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On the relations between the central factor-module and the derived submodule in modules over group rings

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Abstract. A modular analogue of the well-known group theoretical result about finiteness of the derived subgroup in a group with a finite factor by its center has been obtained.

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1. Introduction

The following theorem is one of the classical general results of group theory.

Theorem ZD. *Let G be a group, C a subgroup of the center $\zeta(G)$ such that G/C is finite. Then the derived subgroup $[G,G]$ is finite.*

This result plays a very important role in infinite group theory; it lies at the foundation of many important group-theoretical results. In the form above it first appeared in the work of Neumann [11]. In the conclusion of this work, B. Neumann remarked that R. Baer told him that this theorem is a consequence of a more general result, which was proved by R. Baer in his paper [1]. In fact, in Theorem 3 of this paper it was proved that if a normal subgroup H of a group G has finite index, then the factor $([G,G] \cap H)/[H,G]$ is also finite. However, later R. Baer in his article [2] considered this Theorem ZD in its usual form and supplied it with a new proof. Immediately the natural question on the relations between the order $|G/\zeta(G)| = t$ and the order of the derived subgroup $|[G,G]|$ arose. This question was posed by B. Neumann in the article [11]. He also obtained the first bound for $|[G,G]|$. The best bound here was obtained by J. Wiegold. In his article [17], he showed that $|[G,G]| \leq t^m$ where $m = (\frac{1}{2})(\log_p t - 1)$ and p is the smallest prime divisor of the number t. Also he proved that this bound is attained when $t = p^n$ and p is a prime. If this number t has more than one prime divisor, the situation is much more complicated.

In his famous lectures on nilpotent groups, P. Hall obtained the generalization of Theorem ZD [5, Theorem 8.7]. In these lectures P. Hall called Theorem ZD

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Schur's theorem (note that in this case, P. Hall did not make any specific references). Inheriting Hall, many algebraists started calling Theorem ZD Schur's theorem, while making a reference to the paper of I. Schur [14]. In this old classical paper, I. Schur introduces (only for finite groups!) the concept of the group, which is now called the *Schur multiplicator* or the *Schur multiplier* and obtains some properties of this group. The results of the article [14] were used by B.H. Neumann and J. Wiegold for obtaining new bounds for the order of the derived subgroup. Some analogies of Theorem ZD were obtained in other algebraic branches, such as, for example, Lie algebras and their generalizations [15], [13].

In the article [4], an analogue of Theorem ZD was obtained for linear groups. Let A be a vector space over a field F and G be a subgroup of $GL(F, A)$. The subspace $C_A(G)$ is an analogy of the center and the subspace $[A, G]$ generated by the elements $ag - a$, $a \in A$, $g \in G$, is an analogy of the derived subgroups. In the paper [4], the vector space A such that $\dim_F(A/C_A(G))$ is finite has been considered. Immediately it should be noted, that the finiteness of $\dim_F(A/C_A(G))$ does not always imply the finiteness of $\dim_F([A, G])$. In the paper [4] one can find an example of such situation. Note that in this example, G is an infinite elementary abelian p-group, where p is a prime, and A is a vector space over a field of characteristic p. However if a group G does not have an infinite elementary p-section, $p = char(F)$, then the finiteness of $\dim_F(A/C_A(G))$ implies finiteness of $\dim_F([A, G])$. A similar situation occurs for the case when $char(F) = 0$ (see [4, Theorem A]).

We can consider a vector space A as a module over a group ring FG. Therefore, the next natural step is to consider the situation of the modules over the group ring RG where R is some (commutative) ring. In this case, R-modules having finite composition series are analogues of finite-dimensional vector spaces. Such modules have more characteristics than the dimension. Consider the situation in detail. Let R be a ring and A an R-module. Suppose that A has a finite composition series

$$\langle 0\rangle = C_0 \leq C_1 \leq \ldots \leq C_n = A$$

of submodules. Then $C_j/C_{j-1} = R(c_j + C_{j-1}) \cong_R R/Ann_R(c_j + C_{j-1})$. Since C_j/C_{j-1} is a simple R-module, $Ann_R(c_j + C_{j-1}) = Ann_R(C_j/C_{j-1})$ is a maximal ideal of R. Then the factor-ring $R/Ann_R(c_j + C_{j-1})$ is a field. We recall that every two composition series of A are isomorphic. It follows that the length n of composition series and the sets

$$Spec(A) = \{char(F_j)|F_j = R/Ann_R(C_j/C_{j-1}), 1 \leq j \leq n\},$$

$$Sdim(A) = \{\dim_{F_j}(C_j/C_{j-1})|1 \leq j \leq n\}$$

are invariants of the module A. The length of composition series of A is called the composition length of A and denoted by $c_R(A)$.

We have already noted that the analogue of Theorem ZD for the case when R is a field occurs only when the restrictions on the abelian p-sections of G, where $p = char(F)$, are imposed. Therefore it is natural to keep here the same restrictions.

Let p be a prime. We say that a group G has *finite section p-rank $sr_p(G) = r$* if every elementary abelian p-section of G is finite of order at most p^r and there is an elementary abelian p-section A/B of G such that $|A/B| = p^r$.

And similarly, we say that a group G has *finite section 0-rank $sr_0(G) = r$* if for every torsion-free abelian section U/V of G, $r_{\mathbb{Z}}(U/V) \leq r$ and there is an abelian torsion-free section U/V such that $r_{\mathbb{Z}}(U/V) = r$.

Here $r_{\mathbb{Z}}(A)$ is a $\mathbb{Z}$-rank of an abelian group A (that is a rank A as a $\mathbb{Z}$-module).

We note that if a group G has finite section p-rank for some prime p, then G has finite section 0-rank, moreover $sr_0(G) \leq sr_p(G)$. Indeed, suppose that U/V is a torsion-free abelian section of G. Choose in U/V a free abelian subgroup S/V such that the factor-group U/S is periodic. Then $r_{\mathbb{Z}}(U/V) = r_{\mathbb{Z}}(S/V)$. We have $S/V = Dr_{\lambda\in\Lambda}\langle d_\lambda\rangle$, then $(S/V)^p = Dr_{\lambda\in\Lambda}\langle d_\lambda^p\rangle$, and

$$(S/V)/(S/V)^p = (Dr_{\lambda\in\Lambda}\langle d_\lambda\rangle)/(Dr_{\lambda\in\Lambda}\langle d_\lambda^p\rangle) \cong Dr_{\lambda\in\Lambda}\langle d_\lambda\rangle/\langle d_\lambda^p\rangle.$$

Since $sr_p(G) = r$ is finite, then $(S/V)/(S/V)^p$ is finite and has order at most p^r. On the other hand, then $|(S/V)/(S/V)^p| = p^{|\Lambda|}$, so that $r_{\mathbb{Z}}(U/V) = r_{\mathbb{Z}}(S/V) = |\Lambda| \leq r$. It follows that $sr_0(G) \leq sr_p(G)$.

The group G has finite special rank $r(G) = r$, if every finitely generated subgroup of G can be generated by r elements and r is the least positive integer with this property.

Let G be a group, R a ring and A an RG-module. Put

$$\zeta_{RG}(A) = \{a \in A \mid a(g-1) = 0 \text{ for each element } g \in G\} = C_A(G).$$

Clearly $\zeta_{RG}(A)$ is an RG-submodule of A. This submodule is called the *RG-center of A*.

The analogue of the derived subgroup is as follows. Denote by $[A, G]$ the RG-submodule generated by elements $ag - a$, $a \in A$, $g \in G$. In other words, $[A, G] = A(\omega RG)$ where ωRG is the augmentation ideal of a group ring RG, that is the two-sided ideal generated by all elements $g - 1$, $g \in G$. The submodule $[A, G]$ is called the *derived submodule of A*.

The main result of this paper is the following modular analogue of Theorem ZD.

Theorem 1. *Let R be an integral domain, G be a group and A be an RG-module. Suppose that $A/\zeta_{RG}(A)$ has finite composition series as an R-module. If the group G has finite section p-rank r_p for every $p \in Spec(A/\zeta_{RG}(A))$, then $[A, G]$ has finite R-composition series and $Spec([A, G]) \subseteq Spec(A/\zeta_{RG}(A))$. Moreover, there exists a function κ_6 such that*

$$c_R([A, G]) \leq \kappa_6(r_p, d \mid p \in Spec_R(A/\zeta_{RG}(A)), d \in Sdim(A/\zeta_{RG}(A))).$$

This result is an extension of Theorem A of the paper [4] in the case of modules over group rings with an integral domain as a scalar ring.

2. Linear groups having finite section p-rank.

The first natural step is to consider the case when factor-module $A/\zeta_{RG}(A)$ is a simple RG-module having finite composition series as an R-module. In this case $Ann_R(A/\zeta_{RG}(A))$ is a maximal ideal of R, so that a factor-ring $F = R/Ann_R(A/\zeta_{RG}(A))$ is a field. Moreover, the fact that $A/\zeta_{RG}(A)$ has finite composition series as an R-module implies that $\dim_F(A/\zeta_{RG}(A))$ is finite. Therefore we can think of $G/C_G(A/\zeta_{RG}(A))$ as a subgroup of $GL_n(F)$ where $n = \dim_F(A/\zeta_{RG}(A))$. Thus, we need some information on irreducible linear groups having finite section p-rank where $p = char(F)$.

Lemma 1. *Let p be a prime or $p = 0$ and let G be a group. Suppose that G has finite section p-rank. If U, V are the subgroup of G such that V is a normal subgroup of U, then U/V does not include a non-abelian free subgroup.*

Proof: Suppose the contrary, let U/V include a non-abelian free subgroup F/V. If the free rank of F/V is infinite, then F/V includes a normal subgroup E/V such that F/E is a free abelian group of infinite $\mathbb{Z}$-rank. If $p = 0$, then we obtain a contradiction right away. Suppose that p is a prime. Since F/E is a free abelian group of infinite $\mathbb{Z}$-rank, F/E has an infinite elementary abelian p-factor-group, which implies that F/V has infinite section p-rank. This contradiction shows that F/V has finite free rank. But in this case, $[F/V, F/V]$ is a free subgroup of infinite countable free rank (see, for example, [9, §36]), and using the above arguments, we again come to a contradiction. □

Lemma 2. *Let F be a field of prime characteristic p and G be a periodic subgroup of $GL_n(F)$. If G has finite section p-rank r, then G is abelian-by-finite and has finite special rank at most* $\max\{r, \frac{1}{2}(5n + 1)n\} + 1$.

Proof: We recall that G is locally finite (see, for example, [16, 9.1]). Let P be the Sylow p-subgroup of G. The finiteness of $sr_p(P)$ implies that P has finite special rank r [3, Corollary 2.3]. Then P is a Chernikov subgroup (see, for example, [9, §64]). On the other hand, P is a nilpotent group of finite exponent (see, for example, [16, 9.1]). It follows that P is finite. In the turn out, it follows that G is almost abelian (see, for example, [16, Corollary 9.7]). Let q be a prime such that $q \neq p$ and let Q be a finite q-subgroup of G. Choose in Q a maximal normal abelian subgroup A. Being a finite q-subgroup, Q is nilpotent, therefore $A = C_Q(A)$. By Lemma 2.9 of the paper [4] A has special rank at most $k \leq n$. It follows that Q/A is isomorphic to some q-subgroup of $GL_n(\mathbb{Z}/q^m\mathbb{Z})$ for some positive integer m. Then Q/A has special rank at most $\frac{1}{2}(5n - 1)n$ (see, for example, [6, 25.1.3]). It follows that Q has a special rank at most $n + \frac{1}{2}(5n - 1)n = \frac{1}{2}(5n + 1)n$. Let H be an arbitrary finite subgroup of G. If $q \in \Pi(H)$ and $q \neq p$, then as proved above the Sylow p-subgroup S_q of H has at most $\frac{1}{2}(5n+1)n$ generators. As we remarked above, every Sylow p-subgroup of G has special rank at most r, therefore the Sylow p-subgroup S_p of H has at most r generators. Let $\kappa_1(r, n) = \max\{r, \frac{1}{2}(5n + 1)n\}$. Then H has at most $\kappa_1(r, n) + 1$ generators [10, Theorem 1]. It follows that G has a special rank at most $\kappa_1(r, n) + 1$. □

Lemma 3. *Let F be a field of characteristic 0 and G be a periodic subgroup of $GL_n(F)$. Then G has finite special rank at most $\frac{1}{2}(5n+1)n+1$.*

Proof: We recall that G is locally finite (see, for example, [16, 9.1]). Let q be a prime and let Q be a finite q-subgroup of G. Choose in Q a maximal normal abelian subgroup A. Being a finite q-subgroup, Q is nilpotent, and therefore $A = C_Q(A)$. By Lemma 2.9 of [4], A has special rank at most $k \leq n$. It follows that Q/A is isomorphic to some q-subgroup of $GL_n(\mathbb{Z}/q^m\mathbb{Z})$ for some positive integer m. Then Q/A has special rank at most $\frac{1}{2}(5n-1)n$ (see, for example, [6, 25.1.3]). It follows that Q has a special rank at most $n+\frac{1}{2}(5n-1)n = \frac{1}{2}(5n+1)n$. Let H be an arbitrary finite subgroup of G. If $q \in \Pi(H)$, then as proved above the Sylow p-subgroup S_p of H has at most $\frac{1}{2}(5n+1)n$ generators. It follows that H has at most $\frac{1}{2}(5n+1)n+1$ generators [10, Theorem 1]. Hence G has a special rank at most $\frac{1}{2}(5n+1)n+1$. □

Lemma 4. *Let F be a field, G be a group and A be a simple FG-module. Suppose that G is (locally soluble)-by-finite and $\dim_F(A) = n$ is finite. Then $G/C_G(A)$ is abelian-by-finite.*

Proof: Without loss of generality we can suppose that $C_G(A) = \langle 1 \rangle$. The group G includes a normal locally soluble subgroup S such that G/S is finite. We remark at once that S is soluble (see, for example, [16, Corollary 3.8]). Since $\dim_F(A)$ is finite, A includes a non-zero FS-submodule B, having the least dimension. Then B is a simple FS-submodule and $A = \bigoplus_{1\leq j\leq s} Bg_j$ for some elements $g_1, \dots, g_s \in G$ (see, for example, [8, Lemma 5.4]). Then S includes a normal abelian subgroup U such that S/U is finite (see, for example, [16, Lemma 3.5]). Since G/S is finite, U has finite index in G. □

If G is a group then denote by $Tor(G)$ the maximal normal periodic subgroup of G. The subgroup $Tor(G)$ is called *the periodic part of a group G*. We remark that if a group G is locally nilpotent, then $Tor(G)$ contains all elements having finite order, so that the factor-group $G/Tor(G)$ is torsion-free.

Lemma 5. *Let F be a field of prime characteristic p, G be a group and A be a simple FG-module. Suppose that G has finite section 0-rank r and $\dim_F(A) = n$ is finite. Then $G/C_G(A)$ is abelian-by-finite and has finite special rank. Moreover, there is a function κ_2 such that $r(G/C_G(A)) \leq \kappa_2(r, n)$.*

Proof: Without loss of generality we can suppose again that $C_G(A) = \langle 1 \rangle$. Lemma 1 shows that G does not include a non-abelian free subgroup. Then G includes a normal soluble subgroup S such that G/S is locally finite (see, for example, [16, Theorem 10.17]). Since $\dim_F(A)$ is finite, A includes a non-zero FS-submodule B having the least dimension. Then B is a simple FS-submodule and $A = \bigoplus_{1\leq j\leq s} Bg_j$ for some elements $g_1, \dots, g_s \in G$ (see, for example, [8, Lemma 5.4]). Then S includes a normal abelian subgroup U such that S/U is finite (see, for example, [16, Lemma 3.5]). Let $T = Tor(S)$. An obvious inclusion $Tor(U) \leq Tor(S)$ implies that S/T includes a normal abelian torsion-free subgroup $UT/T \cong U/(U \cap T) = U/Tor(U)$ having finite index m. As we have seen above the fact that G has finite section p-rank implies that UT/T has finite $\mathbb{Z}$-rank, and hence finite special rank. It follows that a subgroup S/T has finite special rank.

Put $V/T = (S/T)^m$, then $V/T \leq UT/T$ and, in particular, V/T is an abelian torsion-free group having finite special rank. The fact that S/T has finite special rank implies that a subgroup V/T has finite index in S/T. Finally the choice of V/T yields that V/T is G-invariant. Since G/S is locally finite, G/V is also locally finite. Put $X/T = C_{G/T}(V/T)$, then by above remark a factor-group G/X is finite. Moreover, the fact that G has finite section p-rank implies that V/T has finite $\mathbb{Z}$-rank, moreover $r_{\mathbb{Z}}(V/T) \leq r$. Lemma 3 shows that G/X has finite special rank at most $\frac{1}{2}(5r+1)r+1$.

The inclusion $V/T \leq \zeta(X/T)$ implies that a derived subgroup $[X/T, X/T]$ is locally finite (see, for example, [12, Corollary to Theorem 4.12]). Then the choice of T yields that $[X/T, X/T] = \langle 1 \rangle$. The choice of T yields also that $Tor(X/T) = \langle 1 \rangle$. The fact that X/T is abelian implies that $(X/T)/Tor(X/T)$ is torsion-free, thus X/T is abelian and torsion-free.

By Lemma 2 a subgroup T is abelian-by-finite and has finite special rank at most $\kappa_1(r,n)+1$. Since X/T is abelian and torsion-free, as we have seen above X/T has finite special rank at most r. Finally, G/X has finite special rank at most $\frac{1}{2}(5r+1)r+1$, thus G has finite special rank at most $\kappa_1(r,n)+1+r+\frac{1}{2}(5r+1)r+1 = \kappa_2(r,n)$.

Since T and G/T are abelian-by-finite, G is soluble-by-finite. An application of Lemma 4 shows that G is abelian-by-finite. □

Lemma 6. *Let F be a field of characteristic 0, G be a group and A be a simple FG-module. Suppose that G has finite section 0-rank r and $\dim_F(A) = n$ is finite. Then $G/C_G(A)$ is abelian-by-finite and has finite special rank. Moreover, there is a function κ_3 such that $r(G/C_G(A)) \leq \kappa_3(r,n)$.*

Proof: Without loss of generality we can suppose that $C_G(A) = \langle 1 \rangle$. Lemma 1 shows that G does not include a non-abelian free subgroup. Then G includes a normal soluble subgroup S such that G/S is finite (see, for example, [16, Theorem 10.17]). Lemma 4 shows that G includes a normal abelian subgroup U such that G/U is finite.

Let $T = Tor(S)$. An obvious inclusion $Tor(U) \leq Tor(S)$ implies that G/T includes a normal abelian torsion-free subgroup $UT/T \cong U/(U \cap T) = U/Tor(U)$ having finite index m. The fact that G has finite section 0-rank implies that UT/T has finite $\mathbb{Z}$-rank. Put $X/T = C_{G/T}(UT/T)$, then the factor-group G/X is finite. Since UT/T is of finite $\mathbb{Z}$-rank at most r, Lemma 3 shows that G/X has finite special rank at most $\frac{1}{2}(5r+1)r+1$.

The inclusion $UT/T \leq \zeta(X/T)$ implies that a derived subgroup $[X/T, X/T]$ is finite (see, for example, [12, Theorem 4.12]). Together with the choice of T it implies the equality $[X/T, X/T] = \langle 1 \rangle$. The choice of T yields also that

$Tor(X/T) = \langle 1\rangle$. The fact that X/T is abelian implies that $(X/T)/Tor(X/T)$ is torsion-free, thus X/T is abelian and torsion-free.

By Lemma 3, the subgroup T is abelian-by-finite and has finite special rank at most $\frac{1}{2}(5n+1)n+1$. As we have seen above, X/T has finite special rank at most r. Finally, G/X has finite special rank at most $\frac{1}{2}(5r+1)r+1$, thus G has finite special rank at most $\frac{1}{2}(5n+1)n+1+r+\frac{1}{2}(5r+1)r+1 = \frac{1}{2}(5n+1)n+\frac{1}{2}(5r+3)r+2 = \kappa_3(r,n)$. □

3. The basic case

Now we will consider the case when the factor-module $A/\zeta_{RG}(A)$ is a simple RG-module having a finite composition series as an R-module. This case is the basic here.

Lemma 7. *Let R be an integral domain, G be a group and A be a non-trivial RG-module. Suppose that $A/\zeta_{RG}(A)$ is a simple RG-module and*

$$F = R/Ann_R(A/\zeta_{RG}(A)).$$

Then $Ann_R(A/\zeta_{RG}(A)) = Ann_R([A,G])$.

Proof: Put $C = \zeta_{RG}(A)$, $D = [A,G]$. Without loss of generality we can suppose that $C_G(A) = \langle 1\rangle$. The mapping $\xi_g : A \mapsto A$, defined by the rule $\xi_g(a) = a(g-1)$, $a \in A$, is R-linear for each element $g \in G$. We have $Im(\xi_g) = A(g-1)$ and $Ker(\xi_g) = C_A(g)$, so that $A(g-1) = Im(\xi_g) \cong_R A/Ker(\xi_g) = A/C_A(g)$. Let $\alpha \in Ann_R(A/C)$, then $\alpha a \in C$ for every element $a \in A$. It follows that $\alpha(a(g-1)) = (\alpha a)(g-1) = 0$, which shows that $Ann_R(A/C) \le Ann_R(D)$. Since $Ann_R(A/C)$ is a maximal ideal of R, then either that $Ann_R(D) = R$ or $Ann_R(D) = Ann_R(A/C)$. In first case $D = \langle 0\rangle$ and A is a trivial RG-module, and we obtain a contradiction. This contradiction proves an equality $Ann_R(A/\zeta_{RG}(A)) = Ann_R([A,G])$. □

Lemma 8. *Let R be an integral domain, G be a group and A be a non-trivial RG-module. Suppose that $A/\zeta_{RG}(A)$ is a simple RG-module and*

$$F = R/Ann_R(A/\zeta_{RG}(A)).$$

If $G = C_G(A/\zeta_{RG}(A))$ and the group G has finite section p-rank r where $p = char(F)$, then $Ann_R(A/\zeta_{RG}(A)) = Ann_R([A,G])$ and $\dim_F([A,G]) \le r$.

Proof: Put $C = \zeta_{RG}(A)$, $D = [A,G]$. Without loss of generality we can suppose that $C_G(A) = \langle 1\rangle$. The equality $G = C_G(A/C)$ implies that $\dim_F(A/C) = 1$. Choose an element $v \in A\setminus C$, then the coset $v+C$ is a basis of A/C. The equality $Ann_R(A/\zeta_{RG}(A)) = Ann_R([A,G])$ follows from Lemma 7.

Consider the mapping $\nu : G \to A$ which is defined by the rule $\nu(g) = v(g-1)$, $g \in G$. We note that $Im(\nu) \le C$. If x is another element of G, then $v(g-1)(x-1) = 0$ and the equality $v(gx-1) = v(g-1)(x-1) + v(g-1) + v(x-1)$ implies that

$$\nu(gx) = v(gx-1) = v(g-1) + v(x-1) = \nu(g) + \nu(x).$$

In other words, ν is a homomorphism of a group G into the additive group of A. We have $Ker(\nu) = C_G(v)$ and $Im(\nu)$ is an additive subgroup of $[A, G]$ generated by all elements $v(g-1)$, $g \in G$.

If $char(F) = p$ is a prime, then $Im(\nu)$ is an elementary abelian p-subgroup. Then the isomorphism $Im(\nu) \cong G/Ker(\nu)$ implies that $G/Ker(\nu)$ is an elementary abelian p-group. The fact that $sr_p(G) = r$ implies that $G/Ker(\nu)$ is a finite elementary abelian p-group of order at most p^r. It follows that $Im(\nu)$ has at most r generators. It follows that a subspace $FIm(\nu) = RIm(\nu)$ is finitely generated and hence finite dimensional. Moreover, $\dim_F(RIm(\nu)) \leq r$.

Consider now the case when $char(F) = 0$. Then $Im(\nu)$ is a torsion-free abelian subgroup. Using again the isomorphism $Im(\nu) \cong G/Ker(\nu)$, we obtain that $G/Ker(\nu)$ is also a torsion-free abelian subgroup. In this case, from $sr_0(G) = r$ we obtain that $G/Ker(\nu)$ is a torsion-free abelian group having finite $\mathbb{Z}$-rank at most r. It follows that $r_{\mathbb{Z}}(Im(\nu)) \leq r$. Let $b_1, \ldots, b_m$ be the maximal $\mathbb{Z}$-independent subset of $Im(\nu)$, $m \leq r$. Every element of $RIm(\nu)$ has the form $\alpha v(y-1)$ where $\alpha \in R$, $y \in G$. There exists an integer s such that $sa(y-1) = t_1b_1 + \ldots + t_mb_m$ for some integers $t_1, \ldots, t_m$. This shows that the subset $\{\alpha a(y-1), b_1, \ldots, b_m\}$ is not independent over F. It follows again that $\dim_F(RIm(\nu)) \leq r$.

If a is an arbitrary element of A, then $a = c + \gamma v$ where $\gamma \in R$, $c \in C$. Then $a(g-1) = (c+\gamma v)(g-1) = \gamma v(g-1) \in RIm(\nu)$. It follows that $[A, G] = RIm(\nu)$, which finishes the proof. $\square$

Using ordinary induction, we derive from this lemma the following

Corollary 1. *Let R be an integral domain, G be a group and A be a non-trivial RG-module. Suppose that A has a series of RG-submodules*

$$\langle 0 \rangle = C_0 \leq C_1 = \zeta_{RG}(A) \leq C_2 \leq \ldots \leq C_{n+1} = A$$

such that the factors C_j/C_{j-1} are simple RG-modules, $2 \leq j \leq n+1$. If $C_G(C_j/C_{j-1}) = G$, $2 \leq j \leq n+1$, and G has finite section p-rank r for every $p \in Spec(A/\zeta_{RG}(A))$, then $Spec([A, G]) \subseteq Spec(A/\zeta_{RG}(A))$ and $c_R([A, G]) \leq rn$.

If G is a group, R is a ring and A is an RG-module $g, x \in G$, then we have

$$a(gx-1) = a(g-1)(x-1) + a(g-1) + a(x-1)$$

for each element $a \in A$. It follows that $A(gx-1) \leq A(g-1) + A(x-1)$. Using ordinary induction we obtain the equalities

$$[A, \langle g \rangle] = A(g-1) \text{ and } [A, \langle g, x \rangle] = A(g-1) + A(x-1).$$

Moreover,

$$[A, \langle g_1, \ldots, g_n \rangle] = A(g_1 - 1) + \ldots + A(g_n - 1).$$

Lemma 9. *Let R be an integral domain, $G = \langle g_1, \ldots, g_n \rangle$ be a finitely generated group and A be an RG-module. Suppose that $A/\zeta_{RG}(A)$ is a simple*

RG-module, $G \neq C_G(A/\zeta_{RG}(A))$ and $\dim_F(A/\zeta_{RG}(A)) = d$ is finite, where $F = R/Ann_R(A/\zeta_{RG}(A))$. Then $Ann_R(A/\zeta_{RG}(A)) = Ann_R([A,G])$ and $\dim_F([A,G]) \leq dn$.

Proof: Put $C = \zeta_{RG}(A)$, $D = [A,G]$. Without loss of generality we can suppose that $C_G(A) = \langle 1 \rangle$. The equality $Ann_R(A/\zeta_{RG}(A)) = Ann_R([A,G])$ follows from Lemma 7. The mapping $\xi_g : A \to A$, defined by the rule $\xi_g(a) = a(g-1)$, $a \in A$, is R-linear for each element $g \in G$. We have $Im(\xi_g) = A(g-1)$ and $Ker(\xi_g) = C_A(g)$, so that $A(g-1) = Im(\xi_g) \cong_R A/Ker(\xi_g) = A/C_A(g)$. The inclusion $C \leq C_A(g)$ implies that $\dim_F(A/Ker(\xi_g)) \leq d$. Thus $dim_F(A(g-1)) \leq d$ for every element $g \leq G$. We have noted above that $[A,G] = A(g_1-1)+\cdots+A(g_n-1)$. It follows that

$$\dim_F([A,G]) \leq \dim_F(A(g_1-1)) + \ldots + \dim_F(A(g_n-1)) \leq dn. \qquad \square$$

Corollary 2. *Let R be an integral domain, G be a group and A be an RG-module. Suppose that the following conditions hold:*

(i) *$G \neq C_G(A/\zeta_{RG}(A))$;*
(ii) *$A/\zeta_{RG}(A)$ is a simple RG-module;*
(iii) *$\dim_F(A/\zeta_{RG}(A)) = d$ is finite, where $F = R/Ann_R(A/\zeta_{RG}(A))$;*
(iv) *group G has finite section p-rank r where $p = char(F)$;*
(v) *there are the elements $g_1, \ldots, g_n$ such that $G = \langle g_1, \ldots, g_n \rangle C_G(A/\zeta_{RG}(A))$.*

Then $Ann_R(A/\zeta_{RG}(A)) = Ann_R([A,G])$ and $dim_F([A,G]) \leq r(n+d)$.

Proof: Put $C = \zeta_{RG}(A)$, $Z = C_G(A/\zeta_{RG}(A))$. Without loss of generality we can suppose that $C_G(A) = \langle 1 \rangle$. The equality $Ann_R(A/\zeta_{RG}(A)) = Ann_R([A,G])$ follows from Lemma 7. By Corollary 1 $\dim_F([A,Z]) \leq rd$. Since Z is a normal subgroup of G, $K = [A,Z]$ is an RG-submodule. Furthermore, $Z \leq C_G(A/K)$, so that a factor-group $G/C_G(A/K)$ is finitely generated and we can apply Lemma 9. Put $L/K = [A/K, G]$, then by Lemma 9 $\dim_F(L/K) \leq rn$. An obvious inclusion $[A,G] \leq L$ shows that

$$\dim_F([A,G]) \leq \dim_F(L) = \dim_F(L/K) + \dim_F(K) \leq rn + rd = r(n+d).$$

$\square$

Corollary 3. *Let R be an integral domain, G be a group and A be an RG-module. Suppose that the following conditions hold:*

(i) *$G \neq C_G(A/\zeta_{RG}(A))$;*
(ii) *$A/\zeta_{RG}(A)$ is a simple RG-module;*
(iii) *$\dim_F(A/\zeta_{RG}(A)) = d$ is finite, where $F = R/Ann_R(A/\zeta_{RG}(A))$;*
(iv) *group G has finite section p-rank r where $p = char(F)$;*
(v) *$G/C_G(A/\zeta_{RG}(A))$ is finite.*

Then $Ann_R(A/\zeta_{RG}(A)) = Ann_R([A,G])$ *and there are functions* κ_3 *and* κ_4 *such that* $\dim_F([A,G]) \leq \kappa_3(r,d)$ *whenever* $p > 0$ *and* $\dim_F([A,G]) \leq \kappa_4(r,d)$ *whenever* $p = 0$.

PROOF: Put $C = \zeta_{RG}(A)$. Without loss of generality we can suppose that $C_G(A) = \langle 1\rangle$. The equality $Ann_R(A/\zeta_{RG}(A)) = Ann_R([A,G])$ follows from Lemma 7. If $p > 0$, then Lemma 2 shows that $G/C_G(A/C)$ has a special rank at most $\kappa_1(r,d)+1$. Being finite $G/C_G(A/C)$ has $\kappa_1(r,d)+1$ generators. Using Corollary 2, we obtain that $\dim_F([A,G]) \leq r(\kappa_1(r,d)+1+d)$.

If $p = 0$, then, by Lemma 2, $G/C_G(A/C)$ has a special rank at most $\frac{1}{2}(5d+1)d+1$. Being finite $G/C_G(A/C)$ has $\frac{1}{2}(5d+1)d+1$ generators. Using Corollary 2, we obtain that $\dim_F([A,G]) \leq r(\frac{1}{2}(5d+1)d+1+d) = r(\frac{1}{2}(5d+3)d+1)$.

Put now $\kappa_3(r,d) = r(\kappa_1(r,d)+1+d)$ and $\kappa_4(r,d) = r(\frac{1}{2}(5d+3)d+1)$. □

Starting from the RG-center, we can construct the upper RG-central series of A

$$\langle 0\rangle = \zeta_{RG,0}(A) \leq \zeta_{RG,1}(A) \leq \zeta_{RG,2}(A) \leq \dots$$
$$\zeta_{RG,\alpha}(A) \leq \zeta_{RG,\alpha+1}(A) \leq \dots \zeta_{RG,\gamma}(A) \leq \dots$$

where $\zeta_{RG,1}(A) = \zeta_{RG}(A)$ is the center of G, and recursively

$$\zeta_{RG,\alpha+1}(A)/\zeta_{RG,\alpha}(A) = \zeta_{RG}(A/\zeta_{RG,\alpha}(A))$$

for all ordinals α, $\zeta_{RG,\lambda}(A) = \bigcup_{\mu<\lambda}\zeta_{RG,\mu}(A)$ for the limit ordinals λ and $\zeta_{RG}(A/\zeta_{RG,\gamma}(A)) = \langle 0\rangle$. The last term $\zeta_{RG,\gamma}(A) = \zeta_{RG,\infty}(A)$ of this series is called the upper RG-hypercenter of A and the ordinal is called the RG-central length of a module A and will denoted by $zl_{RG}(A)$. We observe that $[\zeta_{RG,\alpha+1}(A), G] \leq \zeta_{RG,\alpha}(A)$ for all $\alpha < \gamma$.

If the upper RG-hypercenter of A coincides with A, then A is called RG-hypercentral.

If A is an RG-hypercentral module and $zl_{RG}(A)$ is finite, then we will say that A is RG-nilpotent.

An RG-module A is called locally RG-nilpotent, if the FH-submodule MFH is FH-nilpotent for every finite subset M of A and every finitely generated subgroup H of G.

Let G be a group, R a ring and A an RG-module. If B, C are the RG-submodules of A and $B \leq C$, then a factor C/B is called G-central (respectively G-eccentric), if $G = C_G(C/B)$ (respectively $G \neq C_G(C/B)$).

We say that the RG-module A is G-hypereccentric, if A has an ascending series of RG-submodules

$$\langle 0\rangle = A_0 \leq A_1 \leq \dots A_\alpha \leq A_{\alpha+1} \leq \dots A_\gamma = A$$

whose factors $A_{\alpha+1}/A_\alpha$ are G-eccentric and simple FG-modules for all $\alpha < \gamma$.

We say that the RG-module A has Z-decomposition, if $A = C \times E$ where C is the upper RG-hypercenter of A and E is an RG-submodule which is G-hypereccentric (see, [8, Chapter 10]). We remark that this decomposition is unique (of course, if it exists) [8, Chapter 10].

Lemma 10. *Let R be an integral domain, G be an abelian-by-finite group and A be an RG-module. Suppose that $A/\zeta_{RG}(A)$ is a simple RG-module, $G \neq C_G(A/\zeta_{RG}(A))$ and $\dim_F(A/\zeta_{RG}(A)) = d$ is finite, where*

$$F = R/Ann_R(A/\zeta_{RG}(A)).$$

If a group G has finite section p-rank r where $p = char(F)$ and the factor-group $G/C_G(A/\zeta_{RG}(A))$ is infinite, then $Ann_R(A/\zeta_{RG}(A)) = Ann_R([A,G])$ and $\dim_F([A,G]) \leq d$.

Proof: Put $C = \zeta_{RG}(A)$. Without loss of generality we can suppose that $C_G(A) = \langle 1 \rangle$. The equality $Ann_R(A/\zeta_{RG}(A)) = Ann_R([A,G])$ follows from Lemma 7. Let U be a normal abelian subgroup of G having finite index. Since $\dim_F(A/C)$ is finite, A/C includes a non-zero FU-submodule B/C having the least dimension. Then B/C is a simple FU-submodule and there exist the elements $g_1, \dots, g_s \in G$ such that $A/C = \bigoplus_{1 \leq j \leq s}(B/C)g_j$ (see, for example, [8, Lemma 5.4]). Suppose that $C_U(B/C) = U$. The equation $C_U((B/C)g) = (C_U(B/C))^g$ implies that $C_U((B/C)g) = U^g = U$ for each element $g \in G$. But in this case $C_U(A/C) = U$, so that $U \leq C_G(A/C)$ and the factor-group $G/C_G(A/C)$ is finite, and we obtain a contradiction. This contradiction shows that $C_U(A/C) \neq U$. In this case $C_U((B/C)g) \neq U$ for each element $g \in G$. It follows that A/C is U-hypereccentric. Then the RU-module A has the Z-decomposition $A = \zeta_{RU,\infty}(A) \times E$ where E is an RU-submodule, which is U-hypereccentric [7, Corollary 2.6]. We remark at once that $\zeta_{RU,\infty}(A) = C$. Suppose that E does not include $Ex = B$ for some element $x \in G$. Since U is normal in G, it is not hard to see that B is also U-hypereccentric. Then $(B+E)/E$ is non-zero, therefore it includes a non-zero simple RU-submodule D/E. The isomorphism $(B+E)/E \cong_{RU} B/(B \cap E)$ shows that D/E is RU-isomorphic to some simple RU-factor of B, and it follows that $U/C_U(D/E) \neq U$. On the other hand, $(B+E)/E \leq A/E \leq \zeta_{RU}(A)$, which gives that $U/C_U(D/E) = U$. This contradiction shows that $Eg \leq E$ for every element $g \in G$. Hence E is an RG-submodule. Since $A/E = (C+E)/E$, $G = C_G(A/E)$, which implies an inclusion $[A,G] \leq E$. Then $\dim_F([A,G]) \leq \dim_F(E) = \dim_F(A/C) = d$. □

Corollary 4. *Let R be an integral domain, G be a group and A be an RG-module. Suppose that $A/\zeta_{RG}(A)$ is a simple RG-module, $G \neq C_G(A/\zeta_{RG}(A))$ and $\dim_F(A/\zeta_{RG}(A)) = d$ is finite, where $F = R/Ann_R(A/\zeta_{RG}(A))$. If a group G has finite section p-rank r where $p = char(F)$ and the factor-group*

$$G/C_G(A/\zeta_{RG}(A))$$

is infinite, then $Ann_R(A/\zeta_{RG}(A)) = Ann_R([A,G])$ and $\dim_F([A,G]) \leq d(r+1)$.

Proof: Put $C = \zeta_{RG}(A)$. Without loss of generality we can suppose that $C_G(A) = \langle 1\rangle$. The equality $Ann_R(A/\zeta_{RG}(A)) = Ann_R([A,G])$ follows from Lemma 7. Lemmas 4 and 5 show that $G/C_G(A/C)$ is abelian-by-finite. Put $Z = C_G(A/C)$. By Corollary 1 $\dim_F([A,Z]) \le rd$. Since Z is a normal subgroup of G, $K = [A,Z]$ is an RG-submodule. Furthermore, $Z \le C_G(A/K)$, so that a factor-group $G/C_G(A/K)$ is abelian-by-finite. Put $L/K = [A/K, G]$. Since $G/C_G(A/K)$ is infinite, Lemma 10 shows that $\dim_F(L/K) \le d$. An obvious inclusion $[A,G] \le L$ shows that

$$\begin{aligned}\dim_F([A,G]) &\le \dim_F(L) = \dim_F(L/K) + \dim_F(K)\\ &\le d + rd = d(r+1).\end{aligned}$$ □

Proposition 1. *Let R be an integral domain, G be a group and A be an RG-module. Suppose that $A/\zeta_{RG}(A)$ is a simple RG-module, and $\dim_F(A/\zeta_{RG}(A)) = d$ is finite, where $F = R/Ann_R(A/\zeta_{RG}(A))$. If a group G has finite section p-rank r where $p = char(F)$, then $Ann_R(A/\zeta_{RG}(A)) = Ann_R([A,G])$ and there exists a function κ_5 such that $\dim_F([A,G]) \le \kappa_5(r,d)$.*

Proof: Put $C = \zeta_{RG}(A)$. The equality $Ann_R(A/\zeta_{RG}(A)) = Ann_R([A,G])$ follows from Lemma 7. If $G = C_G(A/C)$, then Lemma 8 shows that $\dim_F([A,G]) \le r$. Hence in this case we put $\kappa_5(r,d) = r$. Suppose now that the factor A/C is G-eccentric. If $G/C_G(A/C)$ is finite and $p > 0$, then Corollary 3 shows that $\dim_F([A,G]) \le \kappa_3(r,d)$. In this case we put $\kappa_5(r,d) = \kappa_3(r,d)$. If $G/C_G(A/C)$ is finite and $p = 0$, then Corollary 3 shows that $\dim_F([A,G]) \le \kappa_4(r,d)$. In this case we put $\kappa_5(r,d) = \kappa_4(r,d)$. Finally, suppose now that $G/C_G(A/C)$ is infinite. Here Corollary 4 shows that $\dim_F([A,G]) \le d(r+1)$, so that for this case we can put $\kappa_5(r,d) = d(r+1)$. □

4. Proof of the main theorem

Put $C = \zeta_{RG}(A)$. Since A/C has finite R-composition series, A has a series of RG-submodules

$$\langle 0\rangle = C_0 \le C_1 = \zeta_{RG}(A) \le C_2 \le \ldots \le C_{n+1} = A$$

such that the factors C_j/C_{j-1} are simple RG-modules, $c_R(C_j/C_{j-1})$ is finite, $2 \le j \le n+1$. We will use induction by $n+1$. If A/C is simple RG-module, the statement follows from Proposition 1. Suppose now that $n > 1$ and consider an RG-submodule $B = C_n$. For this submodule we can use induction hypothesis and obtain that $D = [B,G]$ has a finite R-composition series, $Spec(D) \subseteq Spec(B/\zeta_{RG}(B)) \subseteq Spec(A/\zeta_{RG}(A))$ and there exists a function λ such that

$$c_R(D) \le \lambda(r_p, d \mid p \in Spec_R(B/\zeta_{RG}(B)), d \in Sdim(B/\zeta_{RG}(B))).$$

For the factor-module A/D we have that $B/D \leq \zeta_{RG}(A/D)$, so that $A/D = \zeta_{RG}(A/D)$ or $(A/D)/\zeta_{RG}(A/D)$ is a simple RG-module. Now we can apply Proposition 1.

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