Semicommutativity of the rings relative to prime radical

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Abstract. In this paper, we introduce a new kind of rings that behave like semicommutative rings, but satisfy yet more known results. This kind of rings is called P-semicommutative. We prove that a ring R is P-semicommutative if and only if $R[x]$ is P-semicommutative if and only if $R[x, x^{-1}]$ is P-semicommutative. Also, if $R[[x]]$ is P-semicommutative, then R is P-semicommutative. The converse holds provided that $P(R)$ is nilpotent and R is power serieswise Armendariz. For each positive integer n, R is P-semicommutative if and only if $T_n(R)$ is P-semicommutative. For a ring R of bounded index 2 and a central nilpotent element s, R is P-semicommutative if and only if $K_s(R)$ is P-semicommutative. If T is the ring of a Morita context $(A, B, M, N, \psi, \varphi)$ with zero pairings, then T is P-semicommutative if and only if A and B are P-semicommutative. Many classes of such rings are constructed as well. We also show that the notions of clean rings and exchange rings coincide for P-semicommutative rings.

Keywords: semicommutative ring; P-semicommutative ring; prime radical of a ring

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1. Introduction

Throughout this paper all rings are associative with identity. An element a of a ring R is called *strongly nilpotent* if every sequence $a = a_0, a_1, a_2, \cdots$ such that $a_{i+1} \in a_i R a_i$ is ultimately zero. Obviously, every strongly nilpotent element is nilpotent. The prime radical $P(R)$ of a ring R, i.e., the intersection of all prime ideals, consists of precisely the strongly nilpotent elements (for detail see [2]). For a ring R, as is well known, $P(R) = \{x \in R \mid RxR \text{ is nilpotent}\}$. Recall that a ring R is called *semicommutative* if $ab = 0$ implies $aRb = 0$ for all $a, b \in R$. Mohammadi et al. [14] initiated a version of nil-semicommutative rings as a generalization of semicommutative rings. We call this nil-semicommutative ring nil-semicommutative-I. A ring R is *nil-semicommutative-I* if $ab = 0$ implies $aRb = 0$ for all $a, b \in nil(R)$. In their paper it is shown that in a nil-semicommutative-I ring, $nil(R)$ forms an ideal of R. Every semicommutative ring is nil-semicommutative-I. There are nil-semicommutative-I rings that are not semicommutative. Another type of nil-semicommutative rings is defined in [17] and [6]. Again to get rid of confusion, we call this nil-semicommutative ring nil-semicommutative-II. A ring R is defined to be *nil-semicommutative-II* if

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$ab \in nil(R)$ implies $aRb \subseteq nil(R)$ for all $a, b \in R$. Also another generalization of semicommutative rings is given in [16]. A ring R is called *central semicommutative* if for any $a, b \in R$, $ab = 0$ implies that arb is a central element of R for each $r \in R$. Every semicommutative ring is central semicommutative. But the converse statement need not be true in general. Motivated by these generalizations, in this paper a new kind of rings that behave like semicommutative rings are defined by employing prime radical of the ring, and general properties of this class of rings are investigated. We summarize in short the contents of sections. In Section 2, we investigate general properties of P-semicommutative rings and the interrelations between P-semicommutative rings and the other versions of semicommutativity, such as weakly semicommutative rings, nil-semicommutative-I rings, nil-semicommutative-II rings and central semicommutative rings. A relation between maximal right ideals and idempotents of a P-semicommutative ring is obtained, that is, if M is a maximal right ideal of a P-semicommutative ring R, then $e \in M$ or $1 - e \in M$ for any $e^2 = e \in R$. Also it is proved that the concepts of clean rings and exchange rings are the same for P-semicommutative rings. In Section 3, it is discussed equivalent characterizations of P-semicommutativity of rings with their extensions.

In what follows, by $\mathbb{Z}$ and $\mathbb{Z}_n$ we denote, respectively, integers and the ring of integers modulo n. Also $nil(R)$, $P(R)$ and $J(R)$ stand for the set of nilpotent elements, prime radical and Jacobson radical of a ring R. The symbol $T_n(R)$ stands for the ring of all upper triangular matrices over a ring R, and $M_n(R)$ denotes the $n \times n$ full matrix ring over R.

2. P-semicommutative rings

In this section, we introduce our main concept, namely, P-semicommutative rings, as a generalization of semicommutative rings, and investigate some properties of this class of rings.

Definition 2.1. A ring R is called P-semicommutative if for every $a, b \in R$, $ab = 0$ implies $aRb \subseteq P(R)$.

Clearly, every semicommutative ring is P-semicommutative, and every P-semicommutative semiprime ring is semicommutative. In particular, if $R/P(R)$ is semicommutative, then R is P-semicommutative. Before dealing with examples we introduce the following notion. Let R be a ring and M a bimodule. The *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication,

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2)$$

where r_1, $r_2 \in R$, m_1, $m_2 \in M$. There are P-semicommutative rings that are neither semicommutative nor abelian as the next example shows.

Example 2.2. Let $\mathbb{H}$ denote the Hamilton quaternions over the real number field and R be the trivial extension of $\mathbb{H}$ by $\mathbb{H}$ and S be the trivial extension of R by R.

Then

$$R=\left\{\begin{pmatrix} h & t\\ 0 & h\end{pmatrix} \mid h,t\in\mathbb{H}\right\},\ S=\left\{\begin{pmatrix} a & b & x & y\\ 0 & a & 0 & x\\ 0 & 0 & a & b\\ 0 & 0 & 0 & a\end{pmatrix} \mid a,b,x,y\in\mathbb{H}\right\},$$

$P(S)=\left\{\begin{pmatrix} 0 & b & x & y\\ 0 & 0 & 0 & x\\ 0 & 0 & 0 & b\\ 0 & 0 & 0 & 0\end{pmatrix} \mid b,x,y\in\mathbb{H}\right\}$. For $A=\begin{pmatrix} a & b & x & y\\ 0 & a & 0 & x\\ 0 & 0 & a & b\\ 0 & 0 & 0 & a\end{pmatrix}$, $B=\begin{pmatrix} a' & b' & x' & y'\\ 0 & a' & 0 & x'\\ 0 & 0 & a' & b'\\ 0 & 0 & 0 & a'\end{pmatrix}\in S$ with $AB=0$, we have $aa'=0$. Since $\mathbb{H}$ is a division ring, $a=0$ or $a'=0$. In either case, we have $ASB\subseteq P(S)$. Therefore S is P-semicommutative. In [11, Example 1.7], it is shown that S is not semicommutative. Also, it is obvious that S is not abelian.

We state some properties of prime radical of a ring that we use in the sequel.

Lemma 2.3. *The following hold.*

(1) *Let $\{R_i\}_{i\in\mathcal{I}}$ be a class of rings with prime radicals $P(R_i)$ where $\mathcal{I}=\{1,2,\dots,n\}$. Then $P(\bigoplus_{i\in\mathcal{I}} R_i)=\bigoplus_{i\in\mathcal{I}} P(R_i)$.*
(2) *Let I be an ideal of a ring R. Then $P(I)=I\cap P(R)$ where $P(I)$ is the intersection of all prime ideals of I (as a ring without unit)* (see [10, p. 449, Example 2(a)]).
(3) $P(R)$ is a semiprimary ideal of a ring R, that is, for all $a\in R$, $aRa\subseteq P(R)$ implies $a\in P(R)$.
(4) Let R be a ring with $e^2=e\in R$. Then $P(eRe)=eP(R)e$ (see [13]).

The next result is a useful characterization of P-semicommutative rings. Also, it reveals that every 2-primal ring is P-semicommutative.

Theorem 2.4. *The following are equivalent for a ring R.*

(1) *R is P-semicommutative.*
(2) *For all $a\in R$, if $a^2=0$, then $a\in P(R)$.*
(3) *For all $a\in R$, if $a^2=0$, then $ab-ba\in P(R)$ for any $b\in R$.*

Proof: (1) $\Rightarrow$ (2) Suppose that R is P-semicommutative. Let $a\in R$ with $a^2=0$. By hypothesis, we get $aRa\subseteq P(R)$. By Lemma 2.3(3), $a\in P(R)$.

(2) $\Rightarrow$ (1) Assume that $ab=0$ for $a,b\in R$. For any $r\in R$, $abr=0$, then $(bra)^2=0$ implies $bra\in P(R)$. Since $P(R)$ is an ideal of R, $bras\in P(R)$ for any $s\in R$, that is, $bRaR\subseteq P(R)$. Then $RaR(bRaR)b=(RaRb)(RaRb)\subseteq P(R)$. So $RaRb\subseteq P(R)$. Hence $aRb\subseteq P(R)$. Therefore R is P-semicommutative.

(2) $\Rightarrow$ (3) Let $a\in R$ with $a^2=0$. Then $a\in P(R)$, and so $ab-ba\in P(R)$ for any $b\in R$.

(3) $\Rightarrow$ (2) Assume that (3) holds. Let $a \in R$ with $a^2 = 0$. Then for any $b \in R$, $aba = a(ba - ab) \in P(R)$. Hence $aRa \subseteq P(R)$. Thus $a \in P(R)$ by Lemma 2.3(3). Therefore (2) holds. □

Corollary 2.5. *Let R be a P-semicommutative ring. Then $R/P(R)$ is abelian, that is, $ex - xe \in P(R)$ for all $x \in R$ and $e^2 = e \in R$.*

Proof: Let $e^2 = e \in R$ and $x \in R$. Then $(ex - exe)^2 = 0$ and $(xe - exe)^2 = 0$. By Theorem 2.4, $ex - exe \in P(R)$ and $xe - exe \in P(R)$. Hence $ex - xe \in P(R)$. Thus $R/P(R)$ is abelian. □

An ideal I of a ring R is called *reduced* if it has no nonzero nilpotent elements, and I is said to be (P-)*semicommutative* if it can be considered as a (P-)semicommutative ring without identity. Every reduced ideal is semicommutative.

Theorem 2.6. *Let R be a ring and I be an ideal of R. Suppose that R/I is P-semicommutative. Then R is P-semicommutative if at least one of the following conditions holds.*

(1) $I \subseteq P(R)$.
(2) *I is semicommutative.*

Proof: Assume that (1) holds. Let $a, b \in R$ with $ab = 0$. Then $\overline{a}\overline{b} = \overline{0}$. Hence $\overline{a}\overline{R}\overline{b} \subseteq P(R/I)$. Since $P(R/I) = P(R)/I$ and $I \subseteq P(R)$, we have $aRb \subseteq P(R)$. Therefore R is a P-semicommutative ring.

Assume that (2) holds. Let $x \in R$ such that $x^2 = 0$. Then $\bar{x}^2 = \bar{0}$ in R/I. Since R/I is P-semicommutative, $\bar{x} \in P(\bar{R})$. So there exists $n \in \mathbb{N}$ such that $(\bar{r}\bar{x}\bar{s})^n = \bar{0}$, and then $(rxs)^n \in I$ for any $r, s \in R$. Thus we get $((rxs)^{n+1}rx)(xs(rxs)^{n+1}) = 0$ in R. Since $(rxs)^{n+1}rx \in I$, $xs(rxs)^{n+1} \in I$ and I is semicommutative, it follows that

$$((rxs)^{n+1}rx)sr(xs(rxs)^n rx)sr(xs(rxs)^{n+1}) = 0$$

that is, $(rxs)^{n+2}(rxs)^{n+2}(rxs)^{n+2} = 0$. Hence $(rxs)^{3n+6} = 0$, i.e., rxs is nilpotent for all $r, s \in R$. Hence RxR is nilpotent. Then $x \in P(R)$. By Theorem 2.4, R is P-semicommutative. □

Corollary 2.7. *If I is a nilpotent ideal of a ring R and R/I is a P-semicommutative ring, then R is P-semicommutative.*

The following result is known from [9, Theorem 6]. Here, we prove it by using P-semicommutativity.

Corollary 2.8. *Let I be a reduced ideal of a ring R. If R/I is semicommutative, then R is semicommutative.*

Proof: As I is a reduced ideal of R, I is semicommutative, and so P-semicommutative. Also R/I is P-semicommutative. By Theorem 2.6, R is P-semicommutative. Let $a, b \in R$ with $ab = 0$. Then $aRb \subseteq I$ and $aRb \subseteq P(R)$. Hence $aRb = 0$. This completes the proof. □

Proposition 2.9. *The following statements hold.*

(1) *Every ideal of a P-semicommutative ring is P-semicommutative.*
(2) *Finite direct product of P-semicommutative rings is P-semicommutative.*

Proof: (1) It is clear by noting that for any ideal I of a ring R, $P(I) = I \cap P(R)$ from [10, p. 449, Example 2(a)].

(2) Let $R_1 \times R_2$ be a direct product of P-semicommutative rings R_1 and R_2. Since $P(R_1 \times R_2) = P(R_1) \times P(R_2)$, for any $(a, b), (c, d) \in R_1 \times R_2), (a, b) \subseteq P(R_1 \times R_2)$ if and only if $aR_1c \subseteq P(R_1)$ and $bR_2d \subseteq P(R_2)$. The rest is clear. □

Proposition 2.10. *Let R be a ring and $e^2 = e \in R$. If R is P-semicommutative, then eRe is P-semicommutative.*

Proof: Let $e^2 = e \in R$, $eae, ebe \in eRe$ with $(eae)(ebe) = 0$. Then $(eae)R(ebe) \subseteq P(R)$. Hence $(eae)(eRe)(ebe) \subseteq eP(R)e$. By Lemma 2.3(4), $P(eRe) = eP(R)e$ implies $(eae)(eRe)(ebe) \subseteq P(eRe)$. □

In the next result, we obtain a relevance between maximal right ideals and idempotents of a P-semicommutative ring.

Theorem 2.11. *Let R be a P-semicommutative ring. If M is a maximal right ideal of R, then $e \in M$ or $1 - e \in M$ for any $e^2 = e \in R$.*

Proof: Let M be a maximal right ideal of R. Clearly, $P(R) \subseteq J(R) \subseteq M$. For any idempotent $e \in R$, $e(1-e) = 0$ and $(1-e)e = 0$, and so $eR(1-e) \subseteq P(R) \subseteq M$ and $(1 - e)Re \subseteq P(R) \subseteq M$.

We show that $e \notin M$ implies $1 - e \in M$ for any idempotent $e \in R$. There are two cases:

Case (1). Suppose $eM + M = R$, then $eM(1 - e) + M(1 - e) = R(1 - e)$. Hence $R(1 - e) \subseteq M$, and so $1 - e \in M$.

Case (2). Suppose $eM + M = M$, then $eM \subseteq M$. We claim $(1 - e)M \subseteq M$. By contrary, if $(1-e)M \nsubseteq M$, then $R = (1-e)M+M$. By multiplying from left by e, we have $eR = eM \subseteq M$. Hence $e \in M$. This is a contradiction. So $(1-e)M \subseteq M$. It follows $M = eM+(1-e)M$. Now $R = eR+M = eR+(1-e)M$. By multiplying from left by $1 - e$, we have $(1 - e)R = (1 - e)M$. Being $(1 - e)M \subseteq M$ implies that $1 - e \in M$. □

Proposition 2.12. *Let R be a ring. Then the following are equivalent.*

(1) *R is P-semicommutative.*
(2) *$T_n(R)$ is P-semicommutative for any $n \geq 2$.*
(3) *$R[x]/(x^n)$ is P-semicommutative for any $n \geq 2$ where (x^n) is the ideal generated by x^n in $R[x]$.*

Proof: (1) ⇒ (2) Without loss of generality we may assume $n = 2$. Note that $P(T_2(R)) = \begin{pmatrix} P(R) & R \\ 0 & P(R) \end{pmatrix}$. Let $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $B = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in T_2(R)$ with $AB = 0$. Then $ax = 0$ and $cz = 0$. Hence $aRx \subseteq P(R)$ and $cRz \subseteq P(R)$. Thus $AT_2(R)B \subseteq P(T_2(R))$.

(2) $\Rightarrow$ (1) Let $a, b \in R$ with $ab = 0$. For $A = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix}\right) \in T_2(R)$, $AB = 0$. By (2), we have $AT_2(R)B \subseteq P(T_2(R))$ and so $aRb \subseteq P(R)$.

(1) $\Leftrightarrow$ (3) It is well known that $R[x]/(x^n) \cong V_n$ where V_n is the ring of all matrices of the following form over R:

$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ 0 & a_0 & a_1 & \dots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_0 & \dots & \vdots & a_{n-2} \\ \vdots & \vdots & \vdots & \dots & a_1 & \vdots \\ 0 & 0 & 0 & \dots & a_0 & a_1 \\ 0 & 0 & 0 & \dots & 0 & a_0 \end{pmatrix}$$

Then $P(V_n(R))$ consists of all matrices of the form

$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ 0 & a_0 & a_1 & \dots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_0 & \dots & \vdots & a_{n-2} \\ \vdots & \vdots & \vdots & \dots & a_1 & \vdots \\ 0 & 0 & 0 & \dots & a_0 & a_1 \\ 0 & 0 & 0 & \dots & 0 & a_0 \end{pmatrix}$$

where $a_0 \in P(R), a_i \in R, i = 1, 2, \dots, n$. Let

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ 0 & a_0 & a_1 & \dots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_0 & \dots & \vdots & a_{n-2} \\ \vdots & \vdots & \vdots & \dots & a_1 & \vdots \\ 0 & 0 & 0 & \dots & a_0 & a_1 \\ 0 & 0 & 0 & \dots & 0 & a_0 \end{pmatrix},$$

$$B = \begin{pmatrix} b_0 & b_1 & b_2 & \dots & b_{n-1} & b_n \\ 0 & b_0 & b_1 & \dots & b_{n-2} & b_{n-1} \\ 0 & 0 & b_0 & \dots & \vdots & b_{n-2} \\ \vdots & \vdots & \vdots & \dots & b_1 & \vdots \\ 0 & 0 & 0 & \dots & b_0 & b_1 \\ 0 & 0 & 0 & \dots & 0 & b_0 \end{pmatrix} \in V_n.$$

Assume that R is P-semicommutative and $AB = 0$. Then $a_0b_0 = 0$. By assumption, $a_0Rb_0 \subseteq P(R)$. Hence $AV_nB \subseteq P(V_n)$. Conversely, suppose that V_n is P-semicommutative and let $a, b \in R$ with $ab = 0$. Let $A, B \in V_n$ be such that main diagonal entries of A and B are a and b, other entries of A and B are 0, respectively. Then $AB = 0$ and so $AV_nB \subseteq P(V_n)$. It implies $aRb \subseteq P(R)$. This completes the proof. $\square$

Now we investigate some relations among a ring R, the polynomial ring $R[x]$ and the power series ring $R[[x]]$ in terms of P-semicommutativity. Recall that a ring R is called *Armendariz* if whenever $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_ib_j = 0$ for each i, j.

Proposition 2.13. *Let R be an Armendariz ring. Then R is P-semicommutative if and only if $R[x]$ is P-semicommutative.*

PROOF: Let R be P-semicommutative. Let $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$. Suppose that $f(x)g(x) = 0$. Since R is Armendariz, $a_ib_j = 0$, and so we have $a_iRb_j \subseteq P(R)$ for all i, j. For each $h(x) = \sum_{k=0}^{p} c_k x^k \in R[x]$, we have $f(x)h(x)g(x) = \sum_{s=0}^{m+n+p} (\sum_{i+j+k=s} a_i c_k b_j) x^s \in P(R)[x]$. By the well-known fact that $P(R)[x] = P(R[x])$, $f(x)R[x]g(x) \subseteq P(R[x])$. So $R[x]$ is P-semicommutative. Conversely, assume that $R[x]$ is P-semicommutative. Suppose that $a, b \in R$ with $ab = 0$. Since $R[x]$ is P-semicommutative, $aR[x]b \subseteq P(R[x])$. So $aRb \subseteq P(R)$. Hence R is P-semicommutative. $\square$

Recall that a ring R is called *power serieswise Armendariz* if for every power series $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x]]$, $f(x)g(x) = 0$ implies $a_ib_j = 0$ for all i, j. Note that for any ring R, $P(R[[x]]) \subseteq P(R)[[x]]$ always holds. In [8], it is shown by example that this inclusion is strict and it is proved that if $P(R)$ is nilpotent, then equality $P(R[[x]]) = P(R)[[x]]$ holds.

Theorem 2.14. *Let R be a ring and $R[[x]]$ be the power series ring with coefficients in R. If $R[[x]]$ is P-semicommutative, then R is P-semicommutative. The converse holds in the case that $P(R)$ is nilpotent and R is power serieswise Armendariz.*

PROOF: Assume that $R[[x]]$ is P-semicommutative. Let $a, b \in R$ with $ab = 0$. Then $aR[[x]]b \subseteq P(R[[x]])$. Since $P(R[[x]]) \subseteq P(R)[[x]]$, we have $aRb \subseteq P(R)$. Conversely, suppose that R is P-semicommutative and $P(R)$ is nilpotent. By [8, Theorem 2.9], $P(R[[x]]) = P(R)[[x]]$. Let $f(x) = \sum a_i x^i$, $g(x) = \sum b_j x^j \in R[[x]]$ with $f(x)g(x) = 0$. Then for all i and j, $a_ib_j = 0$. By assumption, $a_iRb_j \subseteq P(R)$. Hence $a_iR[[x]]b_j \subseteq P(R)[[x]]$. Since $P(R[[x]]) = P(R)[[x]]$, $a_iR[[x]]b_j \subseteq P(R[[x]])$. Thus $f(x)R[[x]]g(x) \subseteq P(R[[x]])$. $\square$

Lemma 2.15. *A ring R is P-semicommutative if and only if the trivial extension $T(R, R)$ is P-semicommutative.*

PROOF: Let $A = \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right), B = \left(\begin{smallmatrix} c & d \\ 0 & c \end{smallmatrix}\right) \in T(R, R)$ with $AB = 0$. Then $ac = 0$. Since R is a P-semicommutative ring, $aRc \subseteq P(R)$. Thus $ACB = \left(\begin{smallmatrix} axc & \star \\ 0 & axc \end{smallmatrix}\right) \in P(T(R, R))$ for any $C = \left(\begin{smallmatrix} x & y \\ 0 & x \end{smallmatrix}\right) \in T(R, R)$. Conversely, let $x, y \in R$ with $xy = 0$. Then we have $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)\left(\begin{smallmatrix} y & 0 \\ 0 & y \end{smallmatrix}\right) = 0$. Since $T(R, R)$ is a P-semicommutative ring, for any $r, s \in R$,

$$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} r & s \\ 0 & r \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} \in P(T(R, R)).$$

Hence $xRy \subseteq P(R)$. This means that R is P-semicommutative. $\square$

In the view of Lemma 2.15, one may ask whether for every positive integer n the full matrix ring $M_n(R)$ is P-semicommutative if R is a P-semicommutative ring. The following answer is negative.

Example 2.16. Let R be an integral domain and $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(R)$. Then $AB = 0$. It is obvious $P(M_2(R)) = M_2(P(R)) = 0$. But $AM_2(R)B \neq 0$ since there exists $C \in M_2(R)$ such that $ACB \neq 0$.

Let R be a ring and $\mathcal{U}$ be a multiplicatively closed subset of R consisting of all central regular elements, and let $Q(R) = \{u^{-1}a \mid u \in \mathcal{U}, a \in R\}$. Then $Q(R)$ is a ring. The following Lemma 2.17 is needed in the sequel. For the sake of completeness we give a short proof.

Lemma 2.17. *The prime radical $P(Q(R))$ is given by $P(Q(R)) = \{u^{-1}a \mid u \in \mathcal{U}, a \in P(R)\}$.*

Proof: Let P be a prime ideal of R. Then $\mathcal{U}^{-1}P$ is a prime ideal of $Q(R)$. For if $(u^{-1}a)Q(R)(v^{-1}b) \subseteq \mathcal{U}^{-1}P$, then $aRb \subseteq (u^{-1}a)Q(R)(v^{-1}b) \subseteq \mathcal{U}^{-1}P$. So $a\mathcal{U} \subseteq P$ or $b\mathcal{U} \subseteq P$. Since $\mathcal{U}$ has the identity, we have $a \in P$ or $b \in P$. It follows that $u^{-1}a \in \mathcal{U}^{-1}P$ or $v^{-1}b \in \mathcal{U}^{-1}P$. So $\mathcal{U}^{-1}P(R) \subseteq P(Q(R))$. Let U be a prime ideal of $Q(R)$ and $P = \{r \in R \mid r \in U\}$. Then P is a prime ideal of R and $\mathcal{U}^{-1}P \subseteq U$. Hence $\mathcal{U}^{-1}P(R) = P(Q(R))$. $\square$

For P-semicommutativity of the ring $Q(R)$, we have the next result.

Proposition 2.18. *Let R be a ring. Then the following are equivalent.*

(1) *R is P-semicommutative.*
(2) *$Q(R)$ is P-semicommutative.*

Proof: (1) $\Rightarrow$ (2) Let $(u^{-1}a)(v^{-1}b) = 0$ in $Q(R)$. Then $ab = 0$. By (1), $aRb \subseteq P(R)$. Hence $a\mathcal{U}^{-1}Rb \subseteq \mathcal{U}^{-1}P(R) \subseteq P(Q(R))$. Accordingly, $(u^{-1}a)Q(R)(v^{-1}b) \subseteq P(Q(R))$.

(2) $\Rightarrow$ (1) Let $ab = 0$ in R. By (2), $aQ(R)b \subseteq P(Q(R))$. Then $aRb \subseteq P(Q(R))$. Then $\mathcal{U}aRb \subseteq P(R)$. Hence $aRb \subseteq P(R)$ since $\mathcal{U}$ contains the identity. $\square$

Corollary 2.19. *Let R be an Armendariz ring. Then the following are equivalent.*

(1) *R is P-semicommutative.*
(2) *$R[x, x^{-1}]$ is P-semicommutative.*

Proof: Let $U = \{1, x, x^2, \dots\}$. Then U is a central multiplicatively closed subset of $R[x]$. The proof follows from Proposition 2.13 and Proposition 2.18. $\square$

Lemma 2.20. *Every P-semicommutative ring R is directly finite, that is, $xy = 1$ implies $yx = 1$ where $x, y \in R$.*

PROOF: Let $x, y \in R$ with $xy = 1$. Let $e = yx$ and $z = y(1-e)$. Then $z^2 = 0$. By Theorem 2.4, $z \in P(R)$. Hence $xz \in P(R)$, and so $1 - xz = 1 - xy(1-e) = e$ is invertible. This implies that $e = 1 = yx$. □

Now we give some relations between forenamed generalizations of semicommutativity. A ring R is called *weakly semicommutative* if for any $a, b \in R$, $ab = 0$ implies arb is nilpotent for each $r \in R$ (see [12]). The next result shows that the class of P-semicommutative rings lies between the classes of semicommutative rings and weakly semicommutative rings.

Proposition 2.21. *Every P-semicommutative ring is weakly semicommutative.*

PROOF: Let $a, b \in R$ and $ab = 0$. Since R is a P-semicommutative ring, $aRb \subseteq P(R)$. As $P(R) \subseteq nil(R)$, we have $aRb \subseteq nil(R)$. So R is weakly semicommutative. □

Lemma 2.22. (1) *Every semicommutative ring is nil-semicommutative-II.*
(2) *Every nil-semicommutative-I ring is weakly semicommutative.*
(3) *Every nil-semicommutative-II ring is weakly semicommutative.*

PROOF: (1) Let R be a semicommutative ring and $a, b \in R$ with ab nilpotent, say $(ab)^n = 0$ for some $n \geq 1$. Without loss of generality we may assume that $n = 3$. Then $ababab = 0$ implies $aRbabab = 0$. It also implies $aRbaRbab = 0$. Similarly, $aRbaRbaRb = 0$ and so $(aRb)^3 = 0$. Therefore $aRb \subseteq nil(R)$.

(2) Let R be a nil-semicommutative-I ring and $a, b \in R$ with $ab = 0$. Then $(ba)^2 = 0$. By hypothesis, $baRba = 0$. Multiplying by aR from left and multiplying by Rb from right we have $(aRb)^3 = 0$.

(3) Let R be a nil-semicommutative-II ring and $a, b \in R$ with $ab = 0$. Then ab is nilpotent. Hence $aRb \subseteq nil(R)$. This completes the proof. □

Example 2.23. (1) There are nil-semicommutative-II rings that are not semicommutative.
(2) There are weakly semicommutative rings that are not nil-semicommutative-I.
(3) There are P-semicommutative and weakly semicommutative rings that are not nil-semicommutative-II.

PROOF: (1) Let F be a field and consider the ring R of 2×2 upper triangular matrices over F. It is easy to check that R is nil-semicommutative-II. Let e_{ij} denote 2×2 matrix units and $A = e_{11} + e_{12}$ and $B = e_{12} - e_{22}$ and $C = e_{11} + e_{12} + e_{22}$. Then $AB = 0$ but $ACB \neq 0$.

(2) Let R be a reduced ring and consider the ring

$$S = \left\{ \begin{pmatrix} a & a_1 & a_2 & a_3 \\ 0 & a & a_4 & a_5 \\ 0 & 0 & a & a_6 \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, a_i \in R \ (i = 1, 2, 3, 4, 5, 6) \right\}.$$

By [12, Example 2.1], S is weakly semicommutative but not semicommutative. We show that it is not nil-semicommutative-I. Let e_{ij} denote 4×4 matrix units and $A = e_{12} + 2e_{13}$, $B = -2e_{24} + e_{34} \in S$. Then A and B are nilpotents and $AB = 0$. Let $C = 4e_{23} + 5e_{24}$. Then $ACB = 4e_{14} \neq 0$. Hence S is not nil-semicommutative-I.

(3) Let F be a field, $R = F < x, y >$ be the free algebra on x, y over F and $I = (x^2)^2$ where (x^2) is the ideal of R generated by x^2. Consider the ring $S = R/I$. By the computation as in [7, Example 1], $P(S)$ contains all nilpotent elements of index two. By Theorem 2.4, S is P-semicommutative. Also by Proposition 2.21, S is weakly semicommutative. Let $yx + I$, $x + I \in S$. Then $(yx + I)(x + I)$ is nilpotent in S. But $(yx + I)(y + I)(x + I)$ can not be nilpotent. Hence S is not nil-semicommutative-II. $\square$

Due to [14, Lemma 2.7], every nil-semicommutative-I ring is 2-primal, and so it is P-semicommutative by Theorem 2.4. The diagram below provides an overview of the various containment relationships between the rings mentioned in this paper. An arrow signifies containment of the class of rings at the start of the arrow into the class of rings to which the arrow points.

$$\begin{tikzcd}
 & & \text{central semicommutative} \arrow[drr] & & \\
\text{semicommutative} \arrow[urr] \arrow[rr] \arrow[drr] & & \text{nil-semicommutative-I} \arrow[rr] & & \text{2-primal} \arrow[dll] \arrow[d] \\
 & & \text{nil-semicommutative-II} \arrow[drr] & & P\text{-semicommutative} \arrow[d] \\
 & & & & \text{weakly semicommutative}
\end{tikzcd}$$

Recall that a ring R is called *exchange* if for any $x \in R$, there exists $e^2 = e \in R$ such that $e \in xR$ and $1 - e \in (1-x)R$. A ring R is called *clean* if every element in R is the sum of an idempotent and a unit. These rings are extensively studied by many authors, namely, [1], [3], [4], [5] and [15]. Nicholson proved that every clean ring is exchange. But the reverse statement need not be true in general. Every exchange ring with all idempotents central is clean (see [15] for detail). In [17], it is proved that every nil-semicommutative-II exchange ring is clean. We now extend these results to P-semicommutative rings. Note that by Example 2.2, there are P-semicommutative but not abelian rings, and there are P-semicommutative but not nil-semicommutative-II rings by Example 2.23.

Theorem 2.24. *Let R be a P-semicommutative ring. Then R is clean if and only if R is exchange.*

Proof: By [15], one direction is clear. Conversely, assume that R is an exchange ring. Let $x \in R$. Then there exists $e^2 = e \in R$ such that $e = xy$ and $1-e = (1-x)z$

for some $y = ye$ and $z = z(1-e) \in R$. Hence $(ez)^2 = 0$ and $((1-e)y)^2 = 0$. By Theorem 2.4, $ez \in P(R)$ and $(1-e)y \in P(R)$. So $1-ez$ and $1+ez$ are invertible and $(1-e)y(1+ez) \in P(R)$. So $1-(1-e)y(1+ez)$ is invertible. Now $(x-(1-e))(y-z) = 1-ez-(1-e)y = (1-(1-e)y(1+ez))(1-ez)$. By invoking Lemma 2.20, we conclude that $x-(1-e)$ is invertible. Therefore R is clean. □

3. Equivalent characterizations

In this section, we characterize P-semicommutative rings from various aspects.

Lemma 3.1. *A ring R is P-semicommutative if and only if for any $a, b \in R$, $ab = 0$ implies $ba \in P(R)$.*

Proof: Suppose that R is a P-semicommutative ring. Let $a, b \in R$ with $ab = 0$. Then $aRb \subseteq P(R)$. Hence $(RbaR)^2 = Rb(aRb)aR \subseteq P(R)$. As $P(R)$ is semiprime, $RbaR \subseteq P(R)$. Therefore $ba \in P(R)$. Conversely, let $a, b \in R$ such that $ab = 0$. Then $abr = 0$ for any $r \in R$. By hypothesis, $bra \in P(R)$. Thus $bRa \subseteq P(R)$. Hence $(RaRbR)^2 = RaR(bRa)RbR \subseteq P(R)$. Therefore $RaRbR \subseteq P(R)$, since $P(R)$ is semiprime. Accordingly, $aRb \subseteq P(R)$. □

Lemma 3.2. *Let R be a ring and I, K ideals of R with $I \cap K = 0$. Then $P\big((R/I)\times(R/K)\big) = \big((\bigcap_{i\in I_1} P_i)/I\big)\times\big((\bigcap_{i\in I_2} P_i)/K\big)$ where $P(R) = (\bigcap_{i\in I_1} P_i)\cap(\bigcap_{i\in I_2} P_i)$ and P_i is a prime ideal of R for every $i \in I_1 \cup I_2$ where I_1 and I_2 are index sets for the prime ideals of R containing I and K, respectively.*

Proof: Since $IK \subseteq I \cap K = 0$, $IK \subseteq P_i$ for every prime ideal P_i of R. Then either $I \subseteq P_i$ or $K \subseteq P_i$. Hence $P(R) = (\bigcap_{i\in I_1} P_i)\cap(\bigcap_{i\in I_2} P_i)$ where $I \subseteq P_i$ for every $i \in I_1$ and $K \subseteq P_i$ for every $i \in I_2$. On the other hand, there are one to one correspondences between prime ideals of R/I, R/K and prime ideals of R containing I, K, respectively. Therefore $P((R/I)\times(R/K)) = ((\bigcap_{i\in I_1} P_i)/I)\times((\bigcap_{i\in I_2} P_i)/K)$. □

Theorem 3.3. *Every finite subdirect product of P-semicommutative rings is P-semicommutative.*

Proof: Let R be the subdirect product of two P-semicommutative rings A and B. It will suffice to show that R is P-semicommutative. Clearly, we have epimorphisms $\varphi\colon R \to A$ and $\phi\colon R \to B$ with $Ker(\varphi)\cap Ker(\phi) = 0$. We may assume that $A = R/Ker(\varphi)$ and $B = R/Ker(\phi)$. Let I and K denote $Ker(\varphi)$ and $Ker(\phi)$, respectively. Suppose that $ab = 0$ in R. Then $\varphi(a) = a + I$ and $\phi(b) = b + K$. Hence $[(b+I)(a+I)]^2 = (ba+I)^2 = 0 + I$. Since R/I is P-semicommutative, $ba + I \in P(R/I) = (\bigcap_{i\in I_1} P_i)/I$. It follows that $ba \in \bigcap_{i\in I_1} P_i$. Likewise, we do the same for the ring R/K and $P(R/K) = (\bigcap_{i\in I_2} P_i)/K$, and so we have $ba \in \bigcap_{i\in I_2} P_i$. Therefore $ba \in P(R)$. This completes the proof by Lemma 3.1. □

Lemma 3.4. *Let I and J be ideals of a ring R. If R/I and R/J are P-semicommutative, then*

(1) *$R/(I \cap J)$ is P-semicommutative;*
(2) *$R/(IJ)$ is P-semicommutative.*

Proof: (1) Let φ: $R/(I \cap J) \to R/I$ be given by $x + (I \cap J) \to x + I$ and let ϕ: $R/(I \cap J) \to R/J$ given by $x + (I \cap J) \to x + J$. Then $Ker(\varphi) \cap Ker(\phi) = 0$. Hence $R/(I \cap J)$ is the subdirect product of R/I and R/J. Therefore $R/(I \cap J)$ is P-semicommutative by Theorem 3.3.

(2) Since $IJ \subseteq I \cap J$, we have $R/(I \cap J) \cong \big(R/(IJ)\big)/\big((I \cap J)/(IJ)\big)$. Here, $\big((I \cap J)/(IJ)\big)^2 = 0$. Thus Corollary 2.7 completes the proof. □

Theorem 3.5. *Let R be a ring. Then the following are equivalent.*

(1) *R is P-semicommutative.*
(2) *The ring $S = \{(x, y) \in R \times R \mid x - y \in P(R)\}$ is P-semicommutative.*

Proof: (1) $\Rightarrow$ (2) Let φ: $S \to R$ be given by $(x, y) \to x$ and ϕ: $S \to R$ be given by $(x, y) \mapsto y$. Then φ and ϕ are epimorphisms. Thus $S/Ker(\varphi)$ and $S/Ker(\phi)$ are P-semicommutative. In view of Lemma 3.4, $S/\big(Ker(\varphi) \cap Ker(\phi)\big)$ is P-commutative. But $Ker(\varphi) \cap Ker(\phi) = 0$. Thus S is P-semicommutative.

(2) $\Rightarrow$ (1) Suppose $ab = 0$ in R. Then $(a, a)(b, b) = (0, 0)$ in S. Thus $(b, b)(a, a) \in P(S)$, by Lemma 3.1. Given $ba = x_0, x_1, \cdots, x_n, \cdots$ with $x_{i+1} \in x_iRx_i$ for all i, then $(b, b)(a, a) = (x_0, x_0), (x_1, x_1), \cdots, (x_n, x_n), \cdots$ with $(x_{i+1}, x_{i+1}) \in (x_i, x_i)S(x_i, x_i)$ for all i. Thus we can find some $m \in \mathbb{N}$ such that $(x_m, x_m) = (0, 0)$, and then $x_m = 0$. This shows that $ba \in P(R)$. In view of Lemma 3.1, R is P-semicommutative. □

Corollary 3.6. *Let R be a semicommutative ring. Then the ring $S = \{(x, y) \in R \times R \mid x - y \in P(R)\}$ is P-semicommutative.*

Let A be a ring with an identity 1_A, and let B be a subring with the same identity. Set

$$R[A, B] = \{(a_1, a_2, \cdots, a_n, b, b, \cdots) \mid \text{each } a_i \in A, b \in B, n \geq 1\}.$$

Then $R[A, B]$ is a ring with the identity $(1_A, 1_A, \cdots)$. We now construct more examples of P-semicommutative rings using such rings.

Lemma 3.7. *Let B be a subring of a ring A. Then*

$$P\big(R[A, B]\big) = R[P(A), P(A) \cap P(B)].$$

Proof: Let $x = (a_1, \cdots, a_n, b, b, \cdots) \in R[P(A), P(A) \cap P(B)]$. Given $x = x_0, x_1, \cdots, x_m, \cdots$ in $R[A, B]$ with each $x_{i+1} \in x_iR[A, B]x_i$. Write $x_i = (a_1^{(i)}, a_2^{(i)}, \cdots, a_m^{(i)}, b^{(i)}, b^{(i)}, \cdots)$. Then $a_1 = a_1^{(0)}, a_1^{(1)}, \cdots, a_1^{(s)}, \cdots$ with each $a_1^{(s+1)} \in a_1^{(s)} A a_1^{(s)}$, so $a_1^{(n_1)} = 0$ for some n_1. Similarly, $a_2^{(n_2)} = 0, \cdots, a_k^{(n_k)} = 0$. Then we have some l such that $a_i^{(l)} = 0$ for all i. Further, $b^{(l)} = 0$, so $x_l = 0$. This shows $x \in P\big(R[A, B]\big)$. Thus $R\big[P(A), P(A) \cap P(B)\big] \subseteq P\big(R[A, B]\big)$. Similarly, we show that $P\big(R[A, B]\big) \subseteq R\big[P(A), P(A) \cap P(B)\big]$. □

Theorem 3.8. *Let B be a subring of a ring A. Then the following are equivalent.*
(1) *$R[A,B]$ is P-semicommutative.*
(2) *A and B are P-semicommutative.*

Proof: (1) $\Rightarrow$ (2) Let $a \in A$ with $a^2 = 0$. Then $x := (a, a, \cdots) \in R[A,B]$ with $x^2 = 0$. Hence $x \in P\big(R[A,B]\big)$. This shows that $a \in P(A)$, by Lemma 3.7. Hence A is P-semicommutative by Theorem 2.4. Likewise, B is P-semicommutative.

(2) $\Rightarrow$ (1) Let $x = (a_1, \cdots, a_n, b, b, \cdots) \in R[A,B]$ with $x^2 = 0$. Then each $a_i^2 = 0$ and $b^2 = 0$. Since $a_i^2 = 0$, we get $a_i \in P(A)$, by Theorem 2.4. On the other hand, $b^2 = 0$ in B, so we get $b \in P(B)$. Furthermore, $b^2 = 0$ in A, and so $b \in P(A)$. Therefore $b \in P(A) \cap P(B)$. In view of Lemma 3.7, $x \in P(R[A,B])$. According to Theorem 2.4, $R[A,B]$ is P-semicommutative. $\square$

Example 3.9. Let $A = T_3(\mathbb{Z}_2)$ and $B = \left\{\left(\begin{smallmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{smallmatrix}\right) \mid a, b, c \in \mathbb{Z}_2\right\}$. Then $R[A,B]$ is P-semicommutative.

Proof: As A and B are P-semicommutative, the result follows by Theorem 3.8. $\square$

Let R be a ring, and let s be a central element in R. Let

$$K_s(R) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a, b, c, d \in R\right\},$$

where the addition and multiplication are given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a + a' & b + b' \\ c + c' & d + d' \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + sbc' & ab' + bd' \\ ca' + dc' & scb' + dd' \end{pmatrix}.$$

Then $K_s(R)$ is a ring with the identity $\left(\begin{smallmatrix} 1_R & 0 \\ 0 & 1_R \end{smallmatrix}\right)$.

Recall that a ring R is of *bounded index* (*of nilpotency*) 2 provided that $a^2 = 0$ for all nilpotent elements, e.g., $\mathbb{Z}_4$.

Lemma 3.10. *Let R be a ring of bounded index 2, and let $s \in R$ be central nilpotent. Then $P\big(K_s(R)\big) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a, d \in P(R), b, c \in R\right\}$.*

Proof: Given $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_s(R)$ with $a, d \in P(R), b, c \in R$, then we see that $K_s(R)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)K_s(R)$ is nilpotent. Hence $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in P\big(K_s(R)\big)$. Conversely, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in P\big(K_s(R)\big)$, then

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in P\big(K_s(R)\big).$$

This shows that RaR is nilpotent, and so $a \in P(R)$. Similarly, $d \in P(R)$. Therefore we complete the proof. $\square$

Theorem 3.11. *Let R be a ring of bounded index 2, and let $s \in R$ be central nilpotent. Then the following are equivalent.*

(1) *$K_s(R)$ is P-semicommutative.*

(2) *R is P-semicommutative.*

Proof: (1) $\Rightarrow$ (2) Choose $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in K_s(R)$. Then $e = e^2$ and $R \cong eK_s(R)e$. Thus R is P-semicommutative by Proposition 2.10.

(2) $\Rightarrow$ (1) Given $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^2 = 0$ in $K_s(R)$, then $a^2 + sbc = 0$. Hence $a^4 = 0$. As R is of bounded index 2, we have $a^2 = 0$. Since R is P-semicommutative, we have $a \in P(R)$. Likewise, $d \in P(R)$. Therefore $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in P\big(K_s(R)\big)$ due to Lemma 3.10. Accordingly, $K_s(R)$ is P-semicommutative. $\square$

We conclude this paper by presenting a result on the P-semicommutativity of a Morita context. A Morita context denoted by (A, B, M, N, ψ, ϕ) consists of two rings A and B, two bimodules ${}_AN_B$ and ${}_BM_A$, and a pair of bimodule homomorphisms (called pairings) $\psi : N \bigotimes_B M \to A$ and $\phi : M \bigotimes_A N \to B$ which satisfy the following associativity: $\psi\big(n \bigotimes m\big)n' = n\phi\big(m \bigotimes n'\big)$ and $\phi\big(m \bigotimes n\big)m' = m\psi\big(n \bigotimes m'\big)$ for any $m, m' \in M, n, n' \in N$. This is called the ring of the Morita context. The next lemma is straightforward.

Lemma 3.12. *Let T be the ring of a Morita context $(A, B, M, N, \psi, \varphi)$ with zero pairings. Then*

$$P(T) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a \in P(A), d \in P(B), b \in N, c \in M \right\}.$$

Theorem 3.13. *Let T be the ring of a Morita context $(A, B, M, N, \psi, \varphi)$ with zero pairings. Then the following are equivalent.*

(1) *T is P-semicommutative.*

(2) *A and B are P-semicommutative.*

Proof: (1) $\Rightarrow$ (2) Choose $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in T$. Then $e = e^2$ and $A \cong eTe$. Thus A is P-semicommutative. Similarly, choose $f = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in T$. Then $f = f^2$ and $B \cong fTf$. Therefore B is P-semicommutative.

(2) $\Rightarrow$ (1) Given $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^2 = 0$ in T, then $a^2 = 0$ in A and $d^2 = 0$ in B. Thus $a \in P(A)$ and $d \in P(B)$. In terms of Lemma 3.12, we see that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in P\big(T\big)$. Therefore we complete the proof. $\square$

Example 3.14. Let R be a P-semicommutative ring, and let

$$A = B = \begin{pmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{pmatrix}, \ M = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & R & 0 \end{pmatrix} \ \text{and} \ N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ R & R & 0 \end{pmatrix},$$

and let $\psi : N \bigotimes_B M \to A$, $\psi(n \otimes m) = nm$ and $\phi : M \bigotimes_A N \to B$, $\phi(m, n) = mn$. Then (A, B, M, N, ψ, ϕ) is a Morita context with zero pairings. It follows by Theorem 3.13 that (A, B, M, N, ψ, ϕ) is P-semicommutative. In this case, (A, B, M, N, ψ, ϕ) is not a triangular matrix ring.

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