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# Separable $\aleph_k$-free modules with almost trivial dual

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*Abstract.* An $R$-module $M$ has an almost trivial dual if there are no epimorphisms from $M$ to the free $R$-module of countable infinite rank $R^{(\omega)}$. For every natural number $k > 1$, we construct arbitrarily large separable $\aleph_k$-free $R$-modules with almost trivial dual by means of Shelah's Easy Black Box, which is a combinatorial principle provable in ZFC.

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## 1. Introduction

Almost-free algebraic structures have caught the attention of researchers for many decades. The first known example of such structures, dating back to 1937, is the Baer-Specker group $\mathbb{Z}^\omega$, a non-free $\aleph_1$-free abelian group of cardinality $2^{\aleph_0}$. Other examples from the 1970's are a construction by Griffith [10] of some non-free $\aleph_n$-free abelian groups for all $n < \omega$, and a construction by Hill [12], where he managed to construct such groups of minimal size $\aleph_n$. However, no further algebraic properties of these groups were shown.

During the 1980's, almost-free groups also appeared in the context of constructing groups and modules with prescribed endomorphism ring. For example, Corner, Göbel [1] and Dugas, Göbel [2] used Shelah's Black Box to construct in ZFC torsion-free abelian groups with prescribed endomorphism ring and noticed that these groups were non-free $\aleph_1$-free as well. However, it was clear that these groups were not $\aleph_2$-free, and the question on how to generalize these constructions within ZFC to obtain $\aleph_n$-freeness for $n > 1$ while keeping such additional algebraic properties remained unanswered for many more years. Only in 2007 Shelah [15] introduced a new, more powerful version of his Black Box principle, which allowed him to construct $\aleph_n$-free abelian groups with trivial dual. This breakthrough immediately led to other constructions of $\aleph_n$-free groups and modules with different algebraic properties like almost trivial dual or prescribed endomorphism ring (see for example [6], [7] and [8]).

In [7], the authors construct a class of $\aleph_k$-free $R$-modules $M$ for a fixed natural number $k > 1$, where $R$ is a countable domain, but not a field. Moreover, these modules have *trivial dual*, i.e. $\mathrm{Hom}_R(M, R) = 0$. Generally speaking, one starts

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considering a free $R$-module $B$ and its completion $\widehat{B}$ with respect to a countable multiplicatively closed subset $\mathbb{S}$ of $R$. An $\aleph_k$-free $R$-module $M$ with trivial dual is then realized as the $\mathbb{S}$-pure closure of the module generated by $B$ and a specific family $\mathfrak{F}$ of elements of $\widehat{B}$. By the choice of this family $\mathfrak{F}$, $M$ turns out to be contained in a direct product of copies of the $\mathbb{S}$-completion $\widehat{R}$ of $R$. In this paper we realize the construction of another class of $\aleph_k$-free $R$-modules contained in a direct product of copies of $R$, namely, these modules are $\mathbb{S}$-*separable*. Nevertheless, as in [8], where modules over the ring $J_p$ of $p$-adic integers are considered (and thus, they are separable), we cannot expect a module $M$ from this class to have trivial dual. Instead, $M$ will have *almost trivial dual*, meaning that the free module of countable rank $R^{(\omega)}$ is not an epimorphic image of $M$. This approach refines and significantly enhances the expositions of [7] and [8].

The main result in this work is, for a countable commutative ring $R$ (but not a field), the construction of arbitrarily large $\mathbb{S}$-separable $\aleph_k$-free $R$-modules with no epimorphisms onto $R^{(\omega)}$. So we elaborate a plan to achieve three goals, namely, separability, $\aleph_k$-freeness and eliminating any possible epimorphisms onto $R^{(\omega)}$. We dedicate one section to each one of these goals. In Section 2, the basic notation and notions are given. In Section 3 we construct an $R$-module $M$ which is (only) $\mathbb{S}$-separable. This module is realized as the $\mathbb{S}$-pure closure of the module generated by a free module $B$ and a specific family $\mathfrak{F}$ of elements of a very natural $\mathbb{S}$-pure submodule $\overline{B}$ of its $\mathbb{S}$-completion $\widehat{B}$. In Section 4, we introduce a so called Freeness Proposition and choose the family $\mathfrak{F}$ more carefully towards making our previously constructed module $M$ $\aleph_k$-free. Finally, in Section 5, we introduce Shelah's Easy Black Box from [15], which we use to refine our choice of the family $\mathfrak{F}$ even more in order to eliminate unwanted epimorphisms onto $R^{(\omega)}$. The Easy Black Box 17, the First $\overline{\lambda}$-Black Box 19 and the Second $\overline{\lambda}$-Black Box 21 and their proofs appeared in [7] with other names, but they are also included here for the convenience of the reader.

This work generalizes and is motivated by the first chapter of the second author's Ph.D. thesis. The considerably more technical challenge of realizing endomorphism rings of separable modules is treated in [5]. Our notation matches standard literature and we refer to [4], [9] as background material on the theory of separable modules.

## 2. Notation and basic notions

Let us begin by introducing some notation.

**Notation 1.** (i) Functions will be written on the right side of their argument, so if $f$ is a function with domain $A$ and $a \in A$, then the image of $a$ under $f$ will be written as $af$.

(ii) ${}^{\omega}\lambda$ denotes the set of all functions $\tau : \omega \to \lambda$, while ${}^{\omega\uparrow}\lambda$ is the subset of ${}^{\omega}\lambda$ consisting of all *order preserving* functions $\eta : \omega \to \lambda$, namely

$$^{\omega\uparrow}\lambda = \{\, \eta : \omega \to \lambda \mid m\eta < n\eta \;\text{ for }\; m < n \,\}.$$

Similarly, ${}^{\omega>}\lambda$ denotes the set of all functions $\sigma : n \to \lambda$ with $n < \omega$, while ${}^{\omega\uparrow>}\lambda$ is the subset of ${}^{\omega>}\lambda$ consisting of all *order preserving* functions $\eta : n \to \lambda$ with $n < \omega$.

(iii) If $f : A \to {}^{B}C$, i.e. $af$ is a function for all $a \in A$, then we write $f_a$ instead of $af$.

(iv) If $A$ is a set and $\kappa$ is a cardinal, then $[\,A\,]^{\leq\kappa}$ denotes the set of all $X \subseteq A$ such that $|\,X\,| \leq \kappa$. Analogously we define $[\,A\,]^{<\kappa}$ and $[\,A\,]^{\kappa} = [\,A\,]^{=\kappa}$.

(v) If $\alpha \leq \gamma$ are ordinals, we write $[\,\alpha, \gamma\,] = \{\,\beta \mid \alpha \leq \beta \leq \gamma\,\}$, $(\,\alpha, \gamma\,) = \{\,\beta \mid \alpha < \beta < \gamma\,\}$ and $[\,\alpha, \gamma\,) = \{\,\beta \mid \alpha \leq \beta < \gamma\,\}$.

(vi) Let $\{\,A_i \mid i \in [\,1, m\,]\,\}$ and $\{\,B_i \mid i \in [\,1, n\,]\,\}$ be finite families of sets, $A = A_1 \times \cdots \times A_m$ and $B = B_1 \times \cdots \times B_n$. If $a = (a_1, \ldots, a_m) \in A$ and $b = (b_1, \ldots, b_n) \in B$, we write $a \wedge b = (a_1, \ldots, a_m, b_1, \ldots, b_n)$.

(vii) If $\kappa$ is a cardinal, $\kappa^+$ denotes the successor of $\kappa$, which is the smallest cardinal larger than $\kappa$.

Let $k > 1$ be fixed for the rest of this work. Let $\overline{\lambda} = \langle\,\lambda_1, \ldots, \lambda_k\,\rangle$ be a finite increasing sequence of infinite regular cardinals with the following properties:

(i) $\lambda_1^{\aleph_0} = \lambda_1$;

(ii) for all $f \in [\,1, k-1\,]$, $\lambda_{f+1}^{\lambda_f} = \lambda_{f+1}$.

In particular, for all $f \in [\,1, k\,]$, $\aleph_0 < \lambda_f$ and $\lambda_f^{\aleph_0} = \lambda_f$. For example, one such sequence can be constructed recursively by putting $\lambda_1 = \left(2^{\aleph_0}\right)^+$ and $\lambda_{f+1} = \left(2^{\lambda_f}\right)^+$ for all $f \in [\,1, k-1\,]$.

For $f \in [\,1, k\,]$, we construct the set

$$\Lambda^f = {}^{\omega\uparrow}\lambda_1 \times {}^{\omega\uparrow}\lambda_2 \times \cdots \times {}^{\omega\uparrow}\lambda_{f-1} \times {}^{\omega\uparrow}\lambda_f.$$

In case $f = k$, we simply write $\Lambda$.

Now let $m \in [\,1, f\,]$. We define the set $\Lambda^f_{m*}$, which is obtained by replacing the $m$-th coordinate ${}^{\omega\uparrow}\lambda_m$ of $\Lambda^f$ by the set ${}^{\omega\uparrow>}\lambda_m$, namely,

$$\Lambda^f_{m*} = {}^{\omega\uparrow}\lambda_1 \times \cdots \times {}^{\omega\uparrow>}\lambda_m \times \cdots \times {}^{\omega\uparrow}\lambda_f.$$

Then, let

$$\Lambda^f_* = \dot{\bigcup_{m\in[\,1,k\,]}} \Lambda^f_{m*}.$$

For $f = k$ we simply write $\Lambda_*$ and $\Lambda_{m*}$ for all $m \in [\,1, k\,]$.

**Definition 2.** Let $f \in [\,1, k\,]$ and $m \in [\,1, f\,]$.

(i) If $\eta \in {}^{\omega\uparrow}\lambda_m$, then the *support* of $\eta$ is the set

$$[\,\eta\,] = \{\,\eta \restriction n \mid n < \omega\,\}.$$

(ii) If $\nu \in {}^{\omega\uparrow>}\lambda_m$, then the *support* of $\nu$ is the set

$$[\,\nu\,] = \{\,\nu \restriction \ell \mid \ell \leq \operatorname{dom} \nu\,\}.$$

(iii) If $\overline{\eta} = (\eta_1, \ldots, \eta_f) \in \Lambda^f$ and $n < \omega$, then $\overline{\eta} \upharpoonleft \langle m, n\rangle$ denotes the element of $\Lambda^f_{m*}$ obtained from $\overline{\eta}$ by replacing its component $\eta_m$ by $\eta_m \upharpoonright n$, i.e.

$$\overline{\eta} \upharpoonleft \langle m, n\rangle = (\eta_1, \ldots, \eta_{m-1},\, \eta_m \upharpoonright n,\, \eta_{m+1} \ldots, \eta_f).$$

(iv) For every $\overline{\eta} \in \Lambda^f$ and $m \in [\,1, f\,]$ consider the sets

$$[\,\overline{\eta} \upharpoonleft m\,] = \{\, \overline{\eta} \upharpoonleft \langle m, n\rangle \mid n < \omega \,\}$$

and

$$[\,\overline{\eta}\,] = \dot{\bigcup_{m \in [\,1, f\,]}} [\,\overline{\eta} \upharpoonleft m\,]$$

The set $[\,\overline{\eta}\,]$ is called the *support* of $\overline{\eta}$.

Let $R$ be a countable commutative ring with 1. Given a finite sequence of infinite cardinals $\overline{\lambda}$ and a subset $X_* \subseteq \Lambda_*$, we consider the free $R$-module

$$B_{X_*} = \bigoplus_{\overline{\nu} \in X_*} Re_{\overline{\nu}}.$$

If $X_* = [\,\overline{\eta}\,]$ for some $\overline{\eta} \in \Lambda$, then we simply write $B_{\overline{\eta}}$. The starting point of our construction will be the free $R$-module

$$B = \bigoplus_{\overline{\nu} \in \Lambda_*} Re_{\overline{\nu}}.$$

**Definition 3.** Let $\mathbb{S}$ be a countable multiplicatively closed subset of $R$ such that $1 \in \mathbb{S}$, $0 \notin \mathbb{S}$. We say that an $R$-module $M$ is:

(i) $\mathbb{S}$-*torsion-free* if $sm = 0$ implies $m = 0$ for all $s \in \mathbb{S}$, $m \in M$;
(ii) $\mathbb{S}$-*reduced* if no element of $M$ is divisible by every element of $\mathbb{S}$, i.e. $\bigcap_{s\in\mathbb{S}} sM = 0$;
(iii) If $R$ itself is $\mathbb{S}$-torsion-free and $\mathbb{S}$-reduced, we call it $\mathbb{S}$-*ring*.

Choose a countable multiplicatively closed subset $\mathbb{S}$ of $R$ such that $R$ is an $\mathbb{S}$-ring. Fix an enumeration $\mathbb{S} = \{\, s_n \mid n < \omega \,\}$ such that $s_0 = 1$. For any $n < \omega$, define $q_n = \prod_{i \le n} s_i$. Notice that $q_{n+1} = q_n s_{n+1}$ for all $n < \omega$. Moreover, for all $m, n < \omega$, define

$$\frac{q_n}{q_m} = \prod_{m < i \le n} s_i$$

if $m < n$. Also notice that $\frac{q_n}{q_0} = q_n$ and $\frac{q_n}{q_n} = 1$ for all $n < \omega$.

**Definition 4.** Let $M$ be an $R$-module. An $R$-submodule $N$ of $M$ is $\mathbb{S}$-*pure* if for all $s \in \mathbb{S}$, $N \cap sM = sN$. We write $N \le_* M$.

Consider the $\mathbb{S}$-topology on $B$ generated by the basis $\{\, sB \mid s \in \mathbb{S} \,\}$ of neighborhoods around 0. This topology is Hausdorff since $R$ is an $\mathbb{S}$-ring. Let $\widehat{B}$ be the $\mathbb{S}$-completion of $B$. The elements of $\widehat{B}$ are of the form $b = \sum_{\overline{\nu}\in\Lambda_*} b_{\overline{\nu}} e_{\overline{\nu}} \in \widehat{B}$ for some $b_{\overline{\nu}} \in \widehat{R}$. See [9] for more elementary facts of $\widehat{B}$.

## 3. Separability

Now consider

$$\overline{B} = \widehat{B} \cap \prod_{\overline{\nu}\in\Lambda_*} Re_{\overline{\nu}},$$

which is an $\mathbb{S}$-pure submodule of $\widehat{B}$. Our intention is to choose the family $\mathfrak{F}$ from $\overline{B}$, so that the constructed modules are submodules of a direct product of copies of $R$.

**Definition 5.** Let $R$ be an $\mathbb{S}$-ring. Then $\mathbb{S}$-pure submodules of direct products $R^\kappa$ are called $\mathbb{S}$-*separable*.

**Definition 6.** (i) For $\overline{\eta} \in \Lambda$ and $n < \omega$, we define the *branch element* associated with $\overline{\eta}$ and $n$ as

$$y_{\overline{\eta}n} = \sum_{i\geq n} \frac{q_i}{q_n} \left( \sum_{m=1}^{k} e_{\overline{\eta}\restriction\langle m,i\rangle} \right) \in \overline{B}.$$

We write $y_{\overline{\eta}}$ for $y_{\overline{\eta}0}$.

(ii) Let $\mathfrak{F} = \{\, y_{\overline{\eta}} \mid \overline{\eta} \in \Lambda \,\}$ be the family of branch elements. Define $M$ to be the $\mathbb{S}$-purification of the module generated by $B$ and $\mathfrak{F}$, i.e.

$$M = \langle\, B, y_{\overline{\eta}n} \mid y_{\overline{\eta}} \in \mathfrak{F}, n < \omega \,\rangle = \langle\, B, \mathfrak{F} \,\rangle_* \,.$$

Since $M \leq_* \overline{B} \leq_* \prod_{\overline{\nu}\in\Lambda_*} Re_{\overline{\nu}}$, the $R$-module $M$ is $\mathbb{S}$-separable, so we have achieved the first goal of our plan.

## 4. $\aleph_k$-freeness

We say that a module $M$ over a hereditary ring is $\kappa$-*free*, if every subset of $M$ of cardinality $< \kappa$ is contained in a free submodule of $M$. For non-hereditary rings however, it is necessary to modify this notion. The following more general definition of $\kappa$-freeness is due to Göbel, Herden, Shelah [6], which is a slightly stronger version of that in Eklof, Mekler [3].

**Definition 7.** If $\kappa$ is a regular uncountable cardinal, we say that an $R$-module $M$ is $\kappa$-*free* if there is a family $\mathcal{C}$ of $\mathbb{S}$-pure $R$-submodules of $M$ satisfying:

(a) every element of $\mathcal{C}$ is $< \kappa$-generated and free;
(b) every element of $[\, M\,]^{<\kappa}$ (recall Notation 2(iv)) is contained in an element of $\mathcal{C}$;
(c) $\mathcal{C}$ is closed under unions of well-ordered chains of length $< \kappa$.

The first step to achieve $\aleph_k$-freeness is to prove a Freeness-Proposition, which allows us to enumerate subsets of $\Lambda$ in such a convenient way that we can prove linear independence in the constructed modules. We first have to deal with the notion of being coordinatewise-closed for filtrations, which will appear in the proof of the proposition.

**Lemma 8.** *Let $F : \Lambda \to [\,\Lambda_*\,]^{\leq \aleph_0}$ be any map, $f \in [\,1,k\,]$ and $\Omega \in [\,\Lambda\,]^{\aleph_f}$ together with an $\aleph_f$-filtration $\{\,\mho^\alpha \mid \alpha < \omega_f\,\}$ of $\Omega$ such that $\mho^0 = \emptyset$ and $\left|\,\mho^{\alpha+1} \setminus \mho^\alpha\,\right| = \aleph_{f-1}$ for all $\alpha < \omega_f$. Then it is possible to construct a coordinatewise-closed $\aleph_f$-filtration $\{\,\Omega^\alpha \mid \alpha < \omega_f\,\}$ of $\Omega$, meaning that for all $\overline{\eta} \in \Omega^{\alpha+1}$, if there exist $\overline{\eta}', \overline{\eta}'' \in \Omega^\alpha$ such that*

$$\{\,\eta_m \mid m \in [\,1,k\,]\,\} \subseteq \{\,\eta'_m, \eta''_m \mid m \in [\,1,k\,]\,\} \cup \{\,\nu_m \mid \overline{\nu} \in \overline{\eta}'F \cup \overline{\eta}''F, m \in [\,1,k\,]\,\},$$

*then $\overline{\eta} \in \Omega_\alpha$. This $\aleph_f$-filtration also satisfies $\Omega^0 = \emptyset$ and $\left|\,\Omega^{\alpha+1} \setminus \Omega^\alpha\,\right| = \aleph_{f-1}$ for all $\alpha < \omega_f$.*

Proof: First suppose that we have constructed a coordinatewise-closed $\aleph_f$-filtration $\{\,\Omega^\alpha \mid \alpha < \gamma\,\}$ of $\Omega$ up to some limit ordinal $\gamma < \omega_f$ such that $\Omega^0 = \emptyset$ and $\left|\,\Omega^{\alpha+1} \setminus \Omega^\alpha\,\right| = \aleph_{f-1}$ for all $\alpha + 1 < \gamma$ by means of the original filtration. We define $\Omega^\gamma = \bigcup_{\alpha<\gamma} \Omega^\alpha$ as usual.

Now suppose that the coordinatewise-closed $\aleph_f$-filtration $\{\,\Omega^\alpha \mid \alpha \leq \gamma\,\}$ of $\Omega$ has been constructed up to some successor ordinal $\gamma < \omega_f$ and also satisfies $\left|\,\Omega^{\alpha+1} \setminus \Omega^\alpha\,\right| = \aleph_{f-1}$ for all $\alpha + 1 \leq \gamma$. Let $\beta$ be the minimal ordinal such that $\Omega^\gamma \subseteq \mho^\beta$ and $\left|\,\mho^\beta \setminus \Omega^\gamma\,\right| = \aleph_{f-1}$. Let $\Omega_0^{\gamma+1} = \mho^\beta$ and assume we have constructed $\Omega_n^{\gamma+1}$ for some $n < \omega$. For all $m \in [\,1,k\,]$, let

$$\begin{gathered} {}^{\omega\uparrow}\lambda_m(\Omega_n^{\gamma+1}) = \Big\{\,\eta_\ell \mid \ell \in [\,1,k\,]\,,\, \overline{\eta} \in \Omega_n^{\gamma+1} \cup \bigcup \Omega_n^{\gamma+1}F\,\Big\} \cap {}^{\omega\uparrow}\lambda_m, \\ \Omega_{n+1}^{\gamma+1} = {}^{\omega\uparrow}\lambda_1(\Omega_n^{\gamma+1}) \times \cdots \times {}^{\omega\uparrow}\lambda_k(\Omega_n^{\gamma+1}) \subseteq \Lambda \end{gathered}$$

and

$$\Omega^{\gamma+1} = \Omega \cap \bigcup_{n<\omega} \Omega_n^{\gamma+1}.$$

□

**Definition 9.** (i) For $\eta \in {}^{\omega\uparrow>}\lambda_k \cup {}^{\omega\uparrow}\lambda_k$, we define the *norm* $\|\,\eta\,\|$ of $\eta$ as

$$\|\,\eta\,\| = \sup_{n<\operatorname{dom}\eta} (n\eta + 1) \in \lambda_k;$$

in particular, $\|\,\alpha\,\| = \alpha + 1$ for $\alpha \in \lambda_k$ and $\|\,\emptyset\,\| = 0$.

(ii) For $\overline{\eta} \in \Lambda \cup \Lambda_*$, define $\|\,\overline{\eta}\,\| = \|\,\eta_k\,\|$.

(iii) For $X \subseteq \Lambda$, put $\|\,X\,\| = \sup_{\overline{\eta}\in X} \|\,\overline{\eta}\,\|$. Similarly, $\|\,X\,\| = \sup_{\overline{\nu}\in X} \|\,\overline{\nu}\,\|$ if $X \subseteq \Lambda_*$.

(iv) A function $F : \Lambda \to [\,\Lambda_*\,]^{\leq\aleph_0}$ is *regressive* if $\|\,\overline{\eta}F\,\| < 0\eta_k$ for all $\overline{\eta} \in \Lambda$.

**Freeness-Proposition 10.** *Let $F : \Lambda \to [\Lambda_*]^{\leq \aleph_0}$ be a regressive map, $f \in [1,k]$, $\Omega \in [\Lambda]^{\aleph_{f-1}}$ and $\langle u_{\overline{\eta}} \mid \overline{\eta} \in \Omega \rangle$ be a family of subsets of $[1,k]$ such that $|u_{\overline{\eta}}| \geq f$. Then there exists a bijective enumeration $\{ \overline{\eta}^\alpha \mid \alpha < \zeta \}$ of $\Omega$ for some $\zeta \in [\omega_{f-1}, \omega_f)$ such that, for all $\alpha < \zeta$, there exist $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ and $n_\alpha \in [1, \omega)$ with the property that, for all $n \geq n_\alpha$,*

$$\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{ \overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta < \alpha \} \cup \bigcup \Omega_\alpha F$$

*where $\Omega_\alpha = \{ \overline{\eta}^\beta \mid \beta \leq \alpha \}$.*

Proof: We proceed by induction on $f$. If $f = 1$, then $|\Omega| = \aleph_0$ and $u_{\overline{\eta}} \neq \emptyset$ for all $\overline{\eta} \in \Omega$. For all $\alpha < \lambda_k$, define $U_\alpha = \{ \overline{\eta} \in \Omega \mid 0\eta_k = \alpha \}$. Consider the set $N = \{ \alpha < \lambda_k \mid U_\alpha \neq \emptyset \}$ and enumerate it as $N = \{ \alpha_\beta \mid \beta < \delta \}$ for some $\delta < \omega_1$ in such a way that $\alpha_\beta < \alpha_\gamma$ if and only if $\beta < \gamma < \delta$. Put $\gamma_\beta = \left| U_{\alpha_\beta} \right|$ and $\sigma_\beta = \sum_{\alpha < \beta} \gamma_\alpha$. We enumerate $U_{\alpha_\beta} = \{ \overline{\eta}^\alpha \mid \sigma_\beta \leq \alpha < \sigma_\beta + \gamma_\beta \}$. This results in a bijective enumeration $\{ \overline{\eta}^\alpha \mid \alpha < \zeta \}$ of $\Omega$ such that $\zeta \in [\omega, \omega_1)$ and, for all $\alpha < \zeta$,

$$0\eta_k^\alpha \leq 0\eta_k^{\alpha+1} < 0\eta_k^{\alpha+\omega}.$$

Choose $\ell_\alpha \in u_{\overline{\eta}^\alpha}$ arbitrarily. If $\overline{\eta}^\alpha \in U_\gamma$ and $\beta_0$ is the minimal ordinal such that $\overline{\eta}^{\beta_0} \in U_\gamma$, then we can find some $n_{\alpha,\beta} \in [1, \omega)$ such that $\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \neq \overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle$ for all $\beta \in [\beta_0, \alpha)$ and $n \geq n_{\alpha,\beta}$. Put $n_\alpha = \max_{\beta \in [\beta_0, \alpha)} n_{\alpha,\beta}$. Then, for all $n \geq n_\alpha$, $\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{ \overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta \in [\beta_0, \alpha) \}$. Moreover,

$$\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n \rangle \notin \{ \overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n \rangle \mid \beta < \beta_0 \} \cup \bigcup \Omega_\alpha F$$

since $F$ is regressive and $0\eta_k^\beta < 0\eta_k^\alpha$ for all $\beta < \beta_0$.

Now suppose that the assertion is true for some $f \in [1, k-1]$. Let $\Omega \in [\Lambda]^{\aleph_f}$ and $\langle u_{\overline{\eta}} \mid \overline{\eta} \in \Omega \rangle$ with $|u_{\overline{\eta}}| \geq f+1$. Choose an $\aleph_f$-filtration $\{ \Omega^\alpha \mid \alpha < \omega_f \}$ of $\Omega$ such that $\Omega^0 = \emptyset$ and $\left| \Omega^{\alpha+1} \setminus \Omega^\alpha \right| = \aleph_{f-1}$ for all $\alpha < \omega_f$. By the previous lemma, we can assume that this filtration is coordinatewise-closed. For every $\overline{\eta} \in \Omega^{\alpha+1} \setminus \Omega^\alpha$, consider

$$\begin{aligned} u^*_{\overline{\eta}} = \{ m \in [1,k] \mid \exists \overline{\eta}' \in \Omega^\alpha, n < \omega \\ (\overline{\eta} \upharpoonright \langle m, n \rangle = \overline{\eta}' \upharpoonright \langle m, n \rangle \text{ or } \overline{\eta} \upharpoonright \langle m, n \rangle \in \overline{\eta}' F ) \}. \end{aligned}$$

It follows that $\left| u^*_{\overline{\eta}} \right| \leq 1$, since $\left| u^*_{\overline{\eta}} \right| > 1$ would imply that $\overline{\eta} \in \Omega^\alpha$. Put $u'_{\overline{\eta}} = u_{\overline{\eta}} \setminus u^*_{\overline{\eta}}$ and observe that $\left| u'_{\overline{\eta}} \right| \geq f$. We apply the induction hypothesis on each of the sets $\Omega^{\alpha+1} \setminus \Omega^\alpha$ together with the family $\langle u'_{\overline{\eta}} \mid \overline{\eta} \in \Omega^{\alpha+1} \setminus \Omega^\alpha \rangle$ to obtain an enumeration $\Omega^{\alpha+1} \setminus \Omega^\alpha = \langle \overline{\eta}^\beta \mid \beta < \zeta \rangle$ for some $\zeta \in [\omega_{f-1}, \omega_f)$ with the required property. We induce an enumeration on $\Omega$ with the desired property by ordering these enumerations lexicographically. $\square$

**Definition 11.** (i) For every $b = \sum_{\overline{\nu}\in\Lambda_*} b_{\overline{\nu}} e_{\overline{\nu}} \in \widehat{B}$ with $b_{\overline{\nu}} \in \widehat{R}$, the $\Lambda_*$-*support* of $b$ is the set

$$[\,b\,] = \{\,\overline{\nu} \mid b_{\overline{\nu}} \neq 0\,\}.$$

(ii) For $b \in \overline{B}$, define $\|\,b\,\| = \sup_{\overline{\nu}\in[\,b\,]} \|\,\overline{\nu}\,\|$.

(iii) For $\alpha < \lambda_k$, let $\overline{B}_\alpha = \langle\, b \in \overline{B} \mid \|\,b\,\| < \alpha\,\rangle$.

**Definition 12.** A sequence of elements $(b_{\overline{\eta}n})_{n<\omega} \subseteq \overline{B}$ is called *regressive* with respect to $\overline{\eta} \in \Lambda$, if the following holds:

(i) $\|\,b_{\overline{\eta}0}\,\| < 0\eta_k$,

(ii) $b_{\overline{\eta}n} - s_{n+1}b_{\overline{\eta}\,n+1} \in B$ for all $n < \omega$, i.e. $(b_{\overline{\eta}n})_{n<\omega}$ is a *divisibility chain*,

(iii) $[\,b_{\overline{\eta}n}\,] \subseteq [\,b_{\overline{\eta}0}\,]$ for all $n < \omega$.

Every element $b_{\overline{\eta}} \in \overline{B}$ allows for a suitable sequence $(b_{\overline{\eta}n})_{n<\omega}$ of elements $b_{\overline{\eta}n} \in \overline{B}$ such that conditions (ii) and (iii) hold with $b_{\overline{\eta}0} = b_{\overline{\eta}}$. We fix such a sequence for each $b_{\overline{\eta}} \in \overline{B}$.

**Definition 13.** For an element $b_{\overline{\eta}} \in \overline{B}$ with regressive sequence $(b_{\overline{\eta}n})_{n<\omega} \subseteq \overline{B}$, we define the *branch-like element* associated with $\overline{\eta}$ and $n$ as

$$y'_{\overline{\eta}n} = b_{\overline{\eta}n} + y_{\overline{\eta}n}.$$

We write $y'_{\overline{\eta}}$ for $y'_{\overline{\eta}0}$. We call the element $b_{\overline{\eta}n}$ the *correction* of the branch element $y_{\overline{\eta}n}$.

We fix a map $\delta : \lambda_k \to \overline{B}$ such that $\alpha\delta \in \overline{B}_\alpha$ for all $\alpha < \lambda_k$. Consider the following family $\mathfrak{F} = \{\, y'_{\overline{\eta}} = b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in \Lambda, b_{\overline{\eta}} = 0\eta_k\delta\,\}$ of branch-like elements and construct the module $M$ as in Definition 6(ii).

**Definition 14.** For $g \in M$, define the $\Lambda$-*support* $[\,g\,]_\Lambda$ of $g$ to be the set of elements of $\Lambda$ that contribute to the representation of $g$. More precisely, if $q_m g = b + \sum_{\overline{\eta}\in\Lambda} n_{\overline{\eta}} y'_{\overline{\eta}}$ for some $m \geq 0$, where $b \in B$ and $n_{\overline{\eta}} \in R$ for all $\overline{\eta} \in \Lambda$, then

$$[\,g\,]_\Lambda = \{\,\overline{\eta} \in \Lambda \mid n_{\overline{\eta}} \neq 0\,\}.$$

Obviously, $[\,g\,]_\Lambda$ is finite. For $N \subseteq M$, we define $[\,N\,]_\Lambda = \bigcup_{g\in N} [\,g\,]_\Lambda$.

**Theorem 15.** *The module $M$ defined as above is $\aleph_k$-free.*

PROOF: Suppose that $H$ is a subset of $M$ of cardinality $\aleph_{k-1}$. Let $\sigma : \overline{B} \to [\,\Lambda_*\,]^{\leq\aleph_0}$ be the "$\Lambda_*$-support" function, i.e. $b\sigma = [\,b\,]$ for all $b \in \overline{B}$. Notice that $F : \Lambda \to [\,\Lambda_*\,]^{\leq\aleph_0}$ given by $\overline{\eta}F = 0\eta_k\delta\sigma$ is regressive since $\alpha\delta \in \overline{B}_\alpha$ for all $\alpha < \lambda_k$. Let $\Omega = [\,H\,]_\Lambda$, $\Omega_* = [\,H\,] \setminus ([\,\Omega\,] \cup \bigcup_{\overline{\eta}\in\Omega} [\,\overline{\eta}F\,])$ and observe that $|\,\Omega\,| \leq \aleph_{k-1}$. Then the submodules

$$M_\Omega = \langle\, e_{\overline{\eta}\restriction\langle m,n\rangle}, e_{\overline{\nu}}, y'_{\overline{\eta}} \mid \overline{\eta} \in \Omega, \overline{\nu} \in \overline{\eta}F, m \in [\,1,k\,], n < \omega\,\rangle_*,$$

and

$$M_{\Omega_*\Omega} = B_{\Omega_*} \oplus M_\Omega$$

satisfy $H \subseteq M_{\Omega_*\Omega}$. Our goal is to show that $M_{\Omega_*\Omega}$ is free, for which it suffices to show that $M_\Omega$ is free. Suppose that $|\,\Omega\,| = \aleph_{k-1}$. By taking $u_{\overline{\eta}} = [\,1, k\,]$ for all $\overline{\eta} \in \Omega$, we enumerate $\Omega = \{\, \overline{\eta}^\alpha \mid \alpha < \zeta \,\}$ for some $\zeta \in [\,\omega_{k-1}, \omega_k)$ according to the Freeness Proposition 10 and find $\ell_\alpha \in u_{\overline{\eta}}$ and $n_\alpha < \omega$ such that

$$\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n\rangle \notin \{\, \overline{\eta}^\beta \upharpoonright \langle \ell_\alpha, n\rangle \mid \beta < \alpha \,\} \cup \bigcup \Omega_\alpha F$$

for all $n \geq n_\alpha$. This allows us to write

$$M_\Omega = \langle\, e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}, e_{\overline{\nu}}, y'_{\overline{\eta}^\alpha n} \mid \alpha < \zeta, \overline{\nu} \in \overline{\eta}^\alpha F, m \in [\,1, k\,], n < \omega \,\rangle.$$

Let

$$M_\alpha = \langle\, e_{\overline{\eta}^\beta \upharpoonright \langle m,n\rangle}, e_{\overline{\nu}}, y'_{\overline{\eta}^\beta n} \mid \beta < \alpha, \overline{\nu} \in \overline{\eta}^\beta F, m \in [\,1, k\,], n < \omega)\,\rangle$$

and notice that $M_0 = \{0\}$, $\bigcup_{\alpha<\zeta} M_\alpha = M_\Omega$ and

$$\begin{aligned} M_{\alpha+1} &= M_\alpha + \langle\, e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}, e_{\overline{\nu}}, y'_{\overline{\eta}^\alpha n} \mid \overline{\nu} \in \overline{\eta}^\alpha F, m \in [\,1, k\,], n < \omega)\,\rangle \\ &= M_\alpha + \langle\, e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha,n\rangle} \mid n < n_\alpha)\,\rangle + \langle\, y'_{\overline{\eta}^\alpha n} \mid n \geq n_\alpha \,\rangle + \langle\, e_{\overline{\nu}} \mid \overline{\nu} \in \overline{\eta}^\alpha F \,\rangle \\ &\qquad + \langle\, e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle} \mid m \in [1, k] \setminus \{\ell_\alpha\}, n < \omega)\,\rangle \end{aligned}$$

since

$$e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha,n\rangle} = y'_{\overline{\eta}^\alpha n} - s_{n+1} y'_{\overline{\eta}^\alpha (n+1)} - b_{\overline{\eta}^\alpha n} - \sum_{\substack{m=1 \\ m\neq \ell_\alpha}}^{k} e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}$$

for $n \geq n_\alpha$ and

$$y'_{\overline{\eta}^\alpha n} = \frac{q_{n_\alpha}}{q_n} y'_{\overline{\eta}^\alpha n_\alpha} + \sum_{i=n}^{n_\alpha - 1} \frac{q_i}{q_n} \left( b_{\overline{\eta}^\alpha i} + \sum_{m=1}^{k} e_{\overline{\eta}^\alpha \upharpoonright \langle m,i\rangle} \right)$$

for $n < n_\alpha$. We claim that $M_{\alpha+1}/M_\alpha$ is free. To prove our claim, suppose

$$\underbrace{\sum_{n<n_\alpha} r_n e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha,n\rangle}}_{(1)} + \underbrace{\sum_{n\geq n_\alpha} r_n y'_{\overline{\eta}^\alpha n}}_{(2)} + \underbrace{\sum_{n<\omega} \sum_{\substack{m=1 \\ m\neq \ell_\alpha}}^{k} r_{mn} e_{\overline{\eta}^\alpha \upharpoonright \langle m,n\rangle}}_{(3)} + \underbrace{\sum_{\overline{\nu} \in \overline{\eta}^\alpha F} r_{\overline{\nu}} e_{\overline{\nu}}}_{(4)} \in M_\alpha.$$

It is immediate that the support of term (1) is disjoint from those of the other terms. Then $r_n = 0$ for all $n < n_\alpha$ with $e_{\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha,n\rangle} \notin M_\alpha$. By the Freeness Proposition 10, $\overline{\eta}^\alpha \upharpoonright \langle \ell_\alpha, n\rangle$ neither belongs to the support of the terms (3) and (4) nor to the support of $M_\alpha$ for all $n \geq n_\alpha$, which implies that $r_n = 0$ for all $n \geq n_\alpha$. Since $\|\, b_{\overline{\eta}^\alpha} \,\| < 0\eta_k^\alpha$, it follows that $r_{mn} = r_{\overline{\nu}} = 0$ for all $m \in [\,1, k\,] \setminus \{\, \ell_\alpha \,\}$,

$n < \omega$ and $\overline{\nu} \in \overline{\eta}^\alpha F$ such that $e_{\overline{\eta}^\alpha\upharpoonright\langle m,n\rangle}$, $e_{\overline{\nu}} \notin M_\alpha$. Therefore, $M_{\alpha+1}/M_\alpha$ is freely generated by the set

$$\{\, e_{\overline{\eta}^\alpha\upharpoonright\langle \ell_\alpha,r\rangle},\, y'_{\overline{\eta}^\alpha s},\, e_{\overline{\eta}^\alpha\upharpoonright\langle m,t\rangle},\, e_{\overline{\nu}} \mid \\ r < n_\alpha, s \geq n_\alpha, m \in [\,1,k\,] \setminus \{\,\ell_\alpha\,\}, t < \omega, \overline{\nu} \in \overline{\eta}^\alpha F\,\} \setminus M_\alpha.$$

Since $M_\Omega$ is the union of the continuous chain $\{\, M_\alpha \mid \alpha < \zeta \,\}$ such that $M_0$ is free and every $M_{\alpha+1}/M_\alpha$ is free, $M_\Omega$ itself is free. The $\aleph_k$-freeness of $M$ is now witnessed by the family $\mathcal{C} = \{\, M_{\Omega_*\Omega} \mid \aleph_k > |\,\Omega_*\,|, |\,\Omega\,|\,\}$. □

Hence, we have achieved the second goal of our plan.

## 5. No epimorphisms onto $R^{(\omega)}$

To accomplish the third and final goal of our plan, we need to refine our choice of corrections $b_{\overline{\eta}}$ of the branch elements $y_{\overline{\eta}}$. This is done by means of *Shelah's Easy Black Box*. The Black Box is a combinatorial principle that allows us to partially predict a given map under specific cardinal conditions. Various variants of this principle have been successfully used to realize complicated algebraic constructions (see, for example, [2], [6], [9] and [14] for applications of the General Black Box and the Strong Black Box). Its main feature is the fact that it is provable in ZFC, since prediction of maps is normally the direct consequence of additional set-theoretic assumptions like Martin's Axiom or Jensen's Diamond Principle $\diamondsuit$. Since the Easy Black Box is the central principle behind every version of the Black Box, its current state of development focuses on replacing older versions of the Black Box with the Easy Black Box. See [11] for a more detailed exposition on the advantages and disadvantages of doing this replacement.

**Definition 16.** For an infinite cardinal $\lambda$ and a set $\mathfrak{S}$ of cardinality $\leq \lambda^{\aleph_0}$, a *trap for the Easy Black Box* is a map

$$\varphi_\eta : [\,\eta\,] \to \mathfrak{S}$$

for some $\eta \in {}^{\omega\uparrow}\lambda$.

**The Easy Black Box 17.** *Let $\lambda$ be an infinite cardinal and $\mathfrak{S}$ a set of cardinality $\leq \lambda^{\aleph_0}$. Then there exists a family of traps*

$$\langle\, \varphi_\eta \mid \eta \in {}^{\omega\uparrow}\lambda \,\rangle$$

*that satisfies the following*

Prediction Principle*: for all $\Phi : {}^{\omega\uparrow>}\lambda \to \mathfrak{S}$ and $\nu \in {}^{\omega\uparrow>}\lambda$, there exists $\eta \in {}^{\omega\uparrow}\lambda$ with $\nu \subset \eta$ and $\Phi \upharpoonright [\,\eta\,] = \varphi_\eta$.*

PROOF: Since $|\mathfrak{S}| \leq \lambda^{\aleph_0} = |{}^{\omega}\lambda|$, we can fix an embedding $\pi : \mathfrak{S} \hookrightarrow {}^{\omega}\lambda$. We also fix a map $\mu : \lambda \to {}^{\omega>}\lambda$ such that $\mu^{-1}[\sigma]$ is unbounded in $\lambda$ for all $\sigma \in {}^{\omega>}\lambda$, or equivalently, there is a list

$$ {}^{\omega>}\lambda = \langle \mu_\alpha \mid \alpha < \lambda \rangle $$

with enough repetitions for each $\sigma \in {}^{\omega>}\lambda$. For example, use Solovay's Theorem (see [13, p. 95, Theorem 8.10]) to obtain a family $\{ E_\nu \mid \nu \in {}^{\omega>}\lambda \}$ of disjoint stationary subsets of $\lambda$ such that $\lambda = \dot{\bigcup}_{\nu \in {}^{\omega>}\lambda} E_\nu$, and define $\mu_\alpha = \nu$ for all $\alpha \in E_\nu$.

We would like to identify elements of ${}^{n}\mathfrak{S}$ with those of ${}^{\omega>}\lambda$. For this reason, we define a *coding map* $\pi^n : {}^{n}\mathfrak{S} \to {}^{n^2}\lambda$ for all $n \in (0,\omega)$ such that if $\varphi \in {}^{n}\mathfrak{S}$, then $\pi^n_\varphi$ is given by $(qn+r)\pi^n_\varphi = r\pi_{q\varphi}$ for $q, r \in [0,n)$ (recall Notation 1(iii)). Equivalently (recall Notation 1(viii)),

$$ \varphi\pi^n = (0\varphi\pi \restriction n) \ \wedge \ \cdots \ \wedge \ ((n-1)\varphi\pi \restriction n). $$

We now consider the set

$$ \mathfrak{X} = \{ \eta \in {}^{\omega\uparrow}\lambda \mid \exists \psi \in {}^{\omega}\mathfrak{S} \ \exists i < \omega \ \forall n \geq i \ (\pi^{n+1}_{\psi\restriction(n+1)} = \mu_{n\eta}) \}. $$

Since $\pi$ is an embedding, if $\eta \in \mathfrak{X}$, then $\psi$ is unique, so we call it $\psi_\eta$. We use $\mathfrak{X}$ to construct the family of traps $\langle \varphi_\eta \mid \eta \in {}^{\omega\uparrow}\lambda \rangle$ in the following way: if $\eta \in \mathfrak{X}$, then define $(\eta \restriction n)\varphi_\eta = n\psi_\eta$ for all $n < \omega$, and if $\eta \notin \mathfrak{X}$, choose $\varphi_\eta$ arbitrarily.

We now verify that this family of traps satisfies the Prediction Principle. Let $\Phi : {}^{\omega\uparrow>}\lambda \to \mathfrak{S}$ and $\nu \in {}^{\omega\uparrow>}\lambda$. We start the construction of $\eta \in {}^{\omega\uparrow}\lambda$ extending $\nu$ by setting $n\eta = n\nu$ for all $n \in \operatorname{dom}\nu$. Now assume we have defined $\eta \restriction n$ up to a certain $n \geq \operatorname{dom}\nu$. Consider the element $\varphi^{n+1} \in {}^{n+1}\mathfrak{S}$ given by $m\varphi^{n+1} = (\eta \restriction m)\Phi$ for all $m \leq n$. Then $\pi^{n+1}_{\varphi^{n+1}} \in {}^{\omega>}\lambda$ and $\mu^{-1}\left[\pi^{n+1}_{\varphi^{n+1}}\right]$ is unbounded in $\lambda$. Define $n\eta = \alpha$ to be the least ordinal $\alpha > (n-1)\eta$ such that $\mu_\alpha = \pi^{n+1}_{\varphi^{n+1}}$. This finishes the construction of the extension $\eta \in {}^{\omega\uparrow}\lambda$ of $\nu$. Moreover, let

$$ \varphi = \bigcup_{n \geq \operatorname{dom}\nu} \varphi^n. $$

Since $\eta \in \mathfrak{X}$ is witnessed by $\varphi = \psi_\eta$ and $i = \operatorname{dom}\nu$, it immediately follows that $(\eta \restriction n)\varphi_\eta = (\eta \restriction n)\Phi$ for all $n < \omega$. $\square$

**Definition 18.** Let $\overline{C} = \langle C_1, \ldots, C_k \rangle$ be a sequence of sets such that $|C_m| \leq \lambda_m$ and take $C = \bigcup_{m \in [1,k]} C_m$. A *set-trap for* $\overline{\eta} \in \Lambda$ *and* $\overline{C}$ is a function $\varphi_{\overline{\eta}} : [\overline{\eta}] \to C$.

**The First $\overline{\lambda}$-Black Box 19.** *Let* $\Lambda$ *and* $\Lambda_*$ *be as before,* $\overline{C} = \langle C_1, \ldots, C_k \rangle$ *and* $C$ *as in Definition* 18. *Then there exists a family of set-traps* $\langle \varphi_{\overline{\eta}} \mid \overline{\eta} \in \Lambda \rangle$ *satisfying the following*

Prediction Principle: *If $\varphi : \Lambda_* \to C$ is any map with the trap condition $\Lambda_{m*}\varphi \subseteq C_m$ and $\alpha < \lambda_k$, then there exists $\overline{\eta} \in \Lambda$ such that $\varphi \restriction [\overline{\eta}] = \varphi_{\overline{\eta}}$ and $0\eta_k = \alpha$.*

Proof: We will proceed by induction on the length of $\overline{\lambda}$.

Assume that the length of $\overline{\lambda}$ is 1, so we can simply write $\overline{\lambda} = \langle \lambda \rangle$ and $\overline{C} = \langle C \rangle$. Then $\Lambda = {}^{\omega\uparrow}\lambda$ and $\Lambda_* = {}^{\omega\uparrow>}\lambda$. Put $\mathfrak{S} = C$. Since $|\mathfrak{S}| \leq \lambda = \lambda^{\aleph_0}$, the Easy Black Box 17 provides us with a family of (set-)traps $\langle \varphi_\eta \mid \eta \in {}^{\omega\uparrow}\lambda \rangle$ with $\varphi_\eta : [\eta] \to \mathfrak{S}$. We now verify the prediction principle. Let $\varphi : \Lambda_* \to C$ be a map and let $\alpha < \lambda$. Choose an arbitrary $\nu \in \Lambda_*$ such that $0\nu = \alpha$. By the Easy Black Box 17, there exists $\eta \in \Lambda$ such that $\nu \subset \eta$ and $\varphi_\eta \subseteq \varphi$. This means $\varphi \restriction [\eta] = \varphi_\eta$ and $0\eta = \alpha$.

Now assume that the assertion is true for some $f \in [1, k-1]$ and that the length of $\overline{\lambda}$ is $f+1$. In this case, $\overline{\lambda} = \langle \lambda_1, \ldots, \lambda_{f+1} \rangle$ and $\overline{C} = \langle C_1, \ldots, C_{f+1} \rangle$. We also write $C^m = \bigcup_{i=1}^m C_i$ for $m \in [1, f+1]$. Define $\mathfrak{S} = {}^{\Lambda^f}C_{f+1}$ (the set of all maps from $\Lambda^f$ to $C_{f+1}$) and notice that $|\mathfrak{S}| \leq \lambda_{f+1}^{\lambda_f} = \lambda_{f+1} = \lambda_{f+1}^{\aleph_0}$. Hence, the Easy Black Box 17 provides us with a family $\langle \varphi^\eta \mid \eta \in {}^{\omega\uparrow}\lambda_{f+1} \rangle$ of traps $\varphi^\eta : [\eta] \to \mathfrak{S}$. We would like to define the set-traps $\varphi_{\overline{\eta}} : [\overline{\eta}] \to C^{f+1}$ for all $\overline{\eta} \in \Lambda^{f+1}$. By the induction hypothesis, we already have a family $\langle \psi_{\overline{\eta}} \mid \overline{\eta} \in \Lambda^f \rangle$ of set-traps $\psi_{\overline{\eta}} : [\overline{\eta}] \to C^f$. Given an $\overline{\eta} = (\eta_1, \ldots, \eta_{f+1}) \in \Lambda^{f+1}$, put $\overline{\eta}' = (\eta_1, \ldots, \eta_f) \in \Lambda^f$. Then for every $\overline{\eta} \in \Lambda^{f+1}$ define

$$(\overline{\eta} \upharpoonleft \langle m, n \rangle)\varphi_{\overline{\eta}} = \begin{cases} (\overline{\eta}' \upharpoonleft \langle m, n \rangle)\psi_{\overline{\eta}'}, & \text{if } \ m \in [1, f]; \\ \overline{\eta}' \varphi^{\eta_{f+1}}_{\eta_{f+1}\restriction n}, & \text{if } \ m = f+1. \end{cases}$$

(Recall Notation 1(iii).)

It remains to verify the prediction principle. Let $\varphi : \Lambda_*^{f+1} \to C^{f+1}$ be such that $\Lambda_{m*}^{f+1}\varphi \subseteq C_m$ and let $\alpha < \lambda_{f+1}$. Choose an arbitrary $\nu \in {}^{\omega\uparrow>}\lambda_{f+1}$ with $0\nu = \alpha$. Since $\Lambda_{f+1*}^{f+1}\varphi \subseteq C_{f+1}$, for every $\rho \in {}^{\omega\uparrow>}\lambda_{f+1}$, we can define a function $\Phi_\rho : \Lambda^f \to C_{f+1}$ by $\overline{\eta}'\Phi_\rho = (\overline{\eta}' \wedge \rho)\varphi$. This, in turn, gives us a function $\Phi : {}^{\omega\uparrow>}\lambda_{f+1} \to \mathfrak{S}$. By the Easy Black Box 17, there exists $\eta \in {}^{\omega\uparrow}\lambda_{f+1}$ such that $\nu \subset \eta$ and $\varphi^\eta \subset \Phi$.

We now use $\varphi$ and $\eta$ to define a function $\varphi' : \Lambda_*^f \to C^f$. For every $\overline{\nu} \in \Lambda_*^f$, define $\overline{\nu}\varphi' = (\overline{\nu} \wedge \eta)\varphi$, and observe that $\Lambda_{m*}^f\varphi' \subseteq C_m$ for all $m \in [1, f]$. By induction hypothesis, there exists $\overline{\eta}' \in \Lambda^f$ such that $\varphi' \restriction [\overline{\eta}'] = \psi_{\overline{\eta}'}$. Define $\eta_{f+1} = \eta$ and $\overline{\eta} = \overline{\eta}' \wedge \eta$. In this way, $0\eta_{f+1} = 0\nu = \alpha$. Finally, we must verify that $\varphi \restriction [\overline{\eta}] = \varphi_{\overline{\eta}}$. If $m \in [1, f]$, then

$$(\overline{\eta} \upharpoonleft \langle m, n \rangle)\varphi = (\overline{\eta}' \upharpoonleft \langle m, n \rangle)\varphi' = (\overline{\eta}' \upharpoonleft \langle m, n \rangle)\psi_{\overline{\eta}'} = (\overline{\eta} \upharpoonleft \langle m, n \rangle)\varphi_{\overline{\eta}},$$

and if $m = f+1$, then

$$(\overline{\eta} \upharpoonleft \langle m, n \rangle)\varphi = \overline{\eta}'\Phi_{\eta_{f+1}\restriction n} = \overline{\eta}'\varphi^{\eta_{f+1}}_{\eta_{f+1}\restriction n} = (\overline{\eta} \upharpoonleft \langle m, n \rangle)\varphi_{\overline{\eta}}.$$

This completes the proof. $\square$

**Definition 20.** Let $N$ be an $R$-module. A *trap for $B$ and $N$* is a homomorphism $\varphi_{\overline{\eta}} : B_{\overline{\eta}} \to N$.

**The Second $\overline{\lambda}$-Black Box 21.** *Let $\Lambda$ and $\Lambda_*$ be as before and $N$ an $R$-module such that $|\,N\,| \leq \lambda_1$. Then there exists a family $\langle\, \varphi_{\overline{\eta}} \mid \overline{\eta} \in \Lambda \,\rangle$ of traps for $B$ and $N$ satisfying the following*

Prediction Principle*: If $\varphi : B \to N$ is any homomorphism and $\alpha < \lambda_k$, then there exists $\overline{\eta} \in \Lambda$ such that $\varphi \restriction B_{\overline{\eta}} = \varphi_{\overline{\eta}}$ and $0\eta_k = \alpha$.*

Proof: The members of the sequence $\overline{C} = \langle\, C_1, \ldots, C_k \,\rangle$ where $C_m = N$ for all $m \in [\,1, k\,]$ satisfy $|\,C_m\,| \leq \lambda_m$ since $\overline{\lambda}$ is increasing and $|\,N\,| \leq \lambda_1$. Hence, $C = N$. The First $\overline{\lambda}$-Black Box 19 provides us with a family of set-traps $\langle\, \varphi_{\overline{\eta}} \mid \overline{\eta} \in \Lambda \,\rangle$. Since $[\,\overline{\eta}\,] \subset \Lambda_*$, each $\varphi_{\overline{\eta}}$ can be regarded as a homomorphism $\varphi_{\overline{\eta}} : B_{\overline{\eta}} \to N$. Since any homomorphism $\varphi : B \to N$ is completely determined by its action on the generators $e_{\overline{\nu}}$ of $B$, it can be regarded as a function $\varphi : \Lambda_* \to C$ which also satisfies $\Lambda_m \varphi \subseteq C_m$. Thus, for $\alpha < \lambda_k$, the First $\overline{\lambda}$-Black Box 19 yields that there exists $\overline{\eta} \in \Lambda$ such that $0\eta_k = \alpha$ and $\varphi \restriction [\,\overline{\eta}\,] = \varphi_{\overline{\eta}}$, i.e. $\varphi \restriction B_{\overline{\eta}} = \varphi_{\overline{\eta}}$. $\square$

We now present the Step Lemma. Step Lemmas are the results that allow to choose correctly the elements of the family $\mathfrak{F}$, i.e. to choose properly the corrections $b_{\overline{\eta}}$ for the branch elements $y_{\overline{\eta}}$ in order to eliminate unwanted epimorphisms.

**Step Lemma 22.** *Let $\overline{\eta} \in \Lambda$, $b_{\overline{\eta}} = 0\eta_k\delta$ and $\varphi_{\overline{\eta}}$ from the Second $\overline{\lambda}$-Black Box* 21. *There exists an $\varepsilon_{\overline{\eta}} \in \{\,0, 1\,\}$ such that no homomorphism*

$$\varphi : \left\langle\, B,\, y'_{\overline{\eta}} = \varepsilon_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \,\right\rangle_* \to S,$$

*where $S = \bigoplus_{n<\omega} Re_n$, satisfies both $\varphi \restriction B_{\overline{\eta}} = \varphi_{\overline{\eta}}$ and $b_{\overline{\eta}}\varphi \in \widehat{S} \setminus S$.*

Proof: Suppose towards a contradiction that for both $\varepsilon \in \{0, 1\}$, there exists some $\varphi^\varepsilon : \langle\, B, \varepsilon b_{\overline{\eta}} + y_{\overline{\eta}} \,\rangle_* \to S$ such that $\varphi^\varepsilon \restriction B_{\overline{\eta}} = \varphi_{\overline{\eta}}$ and $b_{\overline{\eta}}\varphi^\varepsilon \in \widehat{S} \setminus S$. On one hand, $(b_{\overline{\eta}} + y_{\overline{\eta}})\varphi^1 - y_{\overline{\eta}}\varphi^0 \in S$. On the other hand, $(b_{\overline{\eta}} + y_{\overline{\eta}})\varphi^1 - y_{\overline{\eta}}\varphi^0 = b_{\overline{\eta}}\varphi^1 \in \widehat{S} \setminus S$, which is the desired contradiction. $\square$

Let $\mathfrak{F} = \{\, y'_{\overline{\eta}} = \varepsilon_{\overline{\eta}} b_{\overline{\eta}} + y_{\overline{\eta}} \mid \overline{\eta} \in \Lambda, b_{\overline{\eta}} = 0\eta_k\delta \,\}$ be the family of branch-like elements obtained after choosing every $\varepsilon_{\overline{\eta}}$ by means of the Step Lemma 22. Define $M$ as in Definition 6(ii).

**Lemma 23.** *If $\varphi : M \to S$ is an epimorphism, then $\overline{B}\varphi \cap \widehat{S} \setminus S \neq \emptyset$.*

Proof: If $\overline{B}\varphi \cap \widehat{S} \setminus S = \emptyset$, then the set $X = \{\, e_{\overline{\nu}} \mid \overline{\nu} \in \Lambda_* \,\} \setminus \ker\varphi$ satisfies $|\,X\,| \leq \aleph_0$, since $|\,S\,| = \aleph_0$. If $|\,X\,| = \aleph_0$, then for any countable subset (with an adequate enumeration) $\{\, e_{\overline{\nu}_n} \mid n < \omega, \overline{\nu}_n \in \Lambda_* \,\} \subseteq X$ we have $(\sum_{n<\omega} q_n e_{\overline{\nu}_n})\varphi \in \overline{B}\varphi \cap \widehat{S} \setminus S$, a contradiction. It follows that $|\,X\,| < \aleph_0$. Therefore, $\varphi$ is not an epimorphism. $\square$

**Theorem 24.** *There exists an $\mathbb{S}$-separable $\aleph_k$-free $R$-module $M$ with no epimorphisms onto $S$.*

Proof: Let $M$ be as before. Notice that $\lambda_k^{\aleph_0} = \lambda_k$ implies that we can fix the map $\delta : \lambda_k \to \overline{B}$ to be surjective. Suppose $\varphi : M \to S$ is an epimorphism. By the previous lemma, $\overline{B}\varphi \cap \widehat{S} \setminus S \neq \emptyset$. Take any $g \in \overline{B}$ such that $g\varphi = s \in \widehat{S} \setminus S$. Then there exists $\alpha < \lambda_k$ such that $\alpha\delta = g$. By the Second $\overline{\lambda}$-Black Box 21, there exists $\overline{\eta} \in \Lambda$ such that $\varphi \restriction B_{\overline{\eta}} = \varphi_{\overline{\eta}}$ and $0\eta_k = \alpha$. But then $\psi = \varphi \restriction \left\langle B, y'_{\overline{\eta}} \right\rangle_*$ satisfies both $\psi \restriction B_{\overline{\eta}} = \varphi_{\overline{\eta}}$ and $b_{\overline{\eta}}\psi = b_{\overline{\eta}}\varphi = (0\eta_k\delta)\varphi = (\alpha\delta)\varphi = g\varphi = s \in \widehat{S} \setminus S$, contradicting the choice of $\varepsilon_{\overline{\eta}}$. Therefore $M$ has no epimorphisms onto $S$. $\square$

In this way, we have achieved the third and last goal of our plan.

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