$ be the $\mathbb{N}$-filtration of A determined by $d(\ )$. Then A has the associated $\mathbb{N}$-graded K-algebra $G(A) = \bigoplus_{p\in\mathbb{N}} G(A)_p$ with $G(A)_0 = F_0A = K$ and $G(A)_p = F_pA/F_{p-1}A$ for $p \geq 1$, where for $\overline{f} = f + F_{p-1}A \in G(A)_p$, $\overline{g} = g + F_{q-1}A$, the multiplication is given by $\overline{f}\,\overline{g} = fg + F_{p+q-1}A \in G(A)_{p+q}$. Another $\mathbb{N}$-graded K-algebra determined by FA is the Rees algebra $\widetilde{A}$ of A, which is defined as $\widetilde{A} = \bigoplus_{p\in\mathbb{N}} \widetilde{A}_p$ with $\widetilde{A}_p = F_pA$, where the multiplication of $\widetilde{A}$ is induced by $F_pAF_qA \subseteq F_{p+q}A$, $p, q \in \mathbb{N}$. For convenience, we fix the following notations once for all.

- If $h \in G(A)_p$ and $h \neq 0$, then we write $d_{\rm gr}(h)$ for the degree of h as a homogeneous element of $G(A)$, i.e., $d_{\rm gr}(h) = p$.
- If $H \in \widetilde{A}_p$ and $H \neq 0$, then we write $d_{\rm gr}(H)$ for the degree of H as a homogeneous element of $\widetilde{A}$, i.e., $d_{\rm gr}(H) = p$.

Concerning the $\mathbb{N}$-graded structure of $G(A)$, if $f \in A$ with $d_{\rm fil}(f) = p$, then by Lemma 3.1.2, the coset $f + F_{p-1}A$ represented by f in $G(A)_p$ is a nonzero homogeneous element of degree p. If we denote this homogeneous element by $\sigma(f)$ (in the literature it is referred to as the principal symbol of f), then $d_{\rm fil}(f) = p = d_{\rm gr}(\sigma(f))$. However, considering the Rees algebra $\widetilde{A}$ of A, we note that a nonzero $f \in F_qA$ represents a homogeneous element of degree q in $\widetilde{A}_q$ on one hand, and on the other hand it represents a homogeneous element of degree q_1 in $\widetilde{A}_{q_1}$, where $q_1 = d_{\rm fil}(f) \leq q$. So, for a nonzero $f \in F_pA$, we denote the corresponding homogeneous element of degree p in $\widetilde{A}_p$ by $h_p(f)$, while we use $\widetilde{f}$ to denote the homogeneous element represented by f in $\widetilde{A}_{p_1}$ with $p_1 = d_{\rm fil}(f) \leq p$. Thus, $d_{\rm gr}(\widetilde{f}) = d_{\rm fil}(f)$, and we see that $h_p(f) = \widetilde{f}$ if and only if $d_{\rm fil}(f) = p$.

Furthermore, if we write Z for the homogeneous element of degree 1 in $\widetilde{A}_1$ represented by the multiplicative identity element 1, then Z is a central regular element of $\widetilde{A}$, i.e., Z is not a divisor of zero and is contained in the center of $\widetilde{A}$. Bringing this homogeneous element Z into play, the homogeneous elements of $\widetilde{A}$ are featured as follows.

If $f \in A$ with $d_{\rm fil}(f) = p_1$ then for all $p \geq p_1$, $h_p(f) = Z^{p-p_1}\widetilde{f}$. In other words, if $H \in \widetilde{A}_p$ is any nonzero homogeneous element of degree p, then there is some $f \in F_pA$ such that $H = Z^{p-d(f)}\widetilde{f} = \widetilde{f} + (Z^{p-d(f)} - 1)\widetilde{f}$.

It follows that by sending H to $f + F_{p-1}A$ and sending H to f respectively, $G(A) \cong \widetilde{A}/\langle Z\rangle$ as $\mathbb{N}$-graded K-algebras and $A \cong \widetilde{A}/\langle 1 - Z\rangle$ as K-algebras (cf. [LVO]).

Since a solvable polynomial algebra A is necessarily a domain (Theorem 1.2.3), we summarize two useful properties concerning the multiplication of $G(A)$ and $\widetilde{A}$ respectively into the following lemma. Notations are as given before.

3.1.3 Lemma. *Let f, g be nonzero elements of A with $d_{\rm fil}(f) = p_1$, $d_{\rm fil}(g) = p_2$. Then*

(i) $\sigma(f)\sigma(g) = \sigma(fg)$;

(ii) $\widetilde{f}\widetilde{g} = \widetilde{fg}$. *If $p_1 + p_2 \le p$, then $h_p(fg) = Z^{p-p_1-p_2}\widetilde{f}\widetilde{g}$.* □

With the preparation made above, the results given in the next theorem, which are analogues of those concerning quadric solvable polynomial algebras in [LW, Section 3], [Li1, Chapter IV], may be derived in a similar way as in loc. cit. Hence the detailed proofs are omitted.

3.1.4 Theorem. *Let $A = K[a_1, \ldots, a_n]$ be a solvable polynomial algebra with the admissible system $(\mathscr{B}, \prec_{gr})$, where $\prec_{gr}$ is a graded monomial ordering on $\mathscr{B}$ with respect to a given positive-degree function $d(\ )$ on A, thereby A is an $\mathbb{N}$-filtered solvable polynomial algebra with respect to the same $d(\ )$ by the foregoing Example (2), and let $FA = \{F_pA\}_{p\in\mathbb{N}}$ be the corresponding $\mathbb{N}$-filtration of A. Considering the associated graded algebra $G(A)$ as well as the Rees algebra $\widetilde{A}$ of A, the following statements hold.*

(i) $G(A) = K[\sigma(a_1), \ldots, \sigma(a_n)]$, *$G(A)$ has the PBW K-basis*

$$\sigma(\mathscr{B}) = \{\sigma(a)^\alpha = \sigma(a_1)^{\alpha_1}\cdots\sigma(a_n)^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\},$$

and, by referring to Definition 1.2.2, for $\sigma(a)^\alpha$, $\sigma(a)^\beta \in \sigma(\mathscr{B})$ such that $a^\alpha a^\beta = \lambda_{\alpha,\beta}a^{\alpha+\beta} + f_{\alpha,\beta}$, where $\lambda_{\alpha,\beta} \in K^*$, if $f_{\alpha,\beta} = 0$ then

$$\sigma(a)^\alpha\sigma(a)^\beta = \lambda_{\alpha,\beta}\sigma(a)^{\alpha+\beta},$$

where

$$\sigma(a)^{\alpha+\beta} = \sigma(a_1)^{\alpha_1+\beta_1}\cdots\sigma(a_n)^{\alpha_n+\beta_n}.$$

In the case where $f_{\alpha,\beta} = \sum_j \mu_j^{\alpha,\beta}a^{\alpha(j)} \ne 0$ with $\mu_j^{\alpha,\beta} \in K$,

$$\sigma(a)^\alpha\sigma(a)^\beta = \lambda_{\alpha,\beta}\sigma(a)^{\alpha+\beta} + \sum_{d(a^{\alpha(k)})=d(a^{\alpha+\beta})} \mu_j^{\alpha,\beta}\sigma(a)^{\alpha(k)}.$$

Moreover, the ordering $\prec_{_{G(A)}}$ defined on $\sigma(\mathscr{B})$ subject to the rule:

$$\sigma(a)^\alpha \prec_{_{G(A)}} \sigma(a)^\beta \Longleftrightarrow a^\alpha \prec_{gr} a^\beta, \quad a^\alpha, a^\beta \in \mathscr{B},$$

is a graded monomial ordering with respect to the positive-degree function $d(\ )$ on $G(A)$ such that $d(\sigma(a_i)) = d(a_i)$ for $1 \le i \le n$, that turns $G(A)$ into an $\mathbb{N}$-graded solvable polynomial algebra.

(ii) $\widetilde{A} = K[\widetilde{a}_1, \ldots, \widetilde{a}_n, Z]$ *where Z is the central regular element of degree 1 in $\widetilde{A}_1$ represented by 1, $\widetilde{A}$ has the PBW K-basis*

$$\widetilde{\mathscr{B}} = \{\widetilde{a}^\alpha Z^m = \widetilde{a}_1^{\alpha_1}\cdots\widetilde{a}_n^{\alpha_n}Z^m \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n, m \in \mathbb{N}\},$$

and, by referring to Definition 1.2.2, for $\widetilde{a}^{\alpha}Z^{s}$, $\cdot\widetilde{a}^{\beta}Z^{t} \in \widetilde{\mathscr{B}}$ *such that* $a^{\alpha}a^{\beta} = \lambda_{\alpha,\beta}a^{\alpha+\beta} + f_{\alpha,\beta}$, *where* $\lambda_{\alpha,\beta} \in K^{*}$, *if* $f_{\alpha,\beta} = 0$ *then*

$$\widetilde{a}^{\alpha}Z^{s}\cdot\widetilde{a}^{\beta}Z^{t} = \lambda_{\alpha,\beta}\widetilde{a}^{\alpha+\beta}Z^{s+t}, \quad \text{where } \widetilde{a}^{\alpha+\beta} = \widetilde{a}_{1}^{\alpha_1+\beta_1}\cdots\widetilde{a}_{n}^{\alpha_n+\beta_n};$$

and in the case where $f_{\alpha,\beta} = \sum_j \mu_j^{\alpha,\beta}a^{\alpha(j)} \neq 0$ *with* $\mu_j^{\alpha,\beta} \in K$,

$$\begin{array}{rcl}\widetilde{a}^{\alpha}Z^{s}\cdot\widetilde{a}^{\beta}Z^{t} & = & \lambda_{\alpha,\beta}\widetilde{a}^{\alpha+\beta}Z^{s+t} + \sum_j \mu_j^{\alpha,\beta}\widetilde{a}^{\alpha(j)}Z^{q-m_j}, \\ & & \text{where } q = d(a^{\alpha+\beta}) + s + t,\ m_j = d(a^{\alpha(j)}).\end{array}$$

Moreover, the ordering $\prec_{\widetilde{A}}$ *defined on* $\widetilde{\mathscr{B}}$ *subject to the rule:*

$$\widetilde{a}^{\alpha}Z^{s} \prec_{\widetilde{A}} \widetilde{a}^{\beta}Z^{t} \Longleftrightarrow a^{\alpha} \prec_{gr} a^{\beta}, \text{ or } a^{\alpha} = a^{\beta} \text{ and } s < t, \quad a^{\alpha}, a^{\beta} \in \mathscr{B},$$

is a monomial ordering on $\widetilde{\mathscr{B}}$ *(which is not necessarily a graded monomial ordering), that turns* $\widetilde{A}$ *into an* $\mathbb{N}$*-graded solvable polynomial algebra with respect to the positive-degree function* $d(\ )$ *on* $\widetilde{A}$ *such that* $d(Z) = 1$ *and* $d(\widetilde{a}_i) = d(a_i)$ *for* $1 \leq i \leq n$. □

By referring to Lemma 3.1.2 and Lemma 3.1.3, the corollary presented below is straightforward and will be very often used in discussing left Gröbner bases and standard bases for submodules of filtered free A-modules and their associated graded free $G(A)$-modules as well the graded free $\widetilde{A}$-modules (Subsection 3.2, Subsection 3.3).

3.1.5 Corollary. *With the assumption and notations as in Theorem* 3.1.4, *if* $f = \lambda a^{\alpha} + \sum_j \mu_j a^{\alpha(j)}$ *with* $d(f) = p$ *and* $\mathbf{LM}(f) = a^{\alpha}$, *then* $p = d_{\rm fil}(f) = d_{\rm gr}(\sigma(f)) = d_{\rm gr}(\widetilde{f})$, *and*

$$\begin{array}{l}\sigma(f) = \lambda\sigma(a)^{\alpha} + \sum_{d(a^{\alpha(j_k)})=p}\mu_{j_k}\sigma(a)^{\alpha(j_k)}; \\ \mathbf{LM}(\sigma(f)) = \sigma(a)^{\alpha} = \sigma(\mathbf{LM}(f)); \\ \widetilde{f} = \lambda\widetilde{a}^{\alpha} + \sum_j \mu_j\widetilde{a}^{\alpha(j)}Z^{p-d(a^{\alpha(j)})}; \\ \mathbf{LM}(\widetilde{f}) = \widetilde{a}^{\alpha} = \widetilde{\mathbf{LM}(f)},\end{array}$$

where $\mathbf{LM}(f)$, $\mathbf{LM}(\sigma(f))$ *and* $\mathbf{LM}(\widetilde{f})$ *are taken with respect to* $\prec_{gr}$, $\prec_{G(A)}$ *and* $\prec_{\widetilde{A}}$ *respectively.*

Let A be an $\mathbb{N}$-filtered solvable polynomial algebra with the filtration $FA = \{F_pA\}_{p\in\mathbb{N}}$ determined by a positive-degree function $d(\ )$ on A, and let $(\mathscr{B}, \prec)$ be a fixed admissible system of A. Consider a free A-module $L = \bigoplus_{i=1}^{s} Ae_i$ with the A-basis $\{e_1, \ldots, e_s\}$. Then L has the K-basis $\mathscr{B}(e) = \{a^{\alpha}e_i \mid a^{\alpha} \in \mathscr{B},\ 1 \leq i \leq s\}$. If $\{b_1, \ldots, b_s\}$ is an *arbitrarily* fixed subset of $\mathbb{N}$, then, with $FL = \{F_qL\}_{q\in\mathbb{N}}$ defined by putting

$$F_qL = \{0\} \ \text{ if } \ q < \min\{b_1, \dots, b_s\}; \ \text{ otherwise } \ F_qL = \sum_{i=1}^{s} \left(\sum_{p_i + b_i \le q} F_{p_i}A \right) e_i,$$

or alternatively, for $q \ge \min\{b_1, \dots, b_s\}$,

$$F_qL = K\text{-span}\{a^\alpha e_i \in \mathscr{B}(e) \mid d(a^\alpha) + b_i \le q\},$$

L forms an $\mathbb{N}$-*filtered free A-module* with respect to the $\mathbb{N}$-filtered structure of A, that is, every F_qL is a K-subspace of L, $F_qL \subseteq F_{q+1}L$ for all $q \in \mathbb{N}$, $L = \bigcup_{q\in\mathbb{N}} F_qL$, $F_pAF_qL \subseteq F_{p+q}L$ for all $p, q \in \mathbb{N}$, and for each $i = 1, \dots, s$,

$$e_i \in F_0L \ \text{ if } \ b_i = 0; \ \text{ otherwise } \ e_i \in F_{b_i}L - F_{b_i-1}L.$$

Convention. Let A be an $\mathbb{N}$-filtered solvable polynomial algebra with respect to a positive-degree function $d(\ )$. Unless otherwise stated, from now on in the subsequent sections if we say that $L = \bigoplus_{i=1}^{s} Ae_i$ is a filtered free A-module with the filtration $FL = \{F_qL\}_{q\in\mathbb{N}}$, then FL is always meant the type as constructed above.

Let $L = \bigoplus_{i=1}^{s} Ae_i$ be a filtered free A-module with the filtration $FL = \{F_qL\}_{q\in\mathbb{N}}$, which is constructed with respect to a given subset $\{b_1, \dots, b_s\} \subset \mathbb{N}$. Then FL is *separated* in the sense that if ξ is a nonzero element of L, then $\xi \in F_qL - F_{q-1}L$ for some q. Thus, to make the discussion on FL compatible with FA, if $\xi \in F_qL - F_{q-1}L$, then we say that ξ has *filtered degree* q and we use $d_{\rm fil}(\xi)$ to denote this degree, i.e.,

$$(\mathbb{P}6) \qquad d_{\rm fil}(\xi) = q \Longleftrightarrow \xi \in F_qL - F_{q-1}L.$$

For instance, we have $d_{\rm fil}(e_i) = b_i$, $1 \le i \le s$. Comparing with Lemma 3.1.2 we first note the following:

3.1.6 Lemma. *Let $\xi \in L$. Then $d_{\rm fil}(\xi) = q$ if and only if $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)} e_j$, where $\lambda_{ij} \in K^*$ and $a^{\alpha(i_j)} \in \mathscr{B}$ with $\alpha(i_j) = (\alpha_{i_{j1}}, \dots, \alpha_{i_{jn}}) \in \mathbb{N}^n$, in which some monomial $a^{\alpha(i_j)} e_j$ satisfies $d(a^{\alpha(i_j)}) + b_j = q$.*

Let $L = \bigoplus_{i=1}^{s} Ae_i$ be a filtered free A-module with the filtration $FL = \{F_qL\}_{q\in\mathbb{N}}$ such that $d_{\rm fil}(e_i) = b_i$, $1 \le i \le s$. Considering the associated $\mathbb{N}$-graded algebra $G(A)$ of A, the filtered free A module L has the *associated $\mathbb{N}$-graded $G(A)$-module* $G(L) = \bigoplus_{q\in\mathbb{N}} G(L)_q$ with $G(L)_q = F_qL/F_{q-1}L$, where for $\overline{f} = f + F_{p-1}A \in G(A)_p$, $\overline{\xi} = \xi + F_{q-1}L \in G(L)_q$, the module action is given by $\overline{f} \cdot \overline{\xi} = f\xi + F_{p+q-1}L \in G(L)_{p+q}$. As with homogeneous elements in $G(A)$, if $h \in G(L)_q$ and $h \ne 0$, then we write $d_{\rm gr}(h)$ for the degree of h as a homogeneous element of $G(L)$, i.e., $d_{\rm gr}(h) = q$. If $\xi \in L$ with $d_{\rm fil}(\xi) = q$, then the coset $\xi + F_{q-1}L$ represented by ξ in $G(L)_q$ is a nonzero homogeneous element of degree q, and if we denote this homogeneous element by $\sigma(\xi)$ (in the literature it is referred to as the principal symbol of ξ) then $d_{\rm gr}(\sigma(\xi)) = q = d_{\rm fil}(\xi)$.

Furthermore, considering the Rees algebra $\widetilde{A}$ of A, the filtration $FL = \{F_qL\}_{q\in\mathbb{N}}$ of L also defines the *Rees module* $\widetilde{L}$ of L, which is the $\mathbb{N}$-graded $\widetilde{A}$-module $\widetilde{L} = \bigoplus_{q\in\mathbb{N}}\widetilde{L}_q$, where $\widetilde{L}_q = F_qL$ and the module action is induced by $F_pAF_qL \subseteq F_{p+q}L$. As with homogeneous elements in $\widetilde{A}$, if $H \in \widetilde{L}_q$ and $H \neq 0$, then we write $d_{\rm gr}(H)$ for the degree of H as a homogeneous element of $\widetilde{L}$, i.e., $d_{\rm gr}(H) = q$. Note that any nonzero $\xi \in F_qL$ represents a homogeneous element of degree q in $\widetilde{L}_q$ on one hand, and on the other hand it represents a homogeneous element of degree q_1 in $\widetilde{L}_{q_1}$, where $q_1 = d_{\rm fil}(\xi) \le q$. So, for a nonzero $\xi \in F_qL$ we denote the corresponding homogeneous element of degree q in $\widetilde{L}_q$ by $h_q(\xi)$, while we use $\widetilde{\xi}$ to denote the homogeneous element represented by ξ in $\widetilde{L}_{q_1}$ with $q_1 = d_{\rm fil}(\xi) \le q$. Thus, $d_{\rm gr}(\widetilde{\xi}) = d_{\rm fil}(\xi)$, and we see that $h_q(\xi) = \widetilde{\xi}$ if and only if $d_{\rm fil}(\xi) = q$.

We also note that if Z denotes the homogeneous element of degree 1 in $\widetilde{A}_1$ represented by the multiplicative identity element 1, then, similar to the discussion given before Theorem 3.1.4, there are A-module isomorphism $L \cong \widetilde{L}/(1-Z)\widetilde{L}$ and graded $G(A)$-module isomorphism $G(L) \cong \widetilde{L}/Z\widetilde{L}$.

3.1.7 Lemma. *With notation as above, the following statements hold.*

(i) $d_{\rm fil}(f\xi) = d(f) + d_{\rm fil}(\xi)$ *holds for all nonzero* $f \in A$ *and nonzero* $\xi \in L$.

(ii) $\sigma(f)\sigma(\xi) = \sigma(f\xi)$ *holds for all nonzero* $f \in A$ *and nonzero* $\xi \in L$.

(iii) *If* $\xi \in L$ *with* $d_{\rm fil}(\xi) = q \le \ell$, *then* $h_\ell(\xi) = Z^{\ell-q}\widetilde{\xi}$. *Furthermore, let* $f \in A$ *with* $d_{\rm fil}(f) = p$, $\xi \in L$ *with* $d_{\rm fil}(\xi) = q$. *Then* $\widetilde{f\xi} = \widetilde{f}\widetilde{\xi}$; *if* $p + q \le \ell$, *then* $h_\ell(f\xi) = Z^{\ell-p-q}\widetilde{f}\widetilde{\xi}$.

PROOF: Since A is a solvable polynomial algebra, $G(A)$ and $\widetilde{A}$ are $\mathbb{N}$-graded solvable polynomial algebras by Theorem 3.1.4, thereby they are necessarily domains (Theorem 1.2.3). By the foregoing ($\mathbb{P}5$), ($\mathbb{P}6$) and Lemma 3.1.6, the verification of (i)–(iii) are then straightforward. $\square$

3.1.8 Proposition. *With notation fixed as before, let* $L = \bigoplus_{i=1}^{s} Ae_i$ *be a filtered free* A*-module with the filtration* $FL = \{F_qL\}_{q\in\mathbb{N}}$ *such that* $d_{\rm fil}(e_i) = b_i$, $1 \le i \le s$. *The following two statements hold.*

(i) $G(L)$ *is an* $\mathbb{N}$*-graded free* $G(A)$*-module with the homogeneous* $G(A)$*-basis* $\{\sigma(e_1), \dots, \sigma(e_s)\}$, *that is,* $G(L) = \bigoplus_{i=1}^{s} G(A)\sigma(e_i) = \bigoplus_{q\in\mathbb{N}} G(L)_q$ *with*

$$G(L)_q = \sum_{p_i+b_i=q} G(A)_{p_i}\sigma(e_i) \quad q \in \mathbb{N}.$$

Moreover, $\sigma(\mathscr{B}(e)) = \{\sigma(a^\alpha e_i) = \sigma(a)^\alpha\sigma(e_i) \mid a^\alpha e_i \in \mathscr{B}(e)\}$ *forms a* K*-basis for* $G(L)$.

(ii) $\widetilde{L}$ *is an* $\mathbb{N}$*-graded free* $\widetilde{A}$*-module with the homogeneous* $\widetilde{A}$*-basis* $\{\widetilde{e}_1, \ldots, \widetilde{e}_s\}$*, that is,* $\widetilde{L} = \bigoplus_{i=1}^{s} \widetilde{A}\widetilde{e}_i = \bigoplus_{q\in\mathbb{N}} \widetilde{L}_q$ *with*

$$\widetilde{L}_q = \sum_{p_i+b_i=q} \widetilde{A}_{p_i}\widetilde{e}_i, \quad q \in \mathbb{N}.$$

Moreover, $\widetilde{\mathscr{B}(e)} = \{\widetilde{a}^{\alpha} Z^m \widetilde{e}_i \mid \widetilde{a}^{\alpha} Z^m \in \widetilde{\mathscr{B}},\ 1 \le i \le s\}$ *forms a* K*-basis for* $\widetilde{L}$*, where* $\widetilde{\mathscr{B}}$ *is the PBW* K*-basis of* $\widetilde{A}$ *determined in Theorem* 4.4(ii).

Proof: Since $d_{\rm fil}(e_i) = b_i$, $1 \le i \le s$, if $\xi = \sum_{i=1}^{s} f_i e_i \in F_q L = \sum_{i=1}^{s}(\sum_{p_i+b_i\le q} F_{p_i}A)e_i$, then $d_{\rm fil}(\xi) \le q$. By Lemma 3.1.7,

$$\sigma(\xi) = \sum_{d(f_i)+b_i=q} \sigma(f_i)\sigma(e_i) \in \sum_{i=1}^{s} G(A)_{q-b_i}\sigma(e_i),$$

$$h_q(\xi) = \sum_{i=1}^{s} Z^{q-d(f_i)-b_i}\widetilde{f}_i\widetilde{e}_i \in \sum_{i=1}^{s} \widetilde{A}_{q-b_i}\widetilde{e}_i.$$

This shows that $\{\sigma(e_1), \ldots, \sigma(e_s)\}$ and $\{\widetilde{e}_1, \ldots, \widetilde{e}_s\}$ generate the $G(A)$-module $G(L)$ and the $\widetilde{A}$-module $\widetilde{L}$, respectively. Next, since each $\sigma(e_i)$ is a homogeneous element of degree b_i, if a degree-q homogeneous element $\sum_{i=1}^{s}\sigma(f_i)\sigma(e_i) = 0$, where $f_i \in A$, $d_{\rm fil}(f_i) + b_i = q$, $1 \le i \le s$, then $\sum_{i=1}^{s} f_i e_i \in F_{q-1}L$ and hence each $f_i \in F_{q-1-b_i}A$ by Lemma 3.1.6, a contradiction. It follows that $\{\sigma(e_1), \ldots, \sigma(e_s)\}$ is linearly independent over $G(A)$. Concerning the linear independence of $\{\widetilde{e}_1, \ldots, \widetilde{e}_s\}$ over $\widetilde{A}$, since each $\widetilde{e}_i$ is a homogeneous element of degree b_i, if a degree-q homogeneous element $\sum_{i=1}^{s} h_{p_i}(f_i)\widetilde{e}_i = 0$, where $f_i \in F_{p_i}A$ and $p_i + b_i = q$, $1 \le i \le s$, then $\sum_{i=1}^{s} f_i e_i = 0$ in F_qL and consequently all $f_i = 0$, thereby $h_{p_i}(f_i) = 0$ as desired. Finally, if $\xi \in F_qL$ with $d_{\rm fil}(\xi) = q$, then by Lemma 3.1.6, $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)} e_j$ with $\lambda_{ij} \in K^*$ and $d(a^{\alpha(i_j)}) + b_j = \ell_{ij} \le q$. It follows from Lemma 3.1.7 that

$$\sigma(\xi) = \sum_{\ell_{ik}=q} \lambda_{ik}\sigma(a)^{\alpha(i_k)}\sigma(e_k),$$

$$\widetilde{\xi} = \sum_{i,j} \lambda_{ij} Z^{q-\ell_{ij}}\widetilde{a}^{\alpha(i_j)}\widetilde{e}_j.$$

Therefore, a further application of Lemma 3.1.6 and Lemma 3.1.7 shows that $\sigma(\mathscr{B}(e))$ and $\widetilde{\mathscr{B}(e)}$ are K-bases for $G(L)$ and $\widetilde{L}$ respectively. □

3.2 Filtered-graded transfer of Gröbner bases for modules. Throughout this subsection, we let $A = K[a_1, \ldots, a_n]$ be a solvable polynomial algebra with the admissible system $(\mathscr{B}, \prec_{gr})$, where $\mathscr{B} = \{a^{\alpha} = a_1^{\alpha_1}\cdots a_n^{\alpha_n} \mid \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\}$ is the PBW K-basis of A and $\prec_{gr}$ is a graded monomial ordering with respect to some given positive-degree function $d(\ )$ on A (see Section 1). Thereby A is turned into an $\mathbb{N}$-filtered solvable polynomial algebra with the filtration

$FA = \{F_pA\}_{p\in\mathbb{N}}$ constructed with respect to the same $d(\ )$ (see Example (2) of Subsection 3.1). In order to compute minimal standard bases by employing both inhomogeneous and homogenous left Gröbner bases in later Subsection 3.4, our aim of the current subsection is to establish the relations between left Gröbner bases in a filtered free (left) A-module L and homogeneous left Gröbner bases in $G(L)$ as well as homogeneous left Gröbner bases in $\widetilde{L}$, which are just module theory analogues of the results on filtered-graded transfer of Gröbner bases for left ideals given in [LW], [Li1]. All notions, notations and conventions introduced in previous sections are maintained.

Let $L = \bigoplus_{i=1}^{s} Ae_i$ be a filtered free A-module with the filtration $FL = \{F_qL\}_{q\in\mathbb{N}}$ such that $d_{\rm fil}(e_i) = b_i$, $1 \le i \le s$. Bearing in mind Lemma 3.1.6, we say that a left monomial ordering on $\mathscr{B}(e)$ is a *graded left monomial ordering*, denoted by $\prec_{e\text{-}gr}$, if for $a^\alpha e_i, a^\beta e_j \in \mathscr{B}(e)$,

$$a^\alpha e_i \prec_{e\text{-}gr} a^\beta e_j \ \text{ implies }\ d_{\rm fil}(a^\alpha e_i) = d(a^\alpha) + b_i \le d(a^\beta) + b_j = d_{\rm fil}(a^\beta e_j).$$

For instance, with respect to the given graded monomial ordering $\prec_{gr}$ on $\mathscr{B}$ and the $\mathbb{N}$-filtration FA of A, if $\{f_1,\dots,f_s\} \subset A$ is a finite subset such that $d(f_i) = b_i = d_{\rm fil}(e_i)$, $1 \le i \le s$, then, by mimicking the Schreyer ordering in the commutative case (see [Sch], or [AL2, p. 166]), one may directly check that the ordering $\prec_{s\text{-}gr}$ on $\mathscr{B}(e)$ induced by $\{f_1,\dots,f_s\}$ subject to the rule: for $a^\alpha e_i, a^\beta e_j \in \mathscr{B}(e)$,

$$a^\alpha e_i \prec_{s\text{-}gr} a^\beta e_j \Longleftrightarrow \begin{cases} \mathbf{LM}(a^\alpha f_i) \prec_{gr} \mathbf{LM}(a^\beta f_j), \\ \text{or} \\ \mathbf{LM}(a^\alpha f_i) = \mathbf{LM}(a^\beta f_j) \ \text{ and }\ i<j, \end{cases}$$

is a graded left monomial ordering on $\mathscr{B}(e)$.

More generally, let $\{\xi_1,\dots,\xi_m\} \subset L$ be a finite subset, where $d_{\rm fil}(\xi_i) = q_i$, $1 \le i \le m$, and let $L_1 = \bigoplus_{i=1}^{m} A\varepsilon_i$ be the filtered free A-module with the filtration $FL_1 = \{F_qL_1\}_{q\in\mathbb{N}}$ such that $d_{\rm fil}(\varepsilon_i) = q_i$, $1 \le i \le m$. Then, given *any* graded left monomial ordering $\prec_{e\text{-}gr}$ on $\mathscr{B}(e)$, the Schreyer ordering $\prec_{s\text{-}gr}$ defined on the K-basis $\mathscr{B}(\varepsilon) = \{a^\alpha\varepsilon_i \mid a^\alpha \in \mathscr{B},\ 1 \le i \le m\}$ of L_1 subject to the rule: for $a^\alpha\varepsilon_i, a^\beta\varepsilon_j \in \mathscr{B}(\varepsilon)$,

$$a^\alpha \varepsilon_i \prec_{s\text{-}gr} a^\beta \varepsilon_j \Longleftrightarrow \begin{cases} \mathbf{LM}(a^\alpha \xi_i) \prec_{e\text{-}gr} \mathbf{LM}(a^\beta \xi_j), \\ \text{or} \\ \mathbf{LM}(a^\alpha \xi_i) = \mathbf{LM}(a^\beta \xi_j) \ \text{ and }\ i<j, \end{cases}$$

is a graded left monomial ordering on $\mathscr{B}(\varepsilon)$.

Comparing with Lemma 3.1.2 and Lemma 3.1.6, the lemma given below reveals the intrinsic property of a graded left monomial ordering employed by a filtered free A-module.

3.2.1 Lemma. *Let $L = \bigoplus_{i=1}^{s} Ae_i$ be a filtered free A-module with the filtration $FL = \{F_qL\}_{q\in\mathbb{N}}$ such that $d_{\rm fil}(e_i) = b_i$, $1 \le i \le s$, and let $\prec_{e\text{-}gr}$ be a graded left monomial ordering on $\mathscr{B}(e)$. Then $\prec_{e\text{-}gr}$ is compatible with the filtration FL of L in the sense that $\xi \in F_qL - F_{q-1}L$, i.e. $d_{\rm fil}(\xi) = q$, if and only if $\mathbf{LM}(\xi) = a^{\alpha}e_i$ with $d_{\rm fil}(a^{\alpha}e_i) = d(a^{\alpha}) + b_i = q$.*

PROOF: Let $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)} e_j \in F_qL - F_{q-1}L$. Then by Lemma 3.1.6, there is some $a^{\alpha(i_\ell)}e_\ell$ such that $d(a^{\alpha(i_\ell)}) + b_\ell = q$. If $\mathbf{LM}(\xi) = a^{\alpha(i_t)}e_t$ with respect to $\prec_{e\text{-}gr}$, then $a^{\alpha(i_k)}e_k \prec_{e\text{-}gr} a^{\alpha(i_t)}e_t$ for all $a^{\alpha(i_k)}e_k$ with $k \ne t$. If $\ell = t$, then $d(a^{\alpha(i_t)}) + b_t = q$; otherwise, since $\prec_{e\text{-}gr}$ is a graded left monomial ordering, we have $d(a^{\alpha(i_k)})+b_k \le d(a^{\alpha(i_t)})+b_t$, in particular, $q = d(a^{\alpha(i_\ell)})+b_\ell \le d(a^{\alpha(i_t)})+b_t \le q$. Hence $d_{\rm fil}(a^{\alpha(i_t)}e_t) = d(a^{\alpha(i_t)}) + b_t = q$, as desired.

Conversely, for $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)} e_j \in L$, if, with respect to $\prec_{e\text{-}gr}$, $\mathbf{LM}(\xi) = a^{\alpha(i_t)}e_t$ with $d_{\rm fil}(a^{\alpha(i_t)}e_t) = d(a^{\alpha(i_t)}) + b_t = q$, then $a^{\alpha(i_k)}e_k \prec_{e\text{-}gr} a^{\alpha(i_t)}e_t$ for all $k \ne t$. Since $\prec_{e\text{-}gr}$ is a graded left monomial ordering, we have $d(a^{\alpha(i_k)}) + b_k \le d(a^{\alpha(i_t)}) + b_t = q$. It follows from Lemma 3.1.6 that $d_{\rm fil}(\xi) = q$, i.e., $\xi \in F_qL - F_{q-1}L$. □

Let $L = \bigoplus_{i=1}^{s} Ae_i$ be a filtered free A-module with the filtration $FL = \{F_qL\}_{q\in\mathbb{N}}$ such that $d_{\rm fil}(e_i) = b_i$, $1 \le i \le s$. Then, by Proposition 3.1.8 we know that the associated graded $G(A)$-module $G(L)$ of L is an $\mathbb{N}$-graded free module, i.e., $G(L) = \bigoplus_{i=1}^{s} G(A)\sigma(e_i)$ with the homogeneous $G(A)$-basis $\{\sigma(e_1), \dots, \sigma(e_s)\}$, and that $G(L)$ has the K-basis $\sigma(\mathscr{B}(e)) = \{\sigma(a^{\alpha}e_i) = \sigma(a)^{\alpha}\sigma(e_i) \mid a^{\alpha}e_i \in \mathscr{B}(e)\}$. Furthermore, let $\prec_{e\text{-}gr}$ be a graded left monomial ordering on $\mathscr{B}(e)$ as defined in the beginning of this section. Then we may define an ordering $\prec_{\sigma(\mathbb{e})\text{-}gr}$ on $\sigma(\mathscr{B}(e))$ subject to the rule:

$$\sigma(a)^{\alpha}\sigma(e_i) \prec_{\sigma(\mathbb{e})\text{-}gr} \sigma(a)^{\beta}\sigma(e_j) \Longleftrightarrow a^{\alpha}e_i \prec_{e\text{-}gr} a^{\beta}e_j, \quad a^{\alpha}e_i, a^{\beta}e_j \in \mathscr{B}(e).$$

3.2.2 Lemma. *With the ordering $\prec_{\sigma(\mathbb{e})\text{-}gr}$ defined above, the following statements hold.*

(i) *$\prec_{\sigma(\mathbb{e})\text{-}gr}$ is a graded left monomial ordering on $\sigma(\mathscr{B}(e))$.*
(ii) (Compare with Corollary 3.1.5.) *$\mathbf{LM}(\sigma(\xi)) = \sigma(\mathbf{LM}(\xi))$ holds for all nonzero $\xi \in L$, where the monomial orderings used for $\mathbf{LM}(\sigma(\xi))$ and $\mathbf{LM}(\xi)$ are $\prec_{\sigma(\mathbb{e})\text{-}gr}$ and $\prec_{e\text{-}gr}$ respectively.*

PROOF: (i) Noticing that the given monomial ordering $\prec_{gr}$ on A is a graded monomial ordering with respect to a positive-degree function $d(\ )$ on A, it follows from Theorem 3.1.4(i) that $G(A)$ is turned into an $\mathbb{N}$-graded solvable polynomial algebra by using the graded monomial ordering $\prec_{_{G(A)}}$ defined on $\sigma(\mathscr{B})$ subject to the rule: $\sigma(a)^{\alpha} \prec_{_{G(A)}} \sigma(a)^{\beta} \Longleftrightarrow a^{\alpha} \prec_{gr} a^{\beta}$, where the positive-degree function on $G(A)$ is given by $d(\sigma(a_i)) = d(a_i)$, $1 \le i \le n$. Moreover, since $\sigma(e_i)$ is a homogeneous element of degree b_i in $G(L)$, $1 \le i \le s$, by Lemma 3.1.7, it is then straightforward to verify that $\prec_{\sigma(\mathbb{e})\text{-}gr}$ is a graded left monomial ordering on $\sigma(\mathscr{B}(e))$.

(ii) Let $\xi = \sum_{i,j}\lambda_{ij}a^{\alpha(i_j)}e_j$, where $\lambda_{ij}\in K^*$ and $a^{\alpha(i_j)}\in\mathscr{B}$ with $\alpha(i_j) = (\alpha_{i_{j1}},\ldots,\alpha_{i_{jn}})\in\mathbb{N}^n$. If $d_{\rm fil}(\xi)=q$, i.e., $\xi\in F_qL - F_{q-1}L$, then by Lemma 3.2.1, $\mathbf{LM}(\xi) = a^{\alpha(i_t)}e_t$ for some t such that $d_{\rm fil}(a^{\alpha(i_t)}e_t) = d(a^{\alpha(i_t)}) + b_t = q$. Since $\prec_{e\text{-}gr}$ is a left graded monomial ordering on $\mathscr{B}(e)$, by Lemma 3.1.7 we have $\sigma(\xi) = \lambda_{it}\sigma(a)^{\alpha(i_t)}\sigma(e_t) + \sum_{d(a^{\alpha(i_k)})+b_k=q}\lambda_{ik}\sigma(a)^{\alpha(i_k)}\sigma(e_k)$. It follows from the definition of $\prec_{\sigma(\mathbb{e})\text{-}gr}$ that $\mathbf{LM}(\sigma(\xi)) = \sigma(a)^{\alpha(i_t)}\sigma(e_t) = \sigma(\mathbf{LM}(\xi))$, as desired. $\square$

3.2.3 Theorem. *Let N be a submodule of the filtered free A-module $L = \bigoplus_{i=1}^s Ae_i$, where L is equipped with the filtration $FL = \{F_qL\}_{q\in\mathbb{N}}$ such that $d_{\rm fil}(e_i) = b_i$, $1\le i\le s$, and let $\prec_{e\text{-}gr}$ be a graded left monomial ordering on $\mathscr{B}(e)$. For a subset $\mathcal{G} = \{g_1,\ldots,g_m\}$ of N, the following two statements are equivalent.*

(i) *$\mathcal{G}$ is a left Gröbner basis of N with respect to $\prec_{e\text{-}gr}$.*

(ii) *Putting $\sigma(\mathcal{G}) = \{\sigma(g_1),\ldots,\sigma(g_m)\}$ and considering the filtration $FN = \{F_qN = F_qL\cap N\}_{q\in\mathbb{N}}$ of N induced by FL, $\sigma(\mathcal{G})$ is a left Gröbner basis for the associated graded $G(A)$-module $G(N)$ of N with respect to the graded left monomial ordering $\prec_{\sigma(\mathbb{e})\text{-}gr}$ defined above.*

Proof: (i) $\Rightarrow$ (ii) Note that any nonzero homogeneous element of $G(N)$ is of the form $\sigma(\xi)$ with $\xi\in N$. If $\mathcal{G}$ is a left Gröbner basis of N, then there exists some $g_i\in\mathcal{G}$ such that $\mathbf{LM}(g_i)|\mathbf{LM}(\xi)$, i.e., there is a monomial $a^\alpha\in\mathscr{B}$ such that $\mathbf{LM}(\xi) = \mathbf{LM}(a^\alpha\mathbf{LM}(g_i))$. Since the given left monomial ordering $\prec_{e\text{-}gr}$ on $\mathscr{B}(e)$ is a graded left monomial ordering, it follows from Lemma 3.1.7 and Lemma 3.2.2 that

$$\begin{aligned}\mathbf{LM}(\sigma(\xi)) &= \sigma(\mathbf{LM}(\xi))\\ &= \sigma(\mathbf{LM}(a^\alpha\mathbf{LM}(g_i)))\\ &= \mathbf{LM}(\sigma(a^\alpha\mathbf{LM}(g_i)))\\ &= \mathbf{LM}(\sigma(a)^\alpha\sigma(\mathbf{LM}(g_i)))\\ &= \mathbf{LM}(\sigma(a)^\alpha\mathbf{LM}(\sigma(g_i))).\end{aligned}$$

This shows that $\mathbf{LM}(\sigma(g_i))|\mathbf{LM}(\sigma(\xi))$, and hence $\sigma(\mathcal{G})$ is a left Gröbner basis for $G(N)$.

(ii) $\Rightarrow$ (i) Suppose that $\sigma(\mathcal{G})$ is a left Gröbner basis of $G(N)$ with respect to $\prec_{\sigma(\mathbb{e})\text{-}gr}$. If $\xi\in N$ and $\xi\ne 0$, then $\sigma(\xi)\ne 0$, and there exists a $\sigma(g_i)\in\sigma(\mathcal{G})$ such that $\mathbf{LM}(\sigma(g_i))|\mathbf{LM}(\sigma(\xi))$, i.e., there is a monomial $\sigma(a)^\alpha\in\sigma(\mathscr{B})$ such that $\mathbf{LM}(\sigma(\xi)) = \mathbf{LM}(\sigma(a)^\alpha\mathbf{LM}(\sigma(g_i)))$. Again as $\prec_{e\text{-}gr}$ is a left graded monomial ordering on $\mathscr{B}(e)$, by Lemma 3.1.7 and Lemma 3.2.2 we have

$$\begin{aligned}\sigma(\mathbf{LM}(\xi)) &= \mathbf{LM}(\sigma(\xi))\\ &= \mathbf{LM}(\sigma(a)^\alpha\mathbf{LM}(\sigma(g_i)))\\ &= \mathbf{LM}(\sigma(a)^\alpha\sigma(\mathbf{LM}(g_i)))\\ &= \mathbf{LM}(\sigma(a^\alpha\mathbf{LM}(g_i)))\\ &= \sigma(\mathbf{LM}(a^\alpha\mathbf{LM}(g_i))).\end{aligned}$$

This shows that $d_{\rm fil}(\mathbf{LM}(\xi)) = d_{\rm fil}(\mathbf{LM}(a^\alpha\mathbf{LM}(g_i)))$. Since both $\mathbf{LM}(\xi)$ and $\mathbf{LM}(a^\alpha\mathbf{LM}(g_i))$ are monomials in $\mathscr{B}(e)$, it follows from the construction of FL

and Lemma 3.2.1 that $\mathbf{LM}(\xi)=\mathbf{LM}(a^\alpha\mathbf{LM}(g_i))$, i.e., $\mathbf{LM}(g_i)|\mathbf{LM}(\xi)$. This shows that $\mathcal{G}$ is a left Gröbner basis for N. □

Similarly, in light of Proposition 3.1.8 we may define an ordering $\prec_{\widetilde{e}}$ on the K-basis $\widetilde{\mathscr{B}(e)}=\{Z^m\widetilde{a}^\alpha\widetilde{e}_i \mid Z^m\widetilde{a}^\alpha\in\widetilde{\mathscr{B}},\ 1\le i\le s\}$ of the $\mathbb{N}$-graded free $\widetilde{A}$-module $\widetilde{L}=\bigoplus_{i=1}^s\widetilde{A}\widetilde{e}_i$ subject to the rule: for $Z^s\widetilde{a}^\alpha\widetilde{e}_i, Z^t\widetilde{a}^\beta\widetilde{e}_j\in\widetilde{\mathscr{B}(e)}$,

$$Z^s\widetilde{a}^\alpha\widetilde{e}_i\prec_{\widetilde{e}} Z^t\widetilde{a}^\beta\widetilde{e}_j \Longleftrightarrow a^\alpha e_i\prec_{e\text{-}gr} a^\beta e_j,\ \text{or}\ a^\alpha e_i=a^\beta e_j\ \text{and}\ s<t.$$

3.2.4 Lemma. *With the ordering $\prec_{\widetilde{e}}$ defined above, the following statements hold.*

(i) *$\prec_{\widetilde{e}}$ is a left monomial ordering on $\widetilde{\mathscr{B}(e)}$.*

(ii) (Compare with Corollary 3.1.5.) $\mathbf{LM}(\widetilde{\xi})=\widetilde{\mathbf{LM}(\xi)}$ *holds for all nonzero $\xi\in L$, where the monomial orderings used for $\mathbf{LM}(\widetilde{\xi})$ and $\mathbf{LM}(\xi)$ are $\prec_{\widetilde{e}}$ and $\prec_{e\text{-}gr}$ respectively.*

Proof: (i) Noticing that the given monomial ordering $\prec_{gr}$ for A is a graded monomial ordering with respect to a positive-degree function $d(\ )$ on A, it follows from Theorem 3.1.4(ii) that $\widetilde{A}$ is turned into an $\mathbb{N}_{\widetilde{\mathscr{B}}}$-graded solvable polynomial algebra by using the monomial ordering $\prec_{_{\widetilde{A}}}$ defined on $\widetilde{\mathscr{B}}$ subject to the rule: $\widetilde{a}^\alpha Z^s\prec_{_{\widetilde{A}}}\widetilde{a}^\beta Z^t\Longleftrightarrow a^\alpha\prec_{gr}a^\beta$, or $a^\alpha=a^\beta$ and $s<t$, $\quad a^\alpha,a^\beta\in\mathscr{B}$, where the positive-degree function on $\widetilde{A}$ is given by $d(\widetilde{a}_i)=d(a_i)$ for $1\le i\le n$, and $d(Z)=1$. Moreover, since $\widetilde{e}_i$ is a homogeneous element of degree b_i in $\widetilde{A}$, $1\le i\le s$, by Lemma 3.1.7, it is then straightforward to verify that $\prec_{\widetilde{e}}$ is a left monomial ordering on $\widetilde{B(e)}$.

(ii) Let $\xi=\sum_{i,j}\lambda_{ij}a^{\alpha(i_j)}e_j$, where $\lambda_{ij}\in K^*$ and $a^{\alpha(i_j)}\in\mathscr{B}$ with $\alpha(i_j)=(\alpha_{i_{j1}},\dots,\alpha_{i_{jn}})\in\mathbb{N}^n$. If $d_{\rm fil}(\xi)=q$, i.e., $\xi\in F_qL-F_{q-1}L$, then by Lemma 3.2.1, $\mathbf{LM}(\xi)=a^{\alpha(i_t)}e_t$ for some t such that $d_{\rm fil}(a^{\alpha(i_t)}e_t)=d(a^{\alpha(i_t)})+b_t=q$. Since $\prec_{e\text{-}gr}$ is a left graded monomial ordering on $\mathscr{B}(e)$, by Lemma 3.1.7 we have $\widetilde{\xi}=\lambda_{it}\widetilde{a}^{\alpha(i_t)}\widetilde{e}_t+\sum_{j\ne t}\lambda_{ij}Z^{q-\ell_{ij}}\widetilde{a}^{\alpha(i_j)}\widetilde{e}_j$, where $\ell_{ij}=d_{\rm fil}(a^{\alpha(i_j)}e_j)=d(a^{\alpha(i_j)})+d_j$. It follows from the definition of $\prec_{\widetilde{e}}$ that $\mathbf{LM}(\widetilde{\xi})=\widetilde{a}^{\alpha(i_t)}\widetilde{e}_t=\widetilde{\mathbf{LM}(\xi)}$, as desired. □

3.2.5 Theorem. *Let N be a submodule of the filtered free A-module $L=\bigoplus_{i=1}^sAe_i$, where L is equipped with the filtration $FL=\{F_qL\}_{q\in\mathbb{N}}$ such that $d_{\rm fil}(e_i)=b_i$, $1\le i\le s$, and let $\prec_{e\text{-}gr}$ be a graded left monomial ordering on $\mathscr{B}(e)$. For a subset $\mathcal{G}=\{g_1,\dots,g_m\}$ of N, the following two statements are equivalent.*

(i) *$\mathcal{G}$ is a left Gröbner basis of N with respect to $\prec_{e\text{-}gr}$.*

(ii) *Putting $\tau(\mathcal{G})=\{\widetilde{g}_1,\dots,\widetilde{g}_m\}$ and considering the filtration $FN=\{F_qN=F_qL\cap N\}_{q\in\mathbb{N}}$ of N induced by FL, $\tau(\mathcal{G})$ is a left Gröbner basis for the Rees module $\widetilde{N}$ of N with respect to the left monomial ordering $\prec_{\widetilde{e}}$ defined above.*

Proof: (i) $\Rightarrow$ (ii) Note that any nonzero homogeneous element of $\widetilde{N}$ is of the form $h_q(\xi)$ for some $\xi\in F_qN$ with $d_{\rm fil}(\xi)=q_1\le q$. By Lemma 3.1.7, $h_q(\xi)=$

$Z^{q-q_1}\widetilde{\xi}$. If $\mathcal{G}$ is a left Gröbner basis of N, then there exists some $g_i \in \mathcal{G}$ such that $\mathbf{LM}(g_i)|\mathbf{LM}(\xi)$, i.e., there is a monomial $a^\alpha \in \mathscr{B}$ such that $\mathbf{LM}(\xi) = \mathbf{LM}(a^\alpha \mathbf{LM}(g_i))$. It follows from Lemma 3.1.7 and Lemma 3.2.4 that

$$\begin{aligned}
\mathbf{LM}(\widetilde{\xi}) &= \widetilde{\mathbf{LM}(\xi)} \\
&= (\mathbf{LM}(a^\alpha \mathbf{LM}(g_i)))^{\sim} \\
&= \mathbf{LM}((a^\alpha \mathbf{LM}(g_i))^{\sim}) \\
&= \mathbf{LM}(\widetilde{a}^\alpha \widetilde{\mathbf{LM}(g_i)}) \\
&= \mathbf{LM}(\widetilde{a}^\alpha \mathbf{LM}(\widetilde{g}_i)).
\end{aligned}$$

Hence, noticing the definition of $\prec_{\widetilde{e}}$ we have

$$\begin{aligned}
\mathbf{LM}(h_q(\xi)) &= \mathbf{LM}(Z^{q-q_1}\widetilde{\xi}) \\
&= Z^{q-q_1}\mathbf{LM}(\widetilde{\xi}) \\
&= Z^{q-q_1}\mathbf{LM}(\widetilde{a}^\alpha \mathbf{LM}(\widetilde{g}_i)) \\
&= \mathbf{LM}(z^{q-q_1}\widetilde{a}^\alpha \mathbf{LM}(\widetilde{g}_i)).
\end{aligned}$$

This shows that $\mathbf{LM}(\widetilde{g}_i)|\mathbf{LM}(h_q(\xi))$, thereby $\tau(\mathcal{G})$ is a left Gröbner basis of $\widetilde{N}$.

(ii) $\Rightarrow$ (i) If $\xi \in N$ and $\xi \neq 0$, then $\widetilde{\xi} \neq 0$ and $\mathbf{LM}(\widetilde{\xi}) = \widetilde{\mathbf{LM}(\xi)}$ by Lemma 3.2.4. Suppose that $\tau(\mathcal{G})$ is a left Gröbner basis of $\widetilde{N}$ with respect to $\prec_{\widetilde{e}}$. Then there exists some $\widetilde{g}_i \in \tau(\mathcal{G})$ such that $\mathbf{LM}(\widetilde{g}_i)|\mathbf{LM}(\widetilde{\xi})$, i.e., there is a monomial $Z^m\widetilde{a}^\gamma \in \widetilde{\mathscr{B}}$ such that $\mathbf{LM}(\widetilde{\xi}) = \mathbf{LM}(Z^m\widetilde{a}^\gamma \mathbf{LM}(\widetilde{g}_i))$. Since the given left monomial ordering $\prec_{e\text{-}gr}$ on $\mathscr{B}(e)$ is a graded left monomial ordering, it follows from Lemma 3.1.7, the definition of $\prec_{\widetilde{e}}$ and Lemma 3.2.2 that

$$\begin{aligned}
\widetilde{a^\alpha e_j} = \widetilde{\mathbf{LM}(\xi)} = \mathbf{LM}(\widetilde{\xi}) &= \mathbf{LM}(Z^m\widetilde{a}^\gamma \mathbf{LM}(\widetilde{g}_i)) \\
&= Z^m(\mathbf{LM}((a^\gamma \mathbf{LM}(g_i))^{\sim})) \\
&= Z^m(\mathbf{LM}(a^\gamma \mathbf{LM}(g_i)))^{\sim}.
\end{aligned}$$

Noticing the discussion on $\widetilde{L}$ and the role played by Z given before Lemma 3.1.7, we must have $m = 0$, thereby $\mathbf{LM}(\xi) = \mathbf{LM}(a^\gamma \mathbf{LM}(g_i))$. This shows that $\mathcal{G}$ is a left Gröbner basis for N. □

Remark. It is known that Gröbner bases for ungraded ideals in both a commutative polynomial algebra and a noncommutative free algebra can be obtained via computing homogeneous Gröbner bases for graded ideals in the corresponding homogenized (graded) algebras (cf. [Fröb], [LS], [Li2]). Similarly for an $\mathbb{N}$-filtered solvable polynomial algebra A with respect to a positive-degree function $d(\ )$, by using a (de)homogenization-like trick with respect to the central regular element Z in $\widetilde{A}$, the discussion on $\widetilde{A}$ and $\widetilde{L}$ presented in Subsection 3.1 indeed enables us to obtain left Gröbner bases of submodules (left ideals) in L (in A) via computing homogeneous left Gröbner bases of graded submodules (graded left ideals) in $\widetilde{L}$ (in $\widetilde{A}$). Since such a topic is beyond the scope of this paper, we omit the detailed discussion here.

3.3 F-bases and standard bases with respect to good filtration. Let $A = K[a_1, \ldots, a_n]$ be an $\mathbb{N}$-filtered solvable polynomial algebra with admissible system $(\mathscr{B}, \prec)$ and the $\mathbb{N}$-filtration $FA = \{F_pA\}_{p\in\mathbb{N}}$ constructed with respect to a given positive-degree function $d(\ )$ on A (see Subsection 3.1). In this section, we introduce F-bases and standard bases respectively for $\mathbb{N}$-filtered left A-modules and their submodules with respect to good filtration, and we show that any two minimal F-bases, respectively any two minimal standard bases have the same number of elements and the same number of elements of the same filtered degree. Moreover, we show that a standard basis for a submodule N of a filtered free A-module L can be obtained via computing a left Gröbner basis of N with respect to a graded left monomial ordering. All notions, notations and conventions used before are maintained.

Let M be an A-module. Recall that M is said to be an $\mathbb{N}$-filtered A-module if M has a filtration $FM = \{F_qM\}_{q\in\mathbb{N}}$, where each F_qM is a K-subspace of M, such that $M = \bigcup_{q\in\mathbb{N}} F_qM$, $F_qM \subseteq F_{q+1}M$ for all $q \in \mathbb{N}$, and $F_pAF_qM \subseteq F_{p+q}M$ for all $p, q \in \mathbb{N}$.

Convention. Unless otherwise stated, from now on in the subsequent sections a filtered A-module M is always meant an $\mathbb{N}$-filtered module with a filtration of the type $FM = \{F_qM\}_{q\in\mathbb{N}}$ as described above.

Let $G(A)$ be the associated graded algebra of A, $\widetilde{A}$ the Rees algebra of A, and Z the homogeneous element of degree 1 in $\widetilde{A}_1$ represented by the multiplicative identity 1 of A (see Subsection 3.1). If M is a filtered A-module with the filtration $FM = \{F_qM\}_{q\in\mathbb{N}}$, then M has the associated graded $G(A)$-module $G(M) = \bigoplus_{q\in\mathbb{N}} G(M)_q$ with $G(M)_0 = F_0M$ and $G(M)_q = F_qM/F_{q-1}M$ for $q \geq 1$, and the Rees module of M is defined as the graded $\widetilde{A}$-module $\widetilde{M} = \bigoplus_{q\in\mathbb{N}} \widetilde{M}_q$ with each $\widetilde{M}_q = F_qM$. As with a filtered free A-module in Subsection 3.1, we have $\widetilde{M}/Z\widetilde{M} \cong G(M)$ as graded $G(A)$-modules, and $\widetilde{M}/(1-Z)\widetilde{M} \cong M$ as A-modules. Moreover, we may also define the filtered degree of a nonzero $\xi \in M$, that is, $d_{\rm fil}(\xi) = q$ if and only if $\xi \in F_qM - F_{q-1}M$. So, actually as in Subsection 3.1, for $\xi \in M$ with $d_{\rm fil}(\xi) = q$, if we write $\sigma(\xi)$ for the nonzero homogeneous element of degree q represented by ξ in $G(M)_q$, $\widetilde{\xi}$ for the degree-q homogeneous element represented by ξ in $\widetilde{M}_q$, and $h_{q'}(\xi)$ for the degree-q' homogeneous element represented by ξ in $\widetilde{M}_{q'}$ with $q < q'$, then $d_{\rm fil}(\xi) = q = d_{\rm gr}(\sigma(\xi)) = d_{\rm gr}(\widetilde{\xi})$, and $d_{\rm gr}(h_{q'}(\xi)) = q'$.

With notation fixed as above, the lemma presented below is a version of [LVO, Chapter I, Lemma 5.4, Theorem 5.7] for $\mathbb{N}$-filtered modules.

3.3.1 Lemma. *Let M be a filtered A-module with the filtration $FM = \{F_qM\}_{q\in\mathbb{N}}$, and $V = \{v_1, \ldots, v_m\}$ a finite subset of nonzero elements in M. The following statements are equivalent:*

(i) *there is a subset* $S = \{n_1, \ldots, n_m\} \subset \mathbb{N}$ *such that*

$$F_qM = \sum_{i=1}^{m} \left(\sum_{p_i+n_i \le q} F_{p_i}A \right) v_i, \quad q \in \mathbb{N};$$

(ii) $G(M) = \sum_{i=1}^{m} G(A)\sigma(v_i)$;

(iii) $\widetilde{M} = \sum_{i=1}^{m} \widetilde{A}\widetilde{v}_i$.

3.3.2 Definition. Let M be a filtered A-module with the filtration $FM = \{F_qM\}_{q\in\mathbb{N}}$, and let $V = \{v_1, \ldots, v_m\} \subset M$ be a finite subset of nonzero elements. If V satisfies one of the equivalent conditions of Lemma 3.3.1, then we call V an *F-basis* of M with respect to FM.

Let M be a filtered A-module with the filtration $FM = \{F_qM\}_{q\in\mathbb{N}}$. If V is an F-basis of M with respect to FM, then it is necessary to note that

(1) since $M = \bigcup_{q\in\mathbb{N}} F_qM$, it is clear that V is certainly a generating set of the A-module M, i.e., $M = \sum_{i=1}^{m} Av_i$;

(2) due to Lemma 3.3.1(i), the filtration FM is usually referred to as a *good filtration* of M in the literature concerning filtered module theory (cf. [LVO]).

Indeed, if an A-module $M = \sum_{i=1}^{t} Au_i$ is finitely generated by the subset $U = \{u_1, \ldots, u_t\}$, and if $S = \{n_1, \ldots, n_t\}$ is an arbitrarily chosen subset of $\mathbb{N}$, then U is an F-basis of M with respect to the good filtration $FM = \{F_qM\}_{q\in\mathbb{N}}$ defined by setting

$$\begin{array}{ll} F_qM = \{0\} \text{ if } q < \min\{n_1, \ldots, n_m\}; & \\ F_qM = \sum_{i=1}^{t} \left(\sum_{p_i+n_i\le q} F_{p_i}A\right) u_i \text{ otherwise.} & q \in \mathbb{N}. \end{array}$$

In particular, if $L = \bigoplus_{i=1}^{s} Ae_i$ is a filtered free A-module with the filtration $FL = \{F_qL\}_{q\in\mathbb{N}}$ as constructed in Subsection 3.1 such that $d_{\rm fil}(e_i) = b_i$, $1 \le i \le s$, then $\{e_1, \ldots, e_s\}$ is an F-basis of L with respect to the good filtration FL.

3.3.3 Definition. Let M be a filtered A-module with the filtration $FM = \{F_qM\}_{q\in\mathbb{N}}$, and suppose that M has an F-basis $V = \{v_1, \ldots, v_m\}$ with respect to FM. If any proper subset of V cannot be an F-basis of M with respect to FM, then we say that V is a *minimal F-basis* of M with respect to FM.

Note that A is an $\mathbb{N}$-filtered K-algebra such that $G(A) = \bigoplus_{p\in\mathbb{N}} G(A)_p$ with $G(A)_0 = K$, $\widetilde{A} = \bigoplus_{p\in\mathbb{N}} \widetilde{A}_p$ with $\widetilde{A}_0 = K$, where K is a field. By Lemma 3.3.1 and the well-known result on graded modules over an $\mathbb{N}$-graded algebra with the degree-0 homogeneous part being a field (cf. [Eis, Chapter 19], [Kr1, Chapter 3], [Li3]), we have immediately the following

3.3.4 Proposition. *Let* M *be a filtered* A*-module with the filtration* $FM = \{F_qM\}_{q\in\mathbb{N}}$, *and* $V = \{v_1, \ldots, v_m\} \subset M$ *a subset of nonzero elements. Then* V

is a minimal F-basis of M with respect to FM if and only if $\sigma(V)=\{\sigma(v_1),\dots,\sigma(v_m)\}$ is a minimal homogeneous generating set of $G(M)$ if and only if $\tau(V)=\{\widetilde{v}_1,\dots,\widetilde{v}_m\}$ is a minimal homogeneous generating set of $\widetilde{M}$. Hence, any two minimal F-bases of M with respect to FM have the same number of elements and the same number of elements of the same filtered degree.

Let M be an $\mathbb{N}$-filtered A-module with the filtration $FM=\{F_qM\}_{q\in\mathbb{N}}$, and let N be a submodule of M with the filtration $FN=\{F_qN=N\cap F_qM\}_{q\in\mathbb{N}}$ induced by FM. Then, as with a filtered free A-module in Subsection 3.1, the associated graded $G(A)$-module $G(N)=\bigoplus_{q\in\mathbb{N}}G(N)_q$ of N with $G(N)_q=F_qN/F_{q-1}N$ is a graded submodule of $G(M)$, and the Rees module $\widetilde{N}=\bigoplus_{q\in\mathbb{N}}\widetilde{N}_q$ of N with $\widetilde{N}_q=F_qN$ is a graded submodule of $\widetilde{M}$.

3.3.5 Definition. Let M be a filtered A-module with the filtration $FM=\{F_qM\}_{q\in\mathbb{N}}$, and let N be a submodule of M. Consider the filtration $FN=\{F_qN=N\cap F_qM\}_{q\in\mathbb{N}}$ of N induced by FM. If $W=\{\xi_1,\dots,\xi_s\}\subset N$ is an F-basis with respect to FN in the sense of Definition 3.3.2, then we call W a *standard basis* of N.

Remark. By referring to Lemma 3.3.1, one may check that our Definition 3.3.5 of a standard basis coincides with the classical Macaulay basis provided $A=K[x_1,\dots,x_n]$ is the commutative polynomial K-algebra (cf. [KR2, Definition 4.2.13, Theorem 4.3.19]), for, taking the $\mathbb{N}$-filtration FA with respect to an arbitrarily chosen positive-degree function $d(\ )$ on A, there are graded algebra isomorphisms $G(A)\cong A$ and $\widetilde{A}\cong K[x_0,x_1,\dots,x_n]$, where $d(x_0)=1$ and x_0 plays the role that the central regular element Z of degree 1 does in $\widetilde{A}$. Moreover, if two-sided ideals of an $\mathbb{N}$-filtered solvable polynomial algebra A are considered, then one may see that our Definition 3.3.5 of a standard basis coincides with the standard basis defined in [Gol].

3.3.6 Definition. Let M be a filtered A-module with the filtration $FM=\{F_qM\}_{q\in\mathbb{N}}$, and N a submodule of M with the filtration $FN=\{F_qN=N\cap F_qM\}_{q\in\mathbb{N}}$ induced by FM. Suppose that N has a standard basis $W=\{\xi_1,\dots,\xi_m\}$ with respect to FN. If any proper subset of W cannot be a standard basis for N with respect to FN, then we call W a *minimal standard basis* of N with respect to FN.

If N is a submodule of some filtered A-module M with filtration FM, then since a standard basis of N is defined as an F-basis of N with respect to the filtration FN induced by FM, the next proposition follows from Proposition 3.3.4.

3.3.7 Proposition. *Let M be a filtered A-module with the filtration $FM=\{F_qM\}_{q\in\mathbb{N}}$, and N a submodule of M with the induced filtration $FN=\{F_qN=N\cap F_qM\}_{q\in\mathbb{N}}$. A finite subset of nonzero elements $W=\{\xi_1,\dots,\xi_s\}\subset N$ is a minimal standard basis of N with respect to FN if and only if $\sigma(W)=\{\sigma(\xi_1),\dots,\sigma(\xi_m)\}$ is a minimal homogeneous generating set of $G(N)$ if and only* *if $\tau(W) = \{\widetilde{\xi}_1, \ldots, \widetilde{\xi}_m\}$ is a minimal homogeneous generating set of $\widetilde{N}$. Hence, any two minimal standard bases of N have the same number of elements and the same number of elements of the same filtered degree.* □

Since A, $G(A)$ and $\widetilde{A}$ are all Noetherian domains (Theorems 1.2.3 and 3.1.4), if a filtered A-module M has an F-basis V with respect to a given filtration FM, then the existence of a standard basis for a submodule N of M follows immediately from Lemma 3.3.1. Our next theorem shows that a standard basis for a submodule N of a filtered free A-module L can be obtained via computing a left Gröbner basis of N with respect to a graded left monomial ordering.

3.3.8 Theorem. *Let $L = \bigoplus_{i=1}^{s} Ae_i$ be a filtered free A-module with the filtration $FL = \{F_qL\}_{q\in\mathbb{N}}$ such that $d_{\rm fil}(e_i) = b_i$, $1 \le i \le s$, and let $\prec_{e\text{-}gr}$ be a graded left monomial ordering on $\mathscr{B}(e)$ (see Subsection* 3.2*). If $\mathcal{G} = \{g_1, \ldots, g_m\} \subset L$ is a left Gröbner basis for the submodule $N = \sum_{i=1}^{m} Ag_i$ of L with respect to $\prec_{e\text{-}gr}$, then $\mathcal{G}$ is a standard basis for N in the sense of Definition* 3.3.5.

Proof: If $\xi \in F_qN = F_qL \cap N$ and $\xi \ne 0$, then $d_{\rm fil}(\xi) \le q$ and ξ has a left Gröbner representation by $\mathcal{G}$, that is, $\xi = \sum_{i,j} \lambda_{ij} a^{\alpha(i_j)} g_j$, where $\lambda_{ij} \in K^*$, $a^{\alpha(i_j)} \in \mathscr{B}$ with $\alpha(i_j) = (\alpha_{i_{j1}}, \ldots, \alpha_{i_{jn}}) \in \mathbb{N}^n$, satisfying $\mathbf{LM}(a^{\alpha(i_j)}g_j) \preceq_{e\text{-}gr} \mathbf{LM}(\xi)$. Suppose $d_{\rm fil}(g_j) = n_j$, $1 \le j \le m$. Since $\prec_{e\text{-}gr}$ is a graded left monomial ordering on $\mathscr{B}(e)$, by Lemma 3.2.1 we may assume that $\mathbf{LM}(g_j) = a^{\beta(j)}e_{t_j}$ with $\beta(j) = (\beta_{j_1}, \ldots, \beta_{j_n}) \in \mathbb{N}^n$ and $1 \le t_j \le s$, such that $d(a^{\beta(j)}) + b_{t_j} = n_j$, where $d(\ )$ is the given positive-degree function on A. Furthermore, by the property ($\mathbb{P}2$) presented in Section 1, we have

$$\mathbf{LM}(a^{\alpha(i_j)}g_j) = \mathbf{LM}(a^{\alpha(i_j)}a^{\beta(j)}e_{t_j}) = a^{\alpha(i_j)+\beta(j)}e_{t_j},$$

and it follows from Lemma 3.1.2, Lemma 3.1.7 and Lemma 3.2.1 that $d(a^{\alpha(i_j)}) + n_j = d(a^{\alpha(i_j)}) + d(a^{\beta(j)}) + b_{t_j} = d(a^{\alpha(i_j)+\beta(j)}) + b_{t_j} \le q$. Hence $\xi \in \sum_{j=1}^{m}(\sum_{p_j+n_j\le q} F_{p_j}A)g_j$. This shows that $F_qN = \sum_{j=1}^{m}\left(\sum_{p_j+n_j\le q} F_{p_j}A\right)g_j$, i.e., $\mathcal{G}$ is a standard basis for N. □

3.4 Computation of minimal F-bases and minimal standard bases. Let $A = K[a_1, \ldots, a_n]$ be an $\mathbb{N}$-filtered solvable polynomial algebra with admissible system $(\mathscr{B}, \prec)$ and the $\mathbb{N}$-filtration $FA = \{F_pA\}_{p\in\mathbb{N}}$ constructed with respect to a positive-degree function $d(\ )$ on A (see Subsection 3.1). In this subsection we show how to algorithmically compute minimal F-bases for quotient modules of a filtered free left A-module L, and how to algorithmically compute minimal standard bases for submodules of L in the case where a graded left monomial ordering $\prec_{e\text{-}gr}$ on L is employed. All notions, notations and conventions used before are maintained.

We start by a little more preparation. Let M and M' be filtered A-modules with the filtration $FM = \{F_qM\}_{q\in\mathbb{N}}$ and $FM' = \{F_qM'\}_{q\in\mathbb{N}}$ respectively. Recall that an A-module homomorphism φ: $M \to M'$ is said to be a *filtered homomorphism* if $\varphi(F_qM) \subseteq F_qM'$ for all $q \in \mathbb{N}$. Let $G(A)$ be the associated $\mathbb{N}$-graded algebra of A and $\widetilde{A}$ the Rees algebra of A. Then naturally, a filtered homomorphism $M \xrightarrow{\varphi} M'$ induces a graded $G(A)$-module homomorphism $G(M) \xrightarrow{G(\varphi)} G(M')$, where if $\xi \in F_qM$ and $\overline{\xi} = \xi + F_{q-1}M$ is the coset represented by ξ in $G(M)_q = F_qM/F_{q-1}M$, then $G(\varphi)(\overline{\xi}) = \varphi(\xi) + F_{q-1}M' \in G(M')_q = F_qM'/F_{q-1}M'$, and φ induces a graded $\widetilde{A}$-module homomorphism $\widetilde{M} \xrightarrow{\widetilde{\varphi}} \widetilde{M}'$, where if $\xi \in F_qM$ and $h_q(\xi)$ is the homogeneous element of degree q in $\widetilde{M}_q = F_qM$, then $\widetilde{\varphi}(h_q(\xi)) = h_q(\varphi(\xi)) \in \widetilde{M}'_q = F_qM'$. Moreover, if $M \xrightarrow{\varphi} M' \xrightarrow{\psi} M''$ is a sequence of filtered homomorphisms, then $G(\psi) \circ G(\varphi) = G(\psi \circ \varphi)$ and $\widetilde{\psi} \circ \widetilde{\varphi} = \widetilde{\psi \circ \varphi}$.

Furthermore, recall that a filtered homomorphism $M \xrightarrow{\varphi} M'$ is called a *strict filtered homomorphism* if $\varphi(F_qM) = \varphi(M) \cap F_qM'$ for all $q \in \mathbb{N}$. Note that if N is a submodule of M and $\overline{M} = M/N$, then, considering the induced filtration $FN = \{F_qN = N \cap F_qM\}_{q\in\mathbb{N}}$ of N and the induced filtration $F(\overline{M}) = \{F_q\overline{M} = (F_qM+N)/N\}_{q\in\mathbb{N}}$ of $\overline{M}$, the inclusion map $N \hookrightarrow M$ and the canonical map $M \to \overline{M}$ are strict filtered homomorphisms. Concerning strict filtered homomorphisms and their associated graded homomorphisms, the next proposition is quoted from [LVO, Chapter I, Section 4].

3.4.1 Proposition. *Given a sequence of filtered homomorphisms*

$$(*) \qquad N \xrightarrow{\varphi} M \xrightarrow{\psi} M',$$

such that $\psi \circ \varphi = 0$, the following statements are equivalent.

(i) *The sequence $(*)$ is exact and φ, ψ are strict filtered homomorphisms.*
(ii) *The sequence $G(N) \xrightarrow{G(\varphi)} G(M) \xrightarrow{G(\psi)} G(M')$ is exact.*
(iii) *The sequence $\widetilde{N} \xrightarrow{\widetilde{\varphi}} \widetilde{M} \xrightarrow{\widetilde{\psi}} \widetilde{M}'$ is exact.*

Let $L = \bigoplus_{i=1}^{m} Ae_i$ be a filtered free A-module with the filtration $FL = \{F_qL\}_{q\in\mathbb{N}}$ such that $d_{\rm fil}(e_i) = b_i$, $1 \le i \le m$. Then as we have noted in Subsection 3.3, $\{e_1, \dots, e_m\}$ is an F-basis of L with respect to the good filtration FL. Let N be a submodule of L, and let the quotient module $M = L/N$ be equipped with the filtration $FM = \{F_qM = (F_qL + N)/N\}_{q\in\mathbb{N}}$ induced by FL. Without loss of generality, we assume that $\overline{e}_i \ne 0$ for $1 \le i \le m$, where each $\overline{e}_i$ is the coset represented by e_i in M. Then we see that $\{\overline{e}_1, \dots, \overline{e}_m\}$ is an F-basis of M with respect to FM.

3.4.2 Lemma. *Let $M = L/N$ be fixed as above, and let $N = \sum_{j=1}^{s} A\xi_j$ be generated by the set of nonzero elements $U = \{\xi_1, \dots, \xi_s\}$, where $\xi_\ell = \sum_{k=1}^{s} f_{k\ell}e_k$ with $f_{k\ell} \in A$ and $d_{\rm fil}(\xi_\ell) = q_\ell$, $1 \le \ell \le s$. The following statements hold.*

(i) *If for some j, ξ_j has a nonzero term $f_{ij}e_i$ such that $d_{\rm fil}(f_{ij}e_i) = d_{\rm fil}(\xi_j) = q_j$ and the coefficient f_{ij} is a nonzero constant, say $f_{ij} = 1$ without loss of generality, then for each $\ell = 1, \dots, j-1, j+1, \dots, s$, the element $\xi'_\ell = \xi_\ell - f_{i\ell}\xi_j$ does not involve e_i. Putting $U' = \{\xi'_1, \dots, \xi'_{j-1}, \xi'_{j+1}, \dots, \xi'_s\}$, $N' = \sum_{\xi'_\ell \in U'} A\xi'_\ell$, and considering the filtered free A-module $L' = \bigoplus_{k\ne i} Ae_k$*

with the filtration $FL' = \{F_qL'\}_{q\in\mathbb{N}}$ *in which each* e_k *has the same filtered degree as it is in* L*, i.e.,* $d_{\rm fil}(e_k) = b_k$*, if the quotient module* $M' = L'/N'$ *is equipped with the filtration* $FM' = \{F_qM' = (F_qL' + N')/N'\}_{q\in\mathbb{N}}$ *induced by* FL'*, then there is a strict filtered isomorphism* φ*:* $M' \cong M$*, i.e.,* φ *is an* A*-module isomorphism such that* $\varphi(F_qM') = F_qM$ *for all* $q \in \mathbb{N}$*.*

(ii) *With the assumptions and notations as in* (i)*, if* $U = \{\xi_1, \ldots, \xi_s\}$ *is a standard basis of* N *with respect to the filtration* FN *induced by* FL*, then* $U' = \{\xi'_1, \ldots, \xi'_{j-1}, \xi'_{j+1}, \ldots, \xi'_s\}$ *is a standard basis of* N' *with respect to the filtration* FN' *induced by* FL'*.*

Proof: (i) Since $f_{ij} = 1$ by the assumption, we see that every $\xi'_\ell = \xi_\ell - f_{i\ell}\xi_j$ does not involve e_i. Let $U' = \{\xi'_1, \ldots, \xi'_{j-1}, \xi'_{j+1}, \ldots, \xi'_s\}$ and $N' = \sum_{\xi'_\ell\in U'} A\xi'_\ell$. Then $N' \subset L' = \bigoplus_{k\neq i} Ae_k \subset L$ and $N = N' + A\xi_j$. Since $\xi_j = e_i + \sum_{k\neq i} f_{kj}e_k$, the naturally defined A-module homomorphism $M' = L'/N' \xrightarrow{\varphi} L/N = M$ with $\varphi(\overline{e}_k) = \overline{e}_k$, $k = 1, \ldots, i-1, i+1, \ldots, m$, is an isomorphism, where, without confusion, $\overline{e}_k$ denotes the coset represented by e_k in M' and M respectively. It remains to see that φ is a strict filtered isomorphism. Note that $\{e_1, \ldots, e_m\}$ is an F-basis of L with respect to FL such that $d_{\rm fil}(e_i) = b_i$, $1 \le i \le m$, i.e.,

$$F_qL = \sum_{i=1}^{m}\left(\sum_{p_i+b_i\le q} F_{p_i}A\right)e_i, \quad q \in \mathbb{N},$$

that $\{e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_m\}$ is an F-basis of L' with respect to FL' such that $d_{\rm fil}(e_k) = b_k$, where $k \neq i$, i.e.,

$$F_qL' = \sum_{k\neq i}\left(\sum_{p_i+b_k\le q} F_{p_i}A\right)e_k, \quad q \in \mathbb{N},$$

and that $\xi_j = e_i + \sum_{k\neq i} f_{kj}e_k$ with $q_j = d_{\rm fil}(\xi_j) = d_{\rm fil}(e_i) = b_i$ such that $d_{\rm fil}(f_{kj}) + b_k \le q_j$ for all $f_{kj} \neq 0$. It follows that $\sum_{k\neq i} f_{kj}\overline{e}_k \in F_{q_j}M'$ and $\varphi(\sum_{k\neq i} f_{kj}\overline{e}_k) = \overline{e}_i \in F_{q_j}M$, thereby $\varphi(F_qM') = F_qM$ for all $q \in \mathbb{N}$, as desired.

(ii) Note that $\xi'_\ell = \xi_\ell - f_{i\ell}\xi_j$. By the assumption on ξ_j, if $f_{i\ell} \neq 0$ and $d_{\rm fil}(f_{i\ell}e_i) = d_{\rm fil}(\xi_\ell) = q_\ell$, then since $d_{\rm fil}(\xi_j) = d_{\rm fil}(e_i)$ we have $d_{\rm fil}(f_{i\ell}\xi_j) = d_{\rm fil}(\xi_\ell) = q_\ell$. It follows that if we equip N with the filtration $FN = \{F_qN = N \cap F_qL\}_{q\in\mathbb{N}}$ induced by FL and consider the associated graded module $G(N)$ of N, then $d_{\rm gr}(\sigma(\xi_\ell)) = d_{\rm gr}(\sigma(f_{i\ell}\xi_j)) = d_{\rm gr}(\sigma(f_{i\ell})\sigma(\xi_j))$ in $G(N)$, i.e., $\sigma(\xi_\ell) - \sigma(f_{i\ell})\sigma(\xi_j) \in G(N)_{q_\ell}$. So, if $\sigma(\xi_\ell) - \sigma(f_{i\ell})\sigma(\xi_j) \neq 0$ then $d_{\rm fil}(\xi'_\ell) = q_\ell$ and thus

$$\sigma(\xi'_\ell) = \sigma(\xi_\ell - f_{i\ell}\xi_j) = \sigma(\xi_\ell) - \sigma(f_{i\ell})\sigma(\xi_j). \tag{1}$$

If $f_{i\ell} \neq 0$ and $d_{\rm fil}(f_{i\ell}e_i) < d_{\rm fil}(\xi_\ell) = q_\ell$, then $\sigma(\xi_\ell) = \sum_{d(f_{k\ell})+b_k=q_\ell} \sigma(f_{k\ell})\sigma(e_k)$ does not involve $\sigma(e_i)$. Also since $d_{\rm fil}(\xi_j) = d_{\rm fil}(e_i)$, we have $d_{\rm fil}(f_{i\ell}\xi_j) < d_{\rm fil}(\xi_\ell) =$

q_ℓ. Hence $d_{\rm fil}(\xi'_\ell) = d_{\rm fil}(\xi_\ell) = q_\ell$ and thus

$$\sigma(\xi'_\ell) = \sigma(\xi_\ell - f_{i\ell}\xi_j) = \sigma(\xi_\ell). \tag{2}$$

If $f_{i\ell} = 0$, then the equality (1) is the same as equality (2). Now, if $U = \{\xi_1, \dots, \xi_s\}$ is a standard basis of N with respect to the induced filtration FN, then $G(N) = \sum_{\ell=1}^{s} G(A)\sigma(\xi_\ell)$ by Lemma 3.3.1. But since we have also $G(N) = \sum_{\xi'_\ell \in U'} G(A)\sigma(\xi'_\ell) + G(A)\sigma(\xi_j)$ where the $\sigma(\xi'_\ell)$ are those nonzero homogeneous elements obtained according to the above equalities (1) and (2), it follows from Lemma 3.3.1 that $U' \cup \{\xi_j\}$ is a standard basis of N with respect to the induced filtration FN.

We next prove that U' is a standard basis of $N' = \sum_{\xi'_\ell \in U'} A\xi'_\ell$ with respect to the filtration $FN' = \{F_qN' = N' \cap F_qL'\}_{q\in\mathbb{N}}$ induced by FL'. Since $\{e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_m\}$ is an F-basis of L' with respect to the filtration FL' such that each e_k has the same filtered degree as it is in L, i.e., $d_{\rm fil}(e_k) = b_k$. It is clear that $F_qL' = L' \cap F_qL$, $q \in \mathbb{N}$, i.e., the filtration FL' is the one induced by FL. Considering the filtration FN' of N' induced by FL', it turns out that

$$F_qN' = N' \cap F_qL' = N' \cap F_qL \subseteq N \cap F_qL = F_qN, \quad q \in \mathbb{N}. \tag{3}$$

If $\xi \in F_qN'$, then since $U' \cup \{\xi_j\}$ is a standard basis of N with respect to the induced filtration FN, the formula (3) entails that

$$\xi = \sum_{\xi'_\ell \in U'} f_\ell\xi'_\ell + f_j\xi_j \text{ with } f_\ell, f_j \in A,\ d(f_\ell) + d_{\rm fil}(\xi'_\ell) \le q,\ d(f_j) + d_{\rm fil}(\xi_j) \le q. \tag{4}$$

Note that every ξ'_ℓ does not involve e_i and consequently ξ does not involve e_i. Hence $f_j = 0$ in (4) by the assumption on ξ_j, and thus

$$\xi \in \sum_{\xi'_\ell \in U'} \Big(\sum_{p_i + q_\ell \le q} F_{p_i}A \Big) \xi'_\ell.$$

Therefore we conclude that U' is a standard basis for N' with respect to the induced filtration FN', as desired. □

Combining Proposition 3.3.4, we now show that for quotient modules of filtered free A-modules, a result similar to Proposition 2.2.11 holds true.

3.4.3 Proposition. *Let $L = \bigoplus_{i=1}^m Ae_i$, $M = L/N$ be as in Lemma 3.4.2, and suppose that $U = \{\xi_1, \dots, \xi_s\}$ is now a standard basis of N with respect to the filtration FN induced by FL. The algorithm presented below computes a subset $\{e_{i_1}, \dots, e_{i_{m'}}\} \subset \{e_1, \dots, e_m\}$ and a subset $V = \{v_1, \dots, v_t\} \subset N \cap L'$, where $m' \le m$ and $L' = \bigoplus_{q=1}^{m'} Ae_{i_q}$ such that*

(i) *there is a strict filtered isomorphism $L'/N' = M' \cong M$, where $N' = \sum_{\ell=1}^t Av_\ell$, and M' has the filtration $FM' = \{F_qM' = (F_qL' + N')/N'\}_{q\in\mathbb{N}}$*

induced by the good filtration FL' determined by the F-basis $\{e_{i_1},\dots,e_{i_{m'}}\}$ of L';

(ii) *$V=\{v_1,\dots,v_t\}$ is a standard basis of $N'=\sum_{\ell=1}^t Av_\ell$ with respect to the filtration FN' induced by FL', such that each $v_\ell=\sum_{k=1}^{m'}h_{k\ell}e_{i_k}$ has the property that $h_{k\ell}\in K^*$ implies $d_{\rm fil}(e_{i_k})=b_{i_k}<d_{\rm fil}(v_\ell)$;*

(iii) *$\{\overline{e}_{i_1},\dots,\overline{e}_{i_{m'}}\}$ is a minimal F-basis of M with respect to the filtration FM.*

Algorithm 5

INPUT : $E=\{e_1,\dots,e_m\};\ U=\{\xi_1,\dots,\xi_s\}$,
  where $\xi_\ell=\sum_{k=1}^m f_{k\ell}e_k$ with $f_{k\ell}\in A$,
  and $d(f_{k\ell})+b_k\le q_\ell=d_{\rm fil}(\xi_\ell),\ 1\le\ell\le s$
OUTPUT : $E'=\{e_{i_1},\dots,e_{i_{m'}}\};\ V=\{v_1,\dots,v_t\}\subset N\cap L'$,
  where $L'=\bigoplus_{k=1}^{m'}Ae_{i_k}$
INITIALIZATION : $t:=s;\ V:=U;\ m':=m;\ E':=E;$
BEGIN
  WHILE there is a $v_j=\sum_{k=1}^{m'}f_{kj}e_k\in V$ and i is the least index
    such that $f_{ij}\in K^*$ with $d(f_{ij})+b_i=d_{\rm fil}(v_j)$ DO
    set $T=\{1,\dots,j-1,j+1,\dots,t\}$ and compute
    $v'_\ell=v_\ell-\frac{1}{f_{ij}}f_{i\ell}v_j,\ \ell\in T$,
    $r=\#\{\ell\mid\ell\in T,\ v_\ell=0\}$
   $t:=t-r-1$
   $V:=\{v_\ell=v'_\ell\mid\ell\in T,\ v'_\ell\ne0\}$
    $=\{v_1,\dots,v_t\}$ (after reordered)
   $m':=m'-1$
   $E':=E'-\{e_i\}=\{e_1,\dots,e_{m'}\}$ (after reordered)
  END
END

Proof: First note that for each $\xi_\ell\in U$, $d_{\rm fil}(\xi_\ell)$ is determined by Lemma 3.1.6. Since the algorithm is clearly finite, the conclusions (i) and (ii) follow from Lemma 3.4.2.

To prove the conclusion (iii), by the strict filtered isomorphism $M'=L'/N'\cong M$ (or the proof of Lemma 3.4.2(i)) it is sufficient to show that $\{\overline{e}_{i_1},\dots,\overline{e}_{i_{m'}}\}$ is a minimal F-basis of M' with respect to the filtration FM'. By the conclusion (ii), $V=\{v_1,\dots,v_t\}$ is a standard basis of the submodule $N'=\sum_{\ell=1}^t Av_\ell$ of L' with respect to the filtration FN' induced by FL' such that each $v_\ell=\sum_{k=1}^{m'}h_{k\ell}e_{i_k}$ has the property that $h_{k\ell}\in K^*$ implies $d_{\rm fil}(e_{i_k})=b_{i_k}<d_{\rm fil}(v_\ell)$. It follows from Lemma 3.3.1 that $G(N')=\sum_{k=1}^t G(A)\sigma(v_k)$ in which each $\sigma(v_k)=\sum_{d(h_{k\ell})+b_{i_k}=d_{\rm fil}(v_k)}\sigma(h_{k\ell})\sigma(e_{i_k})$ and all the coefficients $\sigma(h_{k\ell})$ satisfy $d_{\rm gr}(\sigma(h_{k\ell}))>0$ (see Subsection 3.1). Since $G(A)_0=K$, $G(L')=\bigoplus_{k=1}^{m'}G(A)\sigma(e_{i_k})$ (Proposition 3.1.8) and $G(N')$ is the graded syzygy module of the graded quotient module

$G(L')/G(N')$, by the well-known result on finitely generated graded modules over $\mathbb{N}$-graded algebras with the degree-0 homogeneous part being a field (cf. [Eis, Chapter 19], [Kr1, Chapter 3], [Li3]), we conclude that $\{\overline{\sigma(e_{i_1})}, \ldots, \overline{\sigma(e_{i_{m'}})}\}$ is a minimal homogeneous generating set of $G(L')/G(N')$. On the other hand, considering the naturally formed exact sequence of strict filtered homomorphisms

$$0 \longrightarrow N' \longrightarrow L' \longrightarrow M' = L'/N' \longrightarrow 0,$$

by Proposition 3.4.1 we have the $\mathbb{N}$-graded $G(A)$-module isomorphism

$$G(L')/G(N') \cong G(L'/N') = G(M')$$

with $\overline{\sigma(e_{i_k})} \mapsto \sigma(\overline{e}_{i_k})$, $1 \le k \le m'$. Now, applying Proposition 3.3.4, we conclude that $\{\overline{e}_{i_1}, \ldots, \overline{e}_{i_{m'}}\}$ is a minimal F-basis of M' with respect to the filtration FM', as desired. □

Finally, let $L = \bigoplus_{i=1}^{s} Ae_i$ be a filtered free A-module with the filtration $FL = \{F_qL\}_{q\in\mathbb{N}}$ such that $d_{\mathrm{fil}}(e_i) = b_i$, $1 \le i \le s$, and let $\prec_{e\text{-}gr}$ be a graded left monomial ordering on the K-basis $\mathscr{B}(e) = \{a^\alpha e_i \mid a^\alpha \in \mathscr{B},\ 1 \le i \le s\}$ of L (see Subsection 3.2). Combining Theorem 3.2.3 and Theorem 2.2.8, the next theorem shows how to algorithmically compute a minimal standard basis.

3.4.4 Theorem. *Let $N = \sum_{i=1}^{c} A\theta_i$ be a submodule of L generated by the subset of nonzero elements $\Theta = \{\theta_1, \ldots, \theta_c\}$, and let $FN = \{F_qN = F_qL \cap N\}_{q\in\mathbb{N}}$ be the filtration of N induced by FL. Then a minimal standard basis of N with respect to FN can be obtained by implementing the following procedures:*

Procedure 1. Run **Algorithm 1** presented in Subsection 1.3 with the initial input data $\Theta = \{\theta_1, \ldots, \theta_c\}$ to compute a left Gröbner basis $U = \{\xi_1, \ldots, \xi_m\}$ for N with respect to $\prec_{e\text{-}gr}$ on $\mathscr{B}(e)$.

Procedure 2. Let $G(N)$ be the associated graded $G(A)$-module of N determined by the induced filtration FN. Then $G(N)$ is a graded submodule of the associated grade free $G(A)$-module $G(L)$, and it follows from Theorem 3.2.3 that $\sigma(U) = \{\sigma(\xi_1), \ldots, \sigma(\xi_m)\}$ is a homogeneous left Gröbner basis of $G(N)$ with respect to $\prec_{\sigma(e)\text{-}gr}$ on $\sigma(\mathscr{B}(e))$. Run **Algorithm 3** presented in Theorem 2.2.8 with the initial input data $\sigma(U)$ to compute a minimal homogeneous generating set $\{\sigma(\xi_{j_1}), \ldots, \sigma(\xi_{j_t})\} \subseteq \sigma(U)$ for $G(N)$.

Procedure 3. Write down $W = \{\xi_{j_1}, \ldots, \xi_{j_t}\}$. Then W is a minimal standard basis for N by Proposition 3.3.7. □

Remark. By Theorem 3.2.5 and Proposition 3.3.7 it is clear that we can also obtain a minimal standard basis of the submodule N via computing a minimal homogeneous generating set for the Rees module $\widetilde{N}$ of N, which is a graded submodule of the Rees module $\widetilde{L}$ of L. However, noticing the structure of $\widetilde{A}$ and $\widetilde{L}$ (see Theorem 3.1.4, Proposition 3.1.8) it is equally clear that the cost of working on $\widetilde{A}$ will be much higher than working on $G(N)$.

3.5 Minimal filtered free resolutions and their uniqueness. Let $A = K[a_1, \dots, a_n]$ be an $\mathbb{N}$-filtered solvable polynomial algebra with admissible system $(\mathscr{B}, \prec)$ and the $\mathbb{N}$-filtration $FA = \{F_pA\}_{p\in\mathbb{N}}$ constructed with respect to a positive-degree function $d(\ )$ on A (see Subsection 3.1). In this subsection, by using minimal F-bases and minimal standard bases in the sense of Definition 3.3.3 and Definition 3.3.6, we define minimal filtered free resolutions for finitely generated left A-modules, and we show that such minimal resolutions are unique up to strict filtered isomorphism of chain complexes in the category of filtered A-modules. All notions, notations and conventions used before are maintained.

Let $M = \sum_{i=1}^{m} A\xi_i$ be an arbitrary finitely generated A-module. Then, as we have noted in Subsection 3.4, M may be endowed with a good filtration $FM = \{F_qM\}_{q\in\mathbb{N}}$ with respect to an arbitrarily chosen subset $\{n_1, \dots, n_m\} \subset \mathbb{N}$, where

$$\begin{array}{ll} F_qM = \{0\} \text{ if } q < \min\{n_1, \dots, n_m\}; & \\ F_qM = \sum_{i=1}^{t} \left(\sum_{p_i+n_i\le q} F_{p_i}A\right)\xi_i \text{ otherwise} & (q \in \mathbb{N}). \end{array}$$

If we consider the filtered free A-module $L_0 = \bigoplus_{i=1}^{m} Ae_i$ with the good filtration $FL_0 = \{F_qL_0\}_{q\in\mathbb{N}}$ such that $d_{\rm fil}(e_i) = n_i$, $1 \le i \le m$, then it follows from the construction of FL_0 (see Subsection 3.1) and Proposition 3.4.1 that the following proposition holds.

3.5.1 Proposition. (i) *There is an exact sequence of strict filtered homomorphisms*

$$0 \longrightarrow N \xrightarrow{\ \iota\ } L_0 \xrightarrow{\ \varphi_0\ } M \longrightarrow 0,$$

in which $\varphi_0(e_i) = \xi_i$, $1 \le i \le m$, $N = \mathrm{Ker}\,\varphi_0$ with the induced filtration $FN = \{F_qN = N \cap F_qL_0\}_{q\in\mathbb{N}}$, and ι is the inclusion map. Equipping $\overline{L}_0 = L_0/N$ with the induced filtration

$$F\overline{L}_0 = \{F_q\overline{L}_0 = (F_qL_0 + N)/N\}_{q\in\mathbb{N}},$$

it turns out that the induced A-module isomorphism $\overline{L}_0 \xrightarrow{\ \overline{\varphi}_0\ } M$ is a strict filtered isomorphism, that is, $\overline{\varphi}_0$ satisfies $\overline{\varphi}_0(F_q\overline{L}_0) = F_qM$ for all $q \in \mathbb{N}$.

(ii) *The induced sequence*

$$0 \longrightarrow G(N) \xrightarrow{\ G(\iota)\ } G(L_0) \xrightarrow{\ G(\varphi_0)\ } G(M) \longrightarrow 0$$

is an exact sequence of graded $G(A)$-module homomorphisms, thereby $G(L_0)/G(N) \cong G(M) \cong G(L_0/N)$ as graded $G(A)$-modules.

(iii) *The induced sequence*

$$0 \longrightarrow \widetilde{N} \xrightarrow{\ \widetilde{\iota}\ } \widetilde{L}_0 \xrightarrow{\ \widetilde{\varphi}_0\ } \widetilde{M} \longrightarrow 0$$

is an exact sequence of graded $\widetilde{A}$-module homomorphisms, thereby $\widetilde{L}_0/\widetilde{N} \cong \widetilde{M} \cong \widetilde{L_0/N}$ as graded $\widetilde{A}$-modules.

Proposition 3.5.1(i) enables us to make the following

Convention. In what follows we shall always assume that a finitely generated A-module M is of the form as presented in Proposition 3.5.1(i), i.e., $M = L_0/N$, and M has the good filtration $FM = \{F_qM = (F_qL_0 + N)/N\}_{q\in\mathbb{N}}$.

Comparing with the classical minimal graded free resolutions defined for finitely generated graded modules over $\mathbb{N}$-graded algebras with the degree-0 homogeneous part being a field (cf. [Eis, Chapter 19], [Kr1, Chapter 3], [Li3]), the results obtained in previous sections and the preliminary we made above naturally motivate the following

3.5.2 Definition. Let $L_0 = \bigoplus_{i=1}^m Ae_i$ be a filtered free A-module with the filtration $FL_0 = \{F_qL_0\}_{q\in\mathbb{N}}$ such that $d_{\rm fil}(e_i) = b_i$, $1 \le i \le m$, let N be a submodule of L_0, and let the A-module $M = L_0/N$ be equipped with the filtration $FM = \{F_qM = (F_qL_0 + N)/N\}_{q\in\mathbb{N}}$. A *minimal filtered free resolution* of M is an exact sequence of filtered A-module homomorphisms

$$\mathcal{L}_\bullet \qquad \cdots \xrightarrow{\varphi_{i+1}} L_i \xrightarrow{\varphi_i} \cdots \xrightarrow{\varphi_2} L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \to 0$$

satisfying

(1) φ_0 is the canonical epimorphism, i.e., $\varphi_0(e_i) = \overline{e}_i$ for $e_i \in \mathscr{E}_0 = \{e_1, \ldots, e_m\}$ (where each $\overline{e}_i$ denotes the coset represented by e_i in M), such that $\varphi_0(\mathscr{E}_0)$ is a minimal F-basis of M with respect to FM (in the sense of Definition 3.3.3);

(2) for $i \ge 1$, each L_i is a filtered free A-module with finite A-basis $\mathscr{E}_i$, and each φ_i is a strict filtered homomorphism, such that $\varphi_i(\mathscr{E}_i)$ is a minimal standard basis of $\mathrm{Ker}\,\varphi_{i-1}$ with respect to the filtration induced by FL_{i-1} (in the sense of Definition 3.3.6).

To see that the minimal filtered free resolution introduced above is an appropriate definition of "minimal free resolution" for finitely generated modules over the $\mathbb{N}$-filtered solvable polynomial algebras with filtration determined by positive-degree functions, we now show that such a resolution is unique up to a strict filtered isomorphism of chain complexes in the category of filtered A-modules.

3.5.3 Theorem. *Let $\mathcal{L}_\bullet$ be a minimal filtered free resolution of M as presented in Definition* 3.5.2*. The following statements hold.*

(i) *The associated sequence of graded $G(A)$-modules and graded $G(A)$-module homomorphisms*

$$G(\mathcal{L}_\bullet) \qquad \cdots \xrightarrow{G(\varphi_{i+1})} G(L_i) \xrightarrow{G(\varphi_i)} \cdots \xrightarrow{G(\varphi_1)} G(L_0) \xrightarrow{G(\varphi_0)} G(M) \to 0$$

is a minimal graded free resolution of the finitely generated graded $G(A)$-module $G(M)$.

(ii) *The associated sequence of graded $\widetilde{A}$-modules and graded $\widetilde{A}$-module homomorphisms*

$$\widetilde{\mathcal{L}}_\bullet \quad \cdots \xrightarrow{\widetilde{\varphi}_{i+1}} \widetilde{L}_i \xrightarrow{\widetilde{\varphi}_i} \cdots \xrightarrow{\widetilde{\varphi}_2} \widetilde{L}_1 \xrightarrow{\widetilde{\varphi}_1} \widetilde{L}_0 \xrightarrow{\widetilde{\varphi}_0} \widetilde{M} \rightarrow 0$$

is a minimal graded free resolution of the finitely generated graded $\widetilde{A}$-module $\widetilde{M}$.

(iii) $\mathcal{L}_\bullet$ *is uniquely determined by M in the sense that if M has another minimal filtered free resolution*

$$\mathcal{L}'_\bullet \quad \cdots \xrightarrow{\varphi'_{i+1}} L'_i \xrightarrow{\varphi'_i} \cdots \xrightarrow{\varphi'_2} L'_1 \xrightarrow{\varphi'_1} L_0 \xrightarrow{\varphi_0} M \rightarrow 0$$

then for each $i \geq 1$, there is a strict filtered A-module isomorphism χ_i: $L_i \rightarrow L'_i$ such that the diagram

$$\begin{array}{ccc} L_i & \xrightarrow{\varphi_i} & L_{i-1} \\ \chi_i \Big\downarrow \cong & & \chi_{i-1} \Big\downarrow \cong \\ L'_i & \xrightarrow{\varphi'_i} & L'_{i-1} \end{array}$$

is commutative, thereby $\{\chi_i \mid i \geq 1\}$ gives rise to a strict filtered isomorphism of chain complexes of filtered modules: $\mathcal{L}_\bullet \cong \mathcal{L}'_\bullet$.

Proof: (i) and (ii) follow from Proposition 3.3.4, Proposition 3.3.7, and Proposition 3.5.1.

To prove (iii), let the sequence $\mathcal{L}'_\bullet$ be as presented above. By (ii), $G(\mathcal{L}'_\bullet)$ is another minimal graded free resolution of $G(M)$. It follows from the well-known result on minimal graded free resolutions ([Eis, Chapter 19], [Kr1, Chapter 3], [Li3]) that there is a graded isomorphism of chain complexes $G(\mathcal{L}_\bullet) \cong G(\mathcal{L}'_\bullet)$ in the category of graded $G(A)$-modules, i.e., for each $i \geq 1$, there is a graded $G(A)$-modules isomorphism ψ_i: $G(L_i) \rightarrow G(L'_i)$ such that the diagram

$$\begin{array}{ccc} G(L_i) & \xrightarrow{G(\varphi_i)} & G(L_{i-1}) \\ \psi_i \Big\downarrow \cong & & \psi_{i-1} \Big\downarrow \cong \\ G(L'_i) & \xrightarrow{G(\varphi'_i)} & G(L'_{i-1}) \end{array}$$

is commutative. Our aim below is to construct the desired strict filtered isomorphisms χ_i by using the graded isomorphisms ψ_i carefully. So, starting with L_0, we assume that we have constructed the strict filtered isomorphisms $L_j \xrightarrow{\chi_j} L'_j$, such that $G(\chi_j) = \psi_j$ and $\chi_{j-1}\varphi_j = \varphi'_j\chi_j$, $1 \leq j \leq i-1$. Let $L_i = \bigoplus_{j=1}^{s_i} Ae_{i_j}$. Since each ψ_i is a graded isomorphism, we have $\psi_i(\sigma(e_{i_j})) = \sigma(\xi'_j)$ for some $\xi'_j \in L'_i$ satisfying $d_{\rm fil}(\xi'_j) = d_{\rm gr}(\sigma(\xi'_j)) = d_{\rm gr}(\sigma(e_{i_j})) = d_{\rm fil}(e_{i_j})$. It follows that if we construct the filtered A-module homomorphism $L_i \xrightarrow{\tau_i} L'_i$ by setting $\tau_i(e_{i_j}) = \xi'_j$,

$1 \le j \le s_i$, then $G(\tau_i) = \psi_i$. Hence, τ_i is a strict filtered isomorphism by Proposition 3.4.1. Since $\psi_{i-1} = G(\chi_{i-1})$, $\psi_i = G(\tau_i)$, and thus

$$\begin{aligned} \psi_{i-1}G(\varphi_i) &= G(\chi_{i-1})G(\varphi_i) = G(\chi_{i-1}\varphi_i) \\ G(\varphi_i')\psi_i &= G(\varphi_i')G(\tau_i) = G(\varphi_i'\tau_i), \end{aligned}$$

for each $q \in \mathbb{N}$, by the strictness of the φ_j, φ_j', χ_j and τ_i, we have

$$\begin{aligned} G(\chi_{i-1}\varphi_i)(G(L_i)_q) &= ((\chi_{i-1}\varphi_i)(F_qL_i) + F_{q-1}L_{i-1}')/F_{q-1}L_{i-1}' \\ &= (\chi_{i-1}(\varphi_i(L_i) \cap F_qL_{i-1}) + F_{q-1}L_{i-1}')/F_{q-1}L_{i-1}' \\ &\subseteq ((\chi_{i-1}\varphi_i)(L-i) \cap \chi_{i-1}(F_qL_{i-1}) + F_{q-1}L_{i-1}')/F_{q-1}L_{i-1}' \\ &= ((\chi_{i-1}\varphi_i)(L_i) \cap F_qL_{i-1}' + F_{q-1}L_{i-1}')/F_{q-1}L_{i-1}', \end{aligned}$$

$$\begin{aligned} G(\varphi_i'\tau_i)(G(L_i)_q) &= ((\varphi_i'\tau_i)(F_qL_i) + F_{q-1}L_{i-1}')/F_{q-1}L_{i-1}' \\ &= (\varphi_i'(F_qL_i') + F_{q-1}L_{i-1}')/F_{q-1}L_{i-1}' \\ &= (\varphi_i'(L_i') \cap F_qL_{i-1}' + F_{q-1}L_{i-1}')/F_{q-1}L_{i-1}' \\ &= ((\varphi_i'\tau_i)(L_i) \cap F_qL_{i-1}' + F_{q-1}L_{i-1}')/F_{q-1}L_{i-1}'. \end{aligned}$$

Note that by the exactness of $\mathcal{L}_\bullet$ and $\mathcal{L}_\bullet'$, as well as the commutativity $\chi_{i-2}\varphi_{i-1} = \varphi_{i-1}'\chi_{i-1}$, we have $(\chi_{i-1}\varphi_i)(L_i) \subseteq \varphi_i'(L_i') = (\varphi_i'\tau_i)(L_i)$. Considering the filtration of the submodules $(\chi_{i-1}\varphi_i)(L_i)$ and $(\varphi_i'\tau_i)(L_i)$ induced by the filtration FL_{i-1}' of L_{i-1}', the commutativity $\psi_{i-1}G(\varphi_i) = G(\varphi_i')\psi_i$ and the formulas derived above show that both submodules have the same associated graded module, i.e., $G((\chi_{i-1}\varphi_i)(L_i)) = G((\varphi_i'\tau_i)(L_i))$. It follows from a similar proof of [LVO, Theorem 5.7 on p. 49] that

$$(\chi_{i-1}\varphi_i)(L_i) = (\varphi_i'\tau_i)(L_i). \tag{1}$$

Clearly, the equality (1) does not necessarily mean the commutativity of the diagram

$$\begin{array}{ccc} L_i & \xrightarrow{\varphi_i} & L_{i-1} \\ {\scriptstyle\tau_i}\downarrow{\scriptstyle\cong} & & {\scriptstyle\chi_{i-1}}\downarrow{\scriptstyle\cong} \\ L_i' & \xrightarrow{\varphi_i'} & L_{i-1}'. \end{array}$$

To remedy this problem, we need to further modify the filtered isomorphism τ_i. Since $G(\chi_{i-1}\varphi_i)(\sigma(e_{i_j})) = G(\varphi_i'\tau_i)(\sigma(e_{i_j}))$, $1 \le j \le s_i$, if $d_{\mathrm{fil}}(e_{i_j}) = b_j$, then by the equality (1) and the strictness of τ_i and φ_i we have

$$\begin{array}{rcl} (\chi_{i-1}\varphi_i)(e_{i_j}) - (\varphi_i'\tau_i)(e_{i_j}) & \in & (\varphi_i'\tau_i)(L_i) \cap F_{b_j-1}L_{i-1}' \\ & = & \varphi_i'(L_i') \cap F_{b_j-1}L_{i-1}' \\ & = & \varphi_i'(F_{b_j-1}L_i') \\ & = & (\varphi_i'\tau_i)(F_{b_j-1}L_i), \end{array} \tag{2}$$

and furthermore from (2) we have a $\xi_j \in F_{b_j-1}L_i$ such that $d_{\rm fil}(e_{i_j}-\xi_j)=b_j$ and

$$(\chi_{i-1}\varphi_i)(e_{i_j})=(\varphi'_i\tau_i)(e_{i_j}-\xi_j),\ 1\le j\le s_i. \tag{3}$$

Now, if we construct the filtered homomorphism $L_i \xrightarrow{\chi_i} L'_i$ by setting $\chi_i(e_{i_j})=\tau_i(e_{i_j}-\xi_j)$, $1\le j\le s_i$, then since $\tau_i(\xi_j)\in F_{b_j-1}L'_i$, it turns out that

$$G(\chi_i)(\sigma(e_{i_j}))=G(\tau_i)(\sigma(e_{i_j}-\xi_j))=G(\tau_i)(\sigma(e_{i_j}))=\psi_i(\sigma(e_{i_j})),\ 1\le j\le s_i,$$

thereby $G(\chi_i)=\psi_i$. Hence, χ_i is a strict filtered isomorphism by Proposition 3.4.1, and moreover, it follows from (3) that we have reached the following diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\varphi_{i+1}"] & L_i \arrow[r, "\varphi_i"] \arrow[d, "\chi_i"', "\cong"] & L_{i-1} \arrow[r, "\varphi_{i-1}"] \arrow[d, "\chi_{i-1}"', "\cong"] & \cdots \\
\cdots \arrow[r, "\varphi'_{i+1}"] & L'_i \arrow[r, "\varphi'_i"] & L'_{i-1} \arrow[r, "\varphi'_{i-1}"] & \cdots
\end{tikzcd}$$

in which $\chi_{i-1}\varphi_i=\varphi'_i\chi_i$. Repeating the same process to getting the desired χ_{i+1} and so on, the proof is thus finished. □

3.6 Computation of minimal finite filtered free resolutions. Let $A=K[a_1,\ldots,a_n]$ be a solvable polynomial algebra with the admissible system $(\mathscr{B},\prec_{gr})$ in which $\prec_{gr}$ is a graded monomial ordering with respect to some given positive-degree function $d(\ )$ on A (see Subsection 1.2). Thereby A is turned into an $\mathbb{N}$-filtered solvable polynomial algebra with the filtration $FA=\{F_pA\}_{p\in\mathbb{N}}$ constructed with respect to the same $d(\ )$ (see Example (2) of Subsection 3.1). Note that Theorem 2.3.1, Proposition 2.3.2, and Theorem 2.3.3 hold true for *any* solvable polynomial algebra. Combining the results of Section 2 and previous subsections, we are now able to work out the algorithmic procedures for computing minimal finite filtered free resolutions over A (in the sense of Definition 3.5.2) with respect to any graded left monomial ordering on free left modules. All notions, notations and conventions used before are maintained.

3.6.1 Theorem. *Let $L_0=\bigoplus_{i=1}^m Ae_i$ be a filtered free A-module with the filtration $FL_0=\{F_qL_0\}_{q\in\mathbb{N}}$ such that $d_{\rm fil}(e_i)=b_i$, $1\le i\le m$. If $N=\sum_{i=1}^s A\xi_i$ is a finitely generated submodule of L_0 and the quotient module $M=L_0/N$ is equipped with the filtration $FM=\{F_qM=(F_qL_0+N)/N\}_{q\in\mathbb{N}}$, then M has a minimal filtered free resolution of length $d\le n$:*

$$\mathcal{L}_\bullet \qquad 0\longrightarrow L_d \xrightarrow{\varphi_q} \cdots \xrightarrow{\varphi_2} L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

which can be constructed by implementing the following procedures:

Procedure 1. Fix a graded left monomial ordering $\prec_{e\text{-}gr}$ on the K-basis $\mathscr{B}(e)$ of L_0 (see Subsection 3.2), and run **Algorithm 1** (presented in Subsection 1.3) with the initial input data $U=\{\xi_1,\ldots,\xi_s\}$ to compute a left Gröbner basis

$\mathcal{G} = \{g_1, \ldots, g_z\}$ for N, so that N has the standard basis $\mathcal{G}$ with respect to the induced filtration $FN = \{F_qN = N \cap F_qL_0\}_{q\in\mathbb{N}}$ (Theorem 3.3.8).

Procedure 2. Run **Algorithm 5** of Proposition 3.4.3 with the initial input data $E = \{e_1, \ldots, e_m\}$ and $\mathcal{G} = \{g_1, \ldots, g_z\}$ to compute a subset $\mathscr{E}_0' = \{e_{i_1}, \ldots, e_{i_{m'}}\} \subset \mathscr{E}_0 = \{e_1, \ldots, e_m\}$ and a subset $V = \{v_1, \ldots, v_t\} \subset N \cap L_0'$ such that there is a strict filtered isomorphism $L_0'/N' = M' \cong M$, where $L_0' = \bigoplus_{q=1}^{m'} Ae_{i_q}$ with $m' \leq m$ and $N' = \sum_{k=1}^{t} Av_k$, and such that $\{\overline{e}_{i_1}, \ldots, \overline{e}_{i_{m'}}\}$ is a minimal F-basis of M with respect to the filtration FM.

For convenience, after accomplishing Procedure 2 we may assume that $\mathscr{E}_0 = \mathscr{E}_0'$, $U = V$ and $N = N'$. Accordingly we have the short exact sequence

$$0 \longrightarrow N \longrightarrow L_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

such that $\varphi_0(\mathscr{E}_0) = \{\overline{e}_1, \ldots, \overline{e}_m\}$ is a minimal F-basis of M with respect to the filtration FM.

Procedure 3. With the initial input data $U = V$, implement the procedures presented in Theorem 3.4.4 to compute a minimal standard basis $W = \{\xi_{j_1}, \ldots, \xi_{j_{m_1}}\}$ for N with respect to the induced filtration FN.

Procedure 4. Compute a generating set $U_1 = \{\eta_1, \ldots, \eta_{s_1}\}$ of $N_1 = \mathrm{Syz}(W)$ in the free A-module $L_1 = \bigoplus_{i=1}^{m_1} A\varepsilon_i$ by running **Algorithm 1** with the initial input data W and using Proposition 2.3.2.

Procedure 5. Construct the strict filtered exact sequence

$$0 \longrightarrow N_1 \longrightarrow L_1 \xrightarrow{\varphi_1} L_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

where the filtration FL_1 of L_1 is constructed by setting $d_{\mathrm{fil}}(\varepsilon_k) = d_{\mathrm{fil}}(\xi_{j_k})$, $1 \leq k \leq m_1$, and φ_1 is defined by setting $\varphi_1(\varepsilon_k) = \xi_{j_k}$, $1 \leq k \leq m_1$. If $N_1 \neq 0$, then, with the initial input data $U = U_1$, repeat Procedure 3 – Procedure 5 for N_1 and so on.

By Theorem 3.5.3, a minimal filtered free resolution $\mathcal{L}_\bullet$ of M gives rise to a minimal graded free resolution $G(\mathcal{L}_\bullet)$ of $G(M)$. Since $G(A) = K[\sigma(a_1), \ldots, \sigma(a_n)]$ is a solvable polynomial algebra by Theorem 3.1.4, it follows from Theorem 2.3.4 that $G(\mathcal{L}_\bullet)$ terminates at a certain step, i.e., $\mathrm{Ker}\, G(\varphi_d) = 0$ for some d. But $\mathrm{Ker}\, G(\varphi_d) = G(\mathrm{Ker}\, \varphi_d)$ by Proposition 3.4.1, where $\mathrm{Ker}\, \varphi_d$ has the filtration induced by FL_d, thereby $G(\mathrm{Ker}\, \varphi_d) = 0$. Consequently $\mathrm{Ker}\, \varphi_d = 0$ by classical filtered module theory (cf. [LVO]), thereby a minimal finite filtered free resolution of length $d \leq n$ is achieved for M.

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