A continuum X such that $C(X)$ is not continuously homogeneous

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Abstract. A metric continuum X is said to be continuously homogeneous provided that for every two points $p, q \in X$ there exists a continuous surjective function $f : X \to X$ such that $f(p) = q$. Answering a question by W.J. Charatonik and Z. Garncarek, in this paper we show a continuum X such that the hyperspace of subcontinua of X, $C(X)$, is not continuously homogeneous.

Keywords: continuum; continuously homogeneous; hyperspace

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1. Introduction

A *compactum* is a compact metric space with more than one point. A *continuum* is a connected compactum. A *mapping* is a continuous function. A continuum X is said to be *continuously homogeneous* if for every two points $p, q \in X$, there exists a surjective mapping $f : X \to X$ such that $f(p) = q$.

For the continuum X, we consider its hyperspaces:

$$2^X = \{A \subset X : A \text{ is closed and nonempty}\}, \text{ and}$$
$$C(X) = \{A \in 2^X : A \text{ is connected}\}.$$

The hyperspace 2^X is endowed with the Hausdorff metric H [5, Definition 2.1].

In [2] W.J. Charatonik and Z. Garncarek studied conditions for a hyperspace being continuously homogeneous. They showed that the hyperspace 2^X is continuously homogeneous for an arbitrary continuum X, and the hyperspace $C(X)$ is such if either X is locally connected or X contains an open subset with uncountably many components. They also asked [2, Question 2] the question: Is the hyperspace $C(X)$ continuously homogeneous for every continuum X?

In this paper we answer the question by Charatonik and Garncarek in the negative by showing a continuum Z such $C(Z)$ is not continuously homogeneous.

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2. The example

Given a connected subset S of a continuum X we denote by $C(S)$ the set of subcontinua of X that are contained in S. A compactum X is *connected im kleinen* at a point $p \in X$ provided that for each open subset U of X such that $p \in U$, there exists a subcontinuum M of X such that p belongs to the interior of M and $M \subset U$. The set of points p in X such that X is not connected im kleinen at p is denoted by $N(X)$. We will use the following well known lemma [4, p. 28].

Lemma 1 ([4, p. 28])**.** *Let $f : X \to Y$ be a surjective mapping between compacta. Then $N(Y) \subset f(N(X))$.*

The following lemma can be easily proven by using Theorems 2 and 3 of [4].

Lemma 2. *Let X be a compactification of the ray $[0, \infty)$ with remainder R and $S = X - R$. Then:*

(a) $N(C(X)) \subset C(R)$,

(b) *if* $C(R) \subset \mathrm{cl}_{C(X)}(C(S))$, *then* $N(C(X)) = C(R) - \{R\}$.

In order to construct the continuum Z, we use ideas in the paper [1]. We construct a sequence of continua $Z_1, Z_2, \ldots$ in the Hilbert cube $Q = [-1, 1]^\infty$ in such a way that the complexity of the set of points of non connectedness im kleinen of Z_{n+1} is bigger than the one of Z_n. In this way, we obtain that there is not a mapping from $C(Z_n)$ onto $C(Z_{n+1})$. Each Z_{n+1} is a compactification of the ray $[0, \infty)$ with remainder Z_n and Z_n is a retract of Z_{n+1}. The continua Z_n can also be defined in the Euclidean plane but the description is more complicated.

We start by defining $Z_1 = \{(0, t, 0, 0, \ldots) \in Q : t \in [-1, 1]\}$ and $Z_2 = Z_1 \cup S_1$, where $S_1 = \{(t, \sin(\frac{1}{t}), 0, 0, \ldots) \in Q : t \in (0, 1]\}$.

Define $g_1 : [0, \infty) \to [0, 1]^2$ by $g_1(t) = (\frac{1}{1+t}, \sin(1 + t))$.

Inductively, suppose that $n \geq 2$, $Z_n \subset [-1, 1]^n \times \{0\} \times \{0\} \times \ldots$ has been constructed, $Z_n = Z_{n-1} \cup S_{n-1}$, $Z_{n-1} \cap S_{n-1} = \emptyset$, Z_n is a compactification of $[0, \infty)$ with remainder Z_{n-1}, $h_{n-1} : [0, \infty) \to S_{n-1}$ is a homeomorphism, $h_{n-1} = g_{n-1} \times \{0\} \times \{0\} \times \ldots$, where $g_{n-1} : [0, \infty) \to [-1, 1]^n$ is an embedding. Define $k : [0, \infty) \to [0, \infty)$ by

$$k(t) = \begin{cases} (2m+1)(t - 2m), & \text{if } \ t \in [2m, 2m+1] \ \text{ for some } \ m \geq 0, \\ (2m+1)(2 + 2m - t) & \text{if } \ t \in [2m+1, 2m+2] \ \text{ for some } \ m \geq 0. \end{cases}$$

Define $g_n : [0, \infty) \to [-1, 1]^{n+1}$ by $g_n(t) = (g_{n-1}(k(t)), \frac{1}{t+1})$ and define $h_n : [0, \infty) \to Q$ by

$$h_n = g_n \times \{0\} \times \{0\} \times \ldots.$$

Clearly, h_n is an embedding. Let $S_n = \mathrm{Im}\, h_n$ and $Z_{n+1} = Z_n \cup S_n$. Then Z_{n+1} is a compactification of $[0, \infty)$ with remainder Z_n and the natural projection $r_n : Z_{n+1} \to Z_n$, defined on S_n by $(g_{n-1}(k(t)), \frac{1}{t+1}, 0, \ldots) \longmapsto (g_{n-1}(k(t)), 0, \ldots)$, is a retraction.

Claim 1. For each $n \in \mathbb{N}$, $C(Z_n) \subset \mathrm{cl}_{C(Z_{n+1})}(C(S_n))$.

We prove Claim 1. Clearly, $C(Z_1) \subset \mathrm{cl}_{C(Z_2)}(C(S_1))$. Take $n \geq 2$ and take a subarc J of S_{n-1}, let L be a subinterval of $[0, \infty)$ such that $h_{n-1}(L) = J$. Let $M \in \mathbb{N}$ be such that $L \subset [0, 2M + 1]$. Then for each $m > M$, there exists a subinterval J_m of $[2m, 2m + 1]$ such that $k(J_m) = L$. Thus, the sequence $\{h_n(J_m)\}_{m=M+1}^{\infty}$ is a sequence in S_n such that $\lim h_n(J_m) = J$. We have shown that $C(S_{n-1}) \subset \mathrm{cl}_{C(Z_{n+1})}(C(S_n))$ for every $n \geq 2$.

Notice that $C(Z_2) \subset \mathrm{cl}_{C(Z_2)}(C(S_1)) \subset \mathrm{cl}_{C(Z_3)}(C(S_2))$, so

$$C(Z_2) \subset \mathrm{cl}_{C(Z_3)}(C(S_2)).$$

Since $Z_3 = Z_2 \cup S_2$ is a compactification of the ray $[0, \infty)$ with remainder Z_2, $C(Z_3) = C(Z_2) \cup C(S_2) \cup \{A \in C(Z_3) : Z_2 \subset A\}$. It is easy to prove that $C(S_2) \cup \{A \in C(Z_3) : Z_2 \subset A\} \subset \mathrm{cl}_{C(Z_3)}(C(S_2))$. Since $C(Z_2) \subset \mathrm{cl}_{C(Z_3)}(C(S_2))$, we conclude that $C(Z_3) \subset \mathrm{cl}_{C(Z_3)}(C(S_2))$. By the fact we prove two paragraphs above, $\mathrm{cl}_{C(Z_3)}(C(S_2)) \subset \mathrm{cl}_{C(Z_4)}(C(S_3))$. Hence, $C(Z_3) \subset \mathrm{cl}_{C(Z_4)}(C(S_3))$.

With a similar procedure as the one in the previous paragraph, Claim 1 can be proved for each $n \geq 4$.

The following claim is a consequence of Claim 1 and Lemma 2.

Claim 2. For each $n \in \mathbb{N}$, $N(C(Z_{n+1})) = C(Z_n) - \{Z_n\}$.

Given a subset A of Q and $n \in \mathbb{N}$ let $A(n) = \{(\frac{1}{2^n} + \frac{a}{8^n}, 0) \in Q \times [-1, 1] : a \in A\}$.

Now, we construct the continuum Z as a subspace of the space $Q \times [-1, 1]$. For each $n \geq 2$, let $X_n = Z_n(n)$. Then X_n is a compactification of the ray $[0, \infty)$ with remainder $Z_{n-1}(n)$ and $X_n - Z_{n-1}(n) = S_{n-1}(n)$. Let p_n be the end point of the ray $S_{n-1}(n)$. Choose an arc $L_n \subset Q \times [0, 1]$ with end points p_n and p_{n+1} such that $L_n - \{p_n, p_{n+1}\} \subset Q \times (0, 1]$, $\lim L_n = \{\theta\}$, where $\theta = (0, 0, \ldots)$, and $L_n \cap (\bigcup\{L_m : m \geq 2 \text{ and } m \neq n\}) \subset \{p_n, p_{n+1}\}$. Let $\sigma_n : Z_n \to X_n$ be the homeomorphism given by $\sigma_n(z) = (\frac{1}{2^n} + \frac{z}{8^n}, 0)$.

Finally, define $Z = \{\theta\} \cup (\bigcup\{X_n : n \geq 2\}) \cup (\bigcup\{L_n : n \geq 2\})$. Then Z is a continuum.

The following claim follows from Theorems 2 and 3 of [4].

Claim 3. $N(C(Z)) = \bigcup\{N(C(X_n)) : n \geq 2\}$.

We are going to show that $C(Z)$ is not continuously homogeneous. Suppose the contrary. Then there exists a continuous surjective mapping $f : C(Z) \to C(Z)$ such that $f(\{\theta\}) = \{p_2\}$. Let $\mathcal{U} = C(L_2 \cup X_2)$. Since $L_2 \cup X_2$ contains p_2 in its interior, $\mathcal{U}$ is a neighborhood of $\{p_2\}$ in $C(Z)$. Since $\lim(X_n \cup L_n) = \{\theta\}$, there exists $M \geq 3$ such that

$$f(C(\{\theta\} \cup (\bigcup\{X_n : n > M\}) \cup (\bigcup\{L_n : n > M\}))) \subset \mathcal{U}.$$

Let $Y = (\bigcup\{X_n : n \in \{2, \ldots, M\}\}) \cup (\bigcup\{L_n : n \in \{2, \ldots, M\}\})$. Notice that $N(C(Y)) = \bigcup\{N(C(X_n)) : n \in \{2, \ldots, M\}\}$.

By Claim 3 and Lemma 1,

$$N(C(X_{M+1})) \subset N(C(Z)) \subset f(N(C(Z))) = \bigcup\{f(N(C(X_n))) : n \geq 2\}.$$

Since $\bigcup\{f(C(X_n)) : n > M\} \subset C(L_2 \cup X_2)$ and $C(L_2 \cup X_2) \cap C(X_{M+1}) = \emptyset$, we have that

$$N(C(X_{M+1})) \subset \bigcup\{f(N(C(X_n))) : n \in \{2, \ldots, M\}\}.$$

Since $\{p_{M+1}\} = \mathrm{Bd}_Z(X_{M+1})$, we have that the function $t : Z \to X_{M+1}$ defined by

$$t(z) = \begin{cases} z, & \text{if } \ z \in X_{M+1}, \\ p_{M+1}, & \text{if } \ z \notin X_{M+1}, \end{cases}$$

is a retraction. Then the mapping $T : C(Z) \to C(X_{M+1})$ given by $T(A) = t(A)$ (the image of A under t) is a retraction. Then the inclusion $N(C(X_{M+1})) \subset \bigcup\{f(N(C(X_n))) : n \in \{2, \ldots, M\}\}$ implies that

$$N(C(X_{M+1})) \subset \bigcup\{T(f(N(C(X_n)))) : n \in \{2, \ldots, M\}\}.$$

Consider the homeomorphism $\sigma_{M+1}^{-1} : C(X_{M+1}) \to C(Z_{M+1})$ that sends each $A \in C(X_{M+1})$ to $\sigma_{M+1}^{-1}(A)$ (the image of A under σ_{M+1}^{-1}). Then

$$\sigma_{M+1}^{-1}(N(C(X_{M+1}))) = N(C(Z_{M+1})) = C(Z_M) - \{Z_M\}.$$

Since $r_M : Z_{M+1} \to Z_M$ is a retraction, the mapping $r : C(Z_{M+1}) \to C(Z_M)$ given by $r(A) = r_M(A)$ (the image of A under r_M) is a retraction.

Thus, the function $g = \sigma_{M+1}^{-1} \circ T \circ f : C(Z) \to C(Z_{M+1})$ is a surjective mapping such that

$$N(C(Z_{M+1})) = \sigma_{M+1}^{-1}(N(C(X_{M+1}))) \subset g(\bigcup\{N(C(X_n)) : n \in \{2, \ldots, M\}\}).$$

Hence, $C(Z_M) - \{Z_M\} \subset g(\bigcup\{N(C(X_n)) : n \in \{2, \ldots, M\}\})$. Thus,

$$\begin{aligned} & C(Z_M) - \{Z_M\} \subset r(g(\bigcup\{N(C(X_n)) : n \in \{2, \ldots, M\}\})) \\ & = r(g(\bigcup\{\sigma_n(N(C(Z_n))) : n \in \{2, \ldots, M\}\})) \\ & = r(g(\bigcup\{\sigma_n(C(Z_{n-1}) - \{Z_{n-1}\}) : n \in \{2, \ldots, M\}\})) \\ & \subset r(g(\bigcup\{\sigma_n(C(Z_{n-1})) : n \in \{2, \ldots, M\}\})) \subset C(Z_M). \end{aligned}$$

Since, $r(g(\bigcup\{\sigma_n(C(Z_{n-1})) : n \in \{2, \ldots, M\}\}))$ is compact, we obtain that

$$C(Z_M) = r(g(\bigcup\{\sigma_n(C(Z_{n-1})) : n \in \{2, \ldots, M\}\})).$$

Consider the compactum $W = C(Z_1) \oplus \ldots \oplus C(Z_{M-1})$, which is a disjoint union of the spaces $C(Z_1), \ldots, C(Z_{M-1})$ with the sum topology. Since the subsets $\sigma_2(C(Z_1)), \ldots, \sigma_M(C(Z_{M-1}))$ of Z are compact and pairwise disjoint, $\bigcup\{\sigma_n(C(Z_{n-1})) : n \in \{2, \ldots, M\}\}$ is homeomorphic to W.

We have shown that $C(Z_M)$ is the image under a continuous function φ of the compactum W.

By Lemma 1, $N(C(Z_M)) \subset \varphi(N(W))$. By Claim 2, this implies that

$$\begin{aligned} C(Z_{M-1}) - \{Z_{M-1}\} &\subset \varphi(\emptyset \oplus (C(Z_1) - \{Z_1\}) \oplus \ldots \oplus (C(Z_{M-2}) - \{Z_{M-2}\})) \\ &= \varphi((C(Z_1) - \{Z_1\}) \oplus \ldots \oplus (C(Z_{M-2}) - \{Z_{M-2}\})) \\ &\subset \varphi(C(Z_1) \oplus \ldots \oplus C(Z_{M-2})). \end{aligned}$$

By the compactness of $\varphi(C(Z_1) \oplus \ldots \oplus C(Z_{M-2}))$, we have that

$$C(Z_{M-1}) \subset \varphi(C(Z_1) \oplus \ldots \oplus C(Z_{M-2})).$$

Since the retraction $r_{M-1} : Z_M \to Z_{M-1}$ induces a retraction $R_{M-1} : C(Z_M) \to C(Z_{M-1})$, we obtain that

$$C(Z_{M-1}) = R_{M-1}(\varphi(C(Z_1) \oplus \ldots \oplus C(Z_{M-2}))).$$

This shows that $C(Z_{M-1})$ is the image under a continuous function φ_1 of the compactum $C(Z_1) \oplus \ldots \oplus C(Z_{M-2})$.

Repeating this argument, we conclude that $C(Z_2)$ is a continuous image of the continuum $C(Z_1)$. This is a contradiction since $C(Z_1)$ is locally connected and $C(Z_2)$ is not locally connected. Therefore, $C(Z)$ is not continuously homogeneous.

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