The Rothberger property on $C_p(\Psi(\mathcal{A}), 2)$

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Abstract. A space X is said to have the *Rothberger property* (or simply X is *Rothberger*) if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X, there exists $U_n \in \mathcal{U}_n$ for each $n \in \omega$ such that $X = \bigcup_{n\in\omega} U_n$. For any $n \in \omega$, necessary and sufficient conditions are obtained for $C_p(\Psi(\mathcal{A}), 2)^n$ to have the Rothberger property when $\mathcal{A}$ is a Mrówka mad family and, assuming CH (the Continuum Hypothesis), we prove the existence of a maximal almost disjoint family $\mathcal{A}$ for which the space $C_p(\Psi(\mathcal{A}), 2)^n$ is Rothberger for all $n \in \omega$.

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1. Introduction

There are two classical combinatorial strengthenings of Lindelöfness, namely the Menger and Rothberger properties.

Definition 1.1 ([7])**.** A space X is said to have the *Rothberger property* (or simply X is *Rothberger*) if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X, there exists $U_n \in \mathcal{U}_n$ for each $n \in \omega$ such that $X = \bigcup_{n\in\omega} U_n$.

Definition 1.2 ([4])**.** A space X is said to have the *Menger property* (or simply X is *Menger*) if for every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X, there exists a sequence $\langle \mathcal{F}_n : n \in \omega \rangle$ of finite sets such that $\bigcup_{n\in\omega} \mathcal{F}_n$ is a cover of X and $\mathcal{F}_n \subset \mathcal{U}_n$ for each $n \in \omega$.

These two properties were introduced in studies of strong measure zero and σ-compact metric spaces, respectively. Obviously every Rothberger space has the Menger property.

The author and Á. Tamariz-Mascarúa prove the following in [3, Theorem 8.7].

Theorem 1.3 (CH)**.** *There is a mad family $\mathcal{A}$ such that $C_p(\Psi(\mathcal{A}), 2)$ is Menger.*

This paper is motivated by Problem 8.8 in [3] which asks the following:

Problem 1.4 ([3])**.** Let $\mathcal{A}$ be the mad family from Theorem 1.3. Is $C_p(\Psi(\mathcal{A}), 2)^n$ Menger for every $n \geq 2$?

In this article, we will give a characterization for $C_p(\Psi(\mathcal{A}), 2)^n$ to have the Rothberger property for any $n \in \omega$. Finally, we answer positively Problem 1.4.

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2. Notation and preliminaries

For spaces X and Y, $C_p(X,Y)$ is the subspace of Y^X consisting of the continuous functions from X to Y (i.e., $C(X,Y)$ with the topology of the pointwise convergence). As usual, ω is the discrete space of all non-negative integers and, for each $n \in \omega$, n denotes the subspace of ω consisting of all integers strictly less than n. For any space X and every n-valued continuous function $f : X \to n$, $\mathrm{supp}(f)$ denotes the set $\{x \in X : f(x) \neq 0\}$. The following three basic properties about Rothberger spaces will be useful.

Proposition 2.1 ([5]).

(a) *Every closed subspace of a Rothberger space is Rothberger.*
(b) *The continuous image of a Rothberger space is Rothberger.*
(c) *The countable union of Rothberger spaces is Rothberger.*

3. The Rothberger property on $C_p(\Psi(\mathcal{A}), 2)$

An *almost disjoint family* of subsets of ω is an infinite collection $\mathcal{A}$ of subsets of ω such that each element in $\mathcal{A}$ is infinite, and if $A, B \in \mathcal{A}$ are different, $|A \cap B| < \omega$. An almost disjoint family $\mathcal{A}$ is maximal if it is not a proper subfamily of another almost disjoint family.

For a maximal almost disjoint family (mad) $\mathcal{A}$ on ω, $\Psi(\mathcal{A})$ is the space whose underlying set is $\omega \cup \mathcal{A}$ and its topology is given by the following: All points of ω are isolated, and a neighborhood base at $A \in \mathcal{A}$ consists of all sets $\{A\} \cup A \setminus F$ where F is a finite subset of ω.

Definition 3.1. A mad family $\mathcal{A}$ is *Mrówka* if the Stone-Čech compactification $\beta\Psi(\mathcal{A})$ of $\Psi(\mathcal{A})$ coincides with the one-point compactification of $\Psi(\mathcal{A})$.

For a mad family $\mathcal{A}$, $n \in \omega$ and $j \in n$, we define the subspace

$$^n\sigma^j_m(\mathcal{A}) = \{f \in C_p(\Psi(\mathcal{A}), n) : \forall i \in n\ (i \neq j \to |f^{-1}(i) \cap \mathcal{A}| \leq m)\}$$

of $C_p(\Psi(\mathcal{A}), n)$. It is not hard to see that this subspace is closed.

If $\mathcal{A}$ is a Mrówka mad family, then

$$C_p(\Psi(\mathcal{A}), n) = \bigcup_{m \in \omega, j \in n} {}^n\sigma^j_m(\mathcal{A}).$$

For every $m \in \omega$ and $i, j \in n$, $^n\sigma^i_m(\mathcal{A})$ is homeomorphic to $^n\sigma^j_m(\mathcal{A})$. We are going to write $^n\sigma_m(\mathcal{A})$ instead of $^n\sigma^0_m(\mathcal{A})$. Thus, by Proposition 2.1(a) and 2.1(c):

Lemma 3.2. *Let $\mathcal{A}$ be a Mrówka mad family. Then $C_p(\Psi(\mathcal{A}), n)$ is Rothberger if and only if $^n\sigma_m(\mathcal{A})$ is Rothberger for each $m \in \omega$.*

For each $n \in \omega$, we define

$$\mathrm{Q}(n) = \{g \in n^\omega : |\mathrm{supp}(g)| < \omega\}.$$

With this terminology we introduce the following property, which is a generalization when a mad family concentrates on $[\omega]^{<\omega}$ (see [6], the original definition is equivalent to the case $\bigstar^2_m(\mathcal{A})$). For a mad family $\mathcal{A}$ and $m,n\in\omega$, we define

$\bigstar^n_m(\mathcal{A})$: For each open subset U of n^ω containing $\mathrm{Q}(n)$, there exists a countable subset $\mathcal{B}\subset\mathcal{A}$ such that $\{g\in n^\omega : \exists\,\widehat{g}\in C_p(\Psi(\mathcal{A}),n)(\widehat{g}\restriction\omega=g\wedge \operatorname{supp}(\widehat{g})\cap\mathcal{A}\in[\mathcal{A}\setminus\mathcal{B}]^m)\}\subset U$.

The following generalized version of Theorem 4.2 in [6] holds:

Lemma 3.3. *Let $\mathcal{A}$ be a mad family and let $n,m\in\omega$. If ${}^n\sigma_m(\mathcal{A})$ is Lindelöf, then the property $\bigstar^n_k(\mathcal{A})$ is satisfied for all $k\le m$.*

Proof: Suppose that the property $\bigstar^n_k(\mathcal{A})$ is false for some $k\le m$. So, we may fix an open set U in n^ω, a pairwise disjoint family $\{y_\alpha:\alpha\in\omega_1\}\subset[\mathcal{A}]^k$ and $\{g_\alpha:\alpha\in\omega_1\}\subset C_p(\Psi(\mathcal{A}),n)$ such that

(i) $\mathrm{Q}(n)\subset U$, and
(ii) for each $\alpha\in\omega_1$, $\operatorname{supp}(g_\alpha)\cap\mathcal{A}=y_\alpha$ and $g_\alpha\restriction\omega\notin U$.

Since $\{y_\alpha:\alpha\in\omega_1\}$ are pairwise disjoint, any complete accumulation point of $\{g_\alpha:\alpha\in\omega_1\}$ must be in ${}^n\sigma_0$. Moreover, since U contains $\mathrm{Q}(n)$, there is an open subset V in ${}^n\sigma_m(\mathcal{A})$ containing ${}^n\sigma_0$ such that $f\restriction\omega\in U$ for each $f\in V$. Indeed, we can fix a set $\mathcal{F}$ consisting of finite functions such that $U=\{g\in n^\omega:\exists\, s\in\mathcal{F}(s\subset g)\}$, then $V=\{f\in{}^n\sigma_m(\mathcal{A}):\exists\, s\in\mathcal{F}(s\subset f)\}$ is the required open set.

Thus, the open set V contains any complete accumulation point of $\{g_\alpha:\alpha\in\omega_1\}$ and, by (ii), $g_\alpha\notin V$ for each $\alpha\in\omega_1$. This means that the uncountable set $\{g_\alpha:\alpha\in\omega_1\}$ has no complete accumulation points in ${}^n\sigma_m(\mathcal{A})$, which is a contradiction. $\square$

We need the following terminology for proof of the next lemma. For each $n\in\omega$ and each $t\in\omega^n$ we define

$${}^n\sigma_t(\mathcal{A})=\{f\in C_p(\Psi(\mathcal{A}),n):\forall i\in n\ (i\ne 0\to|f^{-1}(i)\cap\mathcal{A}|\le t(i))\}.$$

The order $\preceq$ will denote the lexicographic order on ω^n. Observe that if $m\in\omega$ and $t\in\omega^n$ is the constant function m, then ${}^n\sigma_m(\mathcal{A})={}^n\sigma_t(\mathcal{A})$.

Lemma 3.4. *Let $\mathcal{A}$ be a mad family, $n\in\omega$, $t_0\in\omega^n$ and $p=\sum_{i=1}^{n-1}t_0(i)$. If $\bigstar^n_p(\mathcal{A})$ is satisfied and ${}^n\sigma_t(\mathcal{A})$ is Rothberger for every $t\prec t_0$, then ${}^n\sigma_{t_0}(\mathcal{A})$ is Rothberger.*

Proof: We adapt, for our purposes, the respective part of the proof of Lemma 8.2 from [3]. The proof depends on two claims.

Claim 1. If V is an open subset of ${}^n\sigma_{t_0}(\mathcal{A})$ containing ${}^n\sigma_t(\mathcal{A})$ for each $t\prec t_0$, then there is a countable subset $\mathcal{B}\subset\mathcal{A}$ such that for any $f\in{}^n\sigma_{t_0}(\mathcal{A})\setminus V$, there is $1\le i<n$ with $f^{-1}(i)\cap\mathcal{B}\ne\emptyset$.

Indeed, since ${}^n\sigma_0(\mathcal{A})$ is a countable subset of ${}^n\sigma_{t_0}(\mathcal{A})$, we can choose a sequence of finite functions $s_k \subset \Psi(\mathcal{A}) \times n$ such that ${}^n\sigma_0(\mathcal{A}) \cap [s_k] \neq \emptyset$ and ${}^n\sigma_0(\mathcal{A}) \subset \bigcup_{k\in\omega}[s_k] \subset V$, where $[s_k] = \{f \in {}^n\sigma_{t_0}(\mathcal{A}) : s_k \subset f\}$ for each $k \in \omega$. Note that $s_k^{-1}(i) \subset \omega$ for each $1 \leq i < n$ and, thus, $s_k \restriction \mathcal{A}$ is the constant zero function for each $k \in \omega$. We define the open subset U of n^ω to be $\bigcup_{k\in\omega}\{f \in n^\omega : s_k \restriction \omega \subset f\}$ and note that $\mathrm{Q}(n) \subset U$. Let $\mathcal{B}'$ be a countable subset of $\mathcal{A}$ given by $\bigstar_p^n(\mathcal{A})$. Let $\mathcal{B} = \mathcal{B}' \cup \bigcup_{k\in\omega}(s_k^{-1}(0) \cap \mathcal{A})$ and let us show that $\mathcal{B}$ is the required set in Claim 1. Let $f \in {}^n\sigma_{t_0}(\mathcal{A}) \setminus V$ and $x = \mathrm{supp}(f) \cap \mathcal{A}$. Since V contains ${}^n\sigma_t(\mathcal{A})$ for each $t \prec t_0$, $|x| = p$. Now, we proceed by contradiction supposing that $x \cap \mathcal{B} = \emptyset$. Then $\mathrm{supp}(f) \cap \mathcal{A} \in [\mathcal{A} \setminus \mathcal{B}]^p$. By the choice of $\mathcal{B}$, $f \restriction \omega \in U$ and consequently, there is $k \in \omega$ such that $s_k \restriction \omega \subset f \restriction \omega$ and, since $x \cap s_k^{-1}(0) = \emptyset$ and $s_k \restriction \mathcal{A}$ is the constant zero, $s_k \subset f$. Thus $f \in V$, which is impossible, and Claim 1 is proved.

Claim 2. If V is an open subset of ${}^n\sigma_{t_0}(\mathcal{A})$ containing ${}^n\sigma_t(\mathcal{A})$ for each $t \prec t_0$, then there is a countable set $Y \subset C_p(\Psi(\mathcal{A}), n)$ such that ${}^n\sigma_{t_0}(\mathcal{A}) \setminus V \subset \bigcup_{h\in Y, t\prec t_0}(h + {}^n\sigma_t(\mathcal{A}))$, where $h + {}^n\sigma_t(\mathcal{A}) = \{h + g : g \in {}^n\sigma_t(\mathcal{A})\}$ and addition is taken mod n.

Let $\mathcal{B}$ be the countable subset of $\mathcal{A}$ given by Claim 1. Fix $1 \leq j < n$ and let $r_j(i)$ be 1 if $i = j$ and 0 otherwise. Define $Y = \bigcup_{j=1}^{n-1}\{f \in {}^n\sigma_{r_j}(\mathcal{A}) : f^{-1}(j) \cap \mathcal{A} \subset \mathcal{B}\}$. It is not difficult to show that Y is countable.

Let $f \in {}^n\sigma_{t_0}(\mathcal{A}) \setminus V$. By the choice of $\mathcal{B}$, there is $1 \leq i < n$ and an element $a \in f^{-1}(i) \cap \mathcal{B}$. We define a continuous function $g : \Psi(\mathcal{A}) \to n$ as follows

$$g(x) = \begin{cases} n - i, & \text{if } \ x \in a \cup \{a\}; \\ 0, & \text{otherwise.} \end{cases}$$

If $t_1 \in \omega^n$ is defined as $t_1(l) = t_0(l)$ if $l \neq i$ and $t_1(i) = t_0(i) - 1$, we obtain that $f + g \in {}^n\sigma_{t_1}(\mathcal{A})$ and $t_1 \prec t_0$. Let $h \in C_p(\Psi(\mathcal{A}), n)$ be the additive inverse function of g. Observe that $h \in Y$. Consequently, $f = h + (f + g) \in \bigcup_{h\in Y, t\prec t_0}(h + {}^n\sigma_t(\mathcal{A}))$. This concludes the proof of Claim 2.

Now, we are going to finish the proof of our lemma. Let $\langle \mathcal{U}_k : k \in \omega \rangle$ be a sequence of covers of ${}^n\sigma_{t_0}(\mathcal{A})$ and $\{P_t : t \preceq t_0\}$ a partition of ω into infinite sets. Since for each $t \prec t_0$, ${}^n\sigma_t(\mathcal{A})$ is Rothberger, there is, for each $k \in P_t$, $U_k \in \mathcal{U}_k$ such that ${}^n\sigma_t(\mathcal{A}) \subset \bigcup_{k\in P_t} U_k = V_t$. Then, by Claim 2, there is a countable set Y such that ${}^n\sigma_{t_0}(\mathcal{A}) \setminus \bigcup_{t\prec t_0} V_t \subset \bigcup_{h\in Y, t\prec t_0}(h + {}^n\sigma_t(\mathcal{A}))$. Since ${}^n\sigma_t(\mathcal{A})$ is homeomorphic to $h + {}^n\sigma_t(\mathcal{A})$ for each $h \in Y$ and Y is countable, $\bigcup_{h\in Y, t\prec t_0}(h + {}^n\sigma_t(\mathcal{A}))$ is Rothberger (see Proposition 2.1(c)). Then, there is $U_k \in \mathcal{U}_k$ for each $k \in P_{t_0}$ such that $\bigcup_{k\in P_{t_0}} U_k$ covers ${}^n\sigma_{t_0}(\mathcal{A}) \setminus \bigcup_{t\prec t_0} V_t$. Therefore, the sequence $\{U_k : k \in \omega\}$ is the required choice. $\square$

Theorem 3.5. *Let $\mathcal{A}$ be a Mrówka mad family and $n \in \omega$. Then, the following statements are equivalent.*

(a) $C_p(\Psi(\mathcal{A}), 2)^n$ *is Lindelöf.*
(b) $C_p(\Psi(\mathcal{A}), 2)^n$ *is Menger.*
(c) $C_p(\Psi(\mathcal{A}), 2)^n$ *is Rothberger.*

(d) *The property* $\bigstar_m^{2^n}(\mathcal{A})$ *is satisfied for all* $m \in \omega$.

Proof: First observe that $C_p(\Psi(\mathcal{A}), 2)^n$ is homeomorphic to $C_p(\Psi(\mathcal{A}), 2^n)$. The implication (d) $\to$ (c) is proved as follows. By Lemma 3.2 it is sufficient to show that ${}^{2^n}\sigma_m(\mathcal{A})$ is Rothberger for each $m \in \omega$. Indeed, fix $m \in \omega$ and $t_m \in \omega^{2^n}$ to be the constant function m. Since ${}^{2^n}\sigma_0$ is countable, this is Rothberger, and if we suppose that ${}^{2^n}\sigma_t$ is Rothberger for each $t \prec t_0$ for some $t_0 \preceq t_m$, by hypothesis and Lemma 3.4, ${}^{2^n}\sigma_{t_0}(\mathcal{A})$ is Rothberger. By induction, ${}^{2^n}\sigma_{t_m}(\mathcal{A}) = {}^{2^n}\sigma_m(\mathcal{A})$ is Rothberger.

The implications (c) $\to$ (b) and (b) $\to$ (a) are clear. Finally, if $C_p(\Psi(\mathcal{A}), 2^n)$ is Lindelöf, the closed subspace ${}^{2^n}\sigma_m(\mathcal{A})$ of $C_p(\Psi(\mathcal{A}), 2^n)$ is Lindelöf for each $m \in \omega$ and, by Lemma 3.3, $\bigstar_m^{2^n}(\mathcal{A})$ is satisfied for each $m \in \omega$. This proves that (a) $\to$ (d). $\square$

As was shown in [6], every finite power of $C_p(\Psi(\mathcal{A}), 2)$ is Lindelöf, where $\mathcal{A}$ is the family constructed in Theorem 1.3. Theorem 3.5 then gives a positive answer to Problem 1.4:

Theorem 3.6 (CH)**.** *There is a Mrówka mad family* $\mathcal{A}$ *such that* $C_p(\Psi(\mathcal{A}), 2)^n$ *is Rothberger for each* $n \in \omega$.

A space X is *ω-monolithic* if $nw(\mathrm{cl}(A)) \leq \omega$ for any $A \subset X$ with $|A| \leq \omega$. In [2] it is proved that if $C_p(X, 2)$ is Lindelöf for a countably compact ω-monolithic X then it is Rothberger. E.A. Reznichenko showed that assuming MA+$\neg$CH, every compact zero-dimensional space X with $C_p(X, \mathbb{R})$ Lindelöf is ω-monolithic (see [1, IV.8.6, IV.8.16]). This leads to the conjecture that, perhaps, strong covering properties of a suitable $C_p(X, Y)$ might imply ω-monolithicity of X. One might, for example, ask whether Reznichenko's result can be generalized.

Question 3.7. Assume X is a zero-dimensional compact space and that $C_p(X, 2)^n$ is Rothberger for every $n \in \omega$. Does this imply that X is ω-monolithic?

The following theorem gives a consistent counterexample.

Theorem 3.8 (CH)**.** *There is a Mrówka mad family* $\mathcal{A}$ *such that* $C_p(\beta\Psi(\mathcal{A}), 2)^n$ *is Rothberger for every* $n \in \omega$.

Proof: It is sufficient to observe that the function

$$\phi_m^{2^n} : {}^{2^n}\sigma_m(\mathcal{A}) \to \{g \in C_p(\beta\Psi(\mathcal{A}), 2^n) : \forall i \in 2^n (i \neq 0 \to |g^{-1}(i) \cap \mathcal{A}| \leq m)\}$$

defined by $\phi_m^{2^n}(f) = \widetilde{f}$ is an onto continuous function where $\widetilde{f}$ is the continuous extension of $f : \Psi(\mathcal{A}) \to 2^n$ to $\beta\Psi(\mathcal{A})$. $\square$

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