A characterization of complex L_1-preduals via a complex barycentric mapping

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Abstract. We provide a complex version of a theorem due to Bednar and Lacey characterizing real L_1-preduals. Hence we prove a characterization of complex L_1-preduals via a complex barycentric mapping.

Keywords: complex Banach spaces; L_1-predual; barycentric mapping

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1. Introduction

A complex (or real) Banach space X is called an L_1*-predual* if its dual space X^* is isometric to a complex (or real) space $L_1(X, \mathcal{S}, \mu)$ for a measure space $(X, \mathcal{S}, \mu)$. This important class of complex Banach spaces was studied e.g. in [6], [16], [9], [17], [4] and recently in [15]. The second author's contribution to this subject can be found in [13], [11], [12] and [10].

An interesting characterization of real L_1-preduals was given by Bednar and Lacey in [2] (see also [8, p. 188]). To state their result we need to introduce a notion of a *barycentric mapping*.

Let $\mathbb{F}$ denote either the field of real or complex numbers. Let K be a compact topological space (all topological spaces will be considered as Hausdorff). We denote by $\mathcal{C}(K, \mathbb{F})$ the space of all $\mathbb{F}$-valued continuous functions on K. By $\mathcal{M}(K, \mathbb{F})$ we denote the space of all $\mathbb{F}$-valued Radon measures on K with the weak* topology given by the duality $(\mathcal{C}(K, \mathbb{F}))^* = \mathcal{M}(K, \mathbb{F})$. By a positive Radon measure on K we mean a finite complete inner regular measure defined at least on all Borel subsets of K. An $\mathbb{F}$-valued measure μ is Radon if its variation $|\mu|$ is Radon. By $\mathcal{M}^1(K)$ we denote the set of all Radon probability measures on K. An $\mathbb{F}$-valued function on K is *universally measurable* if it is μ-measurable with respect to every $\mu \in \mathcal{M}^1(K)$. If $\varphi\colon K \to L$ is a continuous mapping of a compact space K to a compact space L and $\mu \in \mathcal{M}(K, \mathbb{F})$, then $\varphi\mu$ denotes the image measure in $\mathcal{M}(L, \mathbb{F})$.

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Definition 1.1. Let K be a compact space. Following [8, Definition 2, p. 188], we call a mapping $\rho\colon K \to \mathcal{M}(K,\mathbb{R})$, $k \in K \mapsto \rho_k \in \mathcal{M}(K,\mathbb{R})$, a *barycentric mapping* if it satisfies the following three conditions.

(a) For $f \in \mathcal{C}(K,\mathbb{R})$, let $f_\rho(k) = \int_K f(t)\, d\rho_k(t)$, $k \in K$. Then f_ρ is universally measurable for each $f \in \mathcal{C}(K,\mathbb{R})$.
(b) We have $\|\rho_k\| \le 1$ for every $k \in K$.
(c) If $\nu_1, \nu_2 \in \mathcal{M}(K,\mathbb{R})$ are such that $\nu_1(f) = \nu_2(f)$ for every $f \in \mathcal{C}(K,\mathbb{R})$ satisfying $f = f_\rho$, then $\nu_1(f_\rho) = \nu_2(f_\rho)$ for every $f \in \mathcal{C}(K,\mathbb{R})$.

If ρ is a barycentric mapping on a compact space K, we denote

$$A_\rho = \{f \in \mathcal{C}(K,\mathbb{R})\colon f = f_\rho\}.$$

(The term barycentric mapping does not appear in the original article [2] where the term *affine mapping* is used instead.)

The characterization of real L_1-preduals given in [2] as Theorems 3.5 and 3.7 (see also [8, Theorem 6, p. 188, and Theorem 8, p. 216]) then reads as follows:

Theorem 1.2. (a) *Let K be a compact space and let $\rho\colon K \mapsto \mathcal{M}(K,\mathbb{R})$ be a barycentric mapping. Then the Banach space A_ρ is a real L_1-predual.*
(b) *Suppose X is a real L_1-predual. Then there exists a compact space K and a barycentric mapping $\rho\colon K \mapsto \mathcal{M}(K,\mathbb{R})$ such that X is isometric to A_ρ.*

The aim of our paper is to provide a complex version of Theorem 1.2. The proof of the complex variant of Theorem 1.2(a) does not require any new ingredient from its real version. However, for the sake of completeness we present it in Section 3. The proof of the complex variant of assertion (b) in Theorem 1.2 is more complicated and covers the rest of the paper.

2. Results

We start with the crucial definition of a *complex barycentric mapping* which we use throughout the rest of the paper.

Definition 2.1. Let K be a compact space. A mapping $\rho\colon K \mapsto \mathcal{M}(K,\mathbb{C})$, $k \in K \mapsto \rho_k \in \mathcal{M}(K,\mathbb{C})$, is called a *complex barycentric mapping* if it satisfies the following conditions.

(a) For $f \in \mathcal{C}(K,\mathbb{C})$, let $f_\rho(k) = \int_K f(t)\, d\rho_k(t)$, $k \in K$. Then f_ρ is universally measurable for each $f \in \mathcal{C}(K,\mathbb{C})$.
(b) We have $\|\rho_k\| \le 1$ for every $k \in K$.
(c) If $\nu_1, \nu_2 \in \mathcal{M}(K,\mathbb{C})$ are such that $\nu_1(f) = \nu_2(f)$ for every $f \in \mathcal{C}(K,\mathbb{C})$ satisfying $f = f_\rho$, then $\nu_1(f_\rho) = \nu_2(f_\rho)$ for every $f \in \mathcal{C}(K,\mathbb{C})$.

If ρ is a complex barycentric mapping on K, we set

$$A_\rho = \{f \in \mathcal{C}(K,\mathbb{C})\colon f = f_\rho\}.$$

Now we can formulate the complex version of Theorem 1.2.

Theorem 2.2. (a) *Let K be a compact space and let $\rho\colon K \mapsto \mathcal{M}(K,\mathbb{C})$ be a complex barycentric mapping. Then the Banach space A_ρ is a complex L_1-predual.*

(b) *Suppose X is a complex L_1-predual. Then there exists a compact space K and a complex barycentric mapping $\rho\colon K \mapsto \mathcal{M}(K,\mathbb{C})$ such that X is isometric to A_ρ.*

3. Proof of Theorem 2.2(a)

Proof of Theorem 2.2(a) follows that of Theorem 1.2(a) as presented in [8, p. 188]. We thus need the following theorem (see [8, p. 162]).

Theorem 3.1. *Suppose $1 \le p < \infty$ and $p \ne 2$, let (T,Σ,μ) be a measure space and let M be a subspace of $L_p(\mu,\mathbb{C})$. Then the following conditions on M are equivalent.*

(i) *The space M is the range of a contractive projection on $L_p(\mu,\mathbb{C})$.*

(ii) *There is a measure space (Ω,Θ,λ) such that M is isometrically isomorphic to $L_p(\lambda,\mathbb{C})$.*

Now we can prove assertion (a) of Theorem 2.2(a).

Proof of Theorem 2.2(a): Let ρ be a complex barycentric mapping and A_ρ be defined as above. We construct a contractive projection P on $\mathcal{M}(K,\mathbb{C})$ such that the dual $(A_\rho)^*$ is isometric to the range $\operatorname{Rng} P$ of P. Since $\mathcal{M}(K,\mathbb{C})$ is isometric to a space $L_1(\lambda,\mathbb{C})$ for a suitable measure space (Ω,Θ,λ) (see [8, Theorem 3, p. 135]), the proof will follow from Theorem 3.1.

Let $\mu \in \mathcal{M}(K,\mathbb{C})$ be given. The mapping $f \mapsto \int_K f_\rho \, d\mu$ is a continuous linear functional on $\mathcal{C}(K,\mathbb{C})$ and thus there exists a uniquely defined $P\mu \in \mathcal{M}(K,\mathbb{C})$ such that

$$\int_K f \, dP\mu = \int_K f_\rho \, d\mu, \quad f \in \mathcal{C}(K,\mathbb{C}).$$

Then the mapping $P\colon \mu \mapsto P\mu$ is a linear mapping on $\mathcal{M}(K,\mathbb{C})$ with $\|P\| \le 1$. We show that P is a projection. Let $\mu \in \mathcal{M}(K,\mathbb{C})$ be given. If $f \in A_\rho$, then

$$P\mu(f) = \mu(f_\rho) = \mu(f).$$

By (c) of Definition 2.1, $P\mu(f_\rho) = \mu(f_\rho)$ for each $f \in \mathcal{C}(K,\mathbb{C})$. Hence

$$(P\mu)(f) = \mu(f_\rho) = (P\mu)(f_\rho) = (P(P\mu))(f), \quad f \in \mathcal{C}(K,\mathbb{C}).$$

Hence $PP\mu = P\mu$ and P is a projection.

Now we claim that the restriction mapping

$$R\colon \operatorname{Rng} P \to (A_\rho)^*$$

is an isometric surjective isomorphism. To this end, let $x^* \in (A_\rho)^*$ be given. By the Hahn-Banach theorem there exists a measure $\nu \in \mathcal{M}(K,\mathbb{C})$ extending x^*.

Then $R(P\nu) = x^*$, because, for $f \in A_\rho$, we have

$$P\nu(f) = \nu(f_\rho) = \nu(f) = x^*(f).$$

Hence R is surjective.

Let now $\mu \in \operatorname{Rng} P$ be given. Obviously, $\|R\mu\| \le \|\mu\|$. To prove the converse inequality, let $\nu \in \mathcal{M}(K, \mathbb{C})$ be a Hahn-Banach extension of $x^* = R\mu = \mu|_{A_\rho}$. Then

$$(\mu - P\nu)(f) = \mu(f) - \nu(f_\rho) = \mu(f) - \nu(f) = 0, \quad f \in A_\rho.$$

Thus $\mu - P\nu \in (A_\rho)^\perp$, which by virtue of (c) in Definition 2.1 gives $(\mu - P\nu)(f_\rho) = 0$ for every $f \in \mathcal{C}(K, \mathbb{C})$. Hence

$$(\mu - P\nu)(f) = (P\mu - PP\nu)(f) = (\mu - P\nu)(f_\rho) = 0, \quad f \in \mathcal{C}(K, \mathbb{C}),$$

which yields $\mu = P\nu$. Thus

$$\|\mu\| = \|\mu - P\nu + P\nu\| = \|P\nu\| \le \|\nu\| = \|x^*\| = \|R\mu\|.$$

This concludes the proof. $\square$

4. Proof of Theorem 2.2(b)

We start this section with the following definitions. If X is a Banach space, its dual unit ball B_{X^*} will always be considered with the weak* topology.

Definition 4.1. Let X be a complex Banach space. A set $B \subset B_{X^*}$ is called *homogeneous* if $\alpha B = B$ for each $\alpha \in \mathbb{T}$ ($\mathbb{T}$ is the unit circle in the complex plane). A function $f \colon B \to \mathbb{C}$ on a homogeneous set $B \subset B_{X^*}$ is called *homogeneous* (see e.g. [3, p. 53] or [8, p. 240]) if

$$f(\alpha x^*) = \alpha f(x^*), \quad (\alpha, x^*) \in \mathbb{T} \times B.$$

If f is a bounded Borel function defined on a homogeneous set $B \subset B_{X^*}$, we set

$$(\hom f)(x^*) = \int_{\mathbb{T}} \alpha^{-1} f(\alpha x^*)\, d\alpha, \quad x^* \in B,$$

where $d\alpha$ denotes the unit Haar measure on $\mathbb{T}$.

The following lemma presents several useful observations about homogeneous functions. For its extended version along with a proof see [10, Lemma 2.2].

Lemma 4.2. *Let $B \subset B_{X^*}$ be a homogeneous set and f be a bounded complex Borel function on B. Then the following hold.*

(a) *The function* $\hom f$ *is homogeneous on B.*
(b) *The function f is homogeneous if and only if* $\hom f = f$.
(c) *If f is continuous on B, then* $\hom f$ *is continuous on B.*

Definition 4.3. The mapping $\hom\colon \mathcal{C}(B_{X^*},\mathbb{C}) \to \mathcal{C}(B_{X^*},\mathbb{C})$ induces by virtue of Lemma 4.2(c) a mapping (denoted likewise) $\hom\colon \mathcal{M}(B_{X^*},\mathbb{C}) \to \mathcal{M}(B_{X^*},\mathbb{C})$ defined as

$$(\hom\mu)(f) = \mu(\hom f), \quad f \in \mathcal{C}(B_{X^*},\mathbb{C}), \mu \in \mathcal{M}(B_{X^*},\mathbb{C}).$$

We recall that, given a Banach space X and $x^* \in B_{X^*}$, the set $\mathcal{M}^1_{x^*}(B_{X^*})$ denotes the set of all probability measures *representing* x^* (see [14, Definition 2.26] or [8, Definition 1, p. 189]). For any $\mu \in \mathcal{M}^1(B_{X^*})$ there exists a unique point $x^* \in B_{X^*}$ (the *barycenter* of μ) such that $\mu \in \mathcal{M}^1_{x^*}(B_{X^*})$. Further, $\mathcal{M}^1_{\max}(B_{X^*})$ is the set of all probability measures on B_{X^*} which are *maximal* with respect to the Choquet ordering, see [14, Definition 3.57] or [8, p. 192]. A function $f\colon B_{X^*} \to \mathbb{F}$ is *strongly affine* if it is universally measurable and $\mu(f) = f(x^*)$ for each $x^* \in B_{X^*}$ and $\mu \in \mathcal{M}^1_{x^*}(B_{X^*})$. It is not difficult to show that any strongly affine (real or complex) function is bounded (see [7, Satz 2.1]).

Lemma 4.4. *Let $f\colon B_{X^*} \to \mathbb{C}$ be bounded. Then f is strongly affine if and only if both* $\operatorname{Re} f$ *and* $\operatorname{Im} f$ *are strongly affine.*

Proof: Assume that f is strongly affine, $x^* \in B_{X^*}$ and $\mu \in \mathcal{M}^1_{x^*}(B_{X^*})$. Since f is μ-measurable, it is easy to observe that both $\operatorname{Re} f$ and $\operatorname{Im} f$ are also μ-measurable. Further

$$(\operatorname{Re} f)(x^*) + i(\operatorname{Im} f)(x^*) = f(x^*) = \mu(f) = \mu(\operatorname{Re} f) + i\mu(\operatorname{Im} f),$$

and thus

$$(\operatorname{Re} f)(x^*) = \mu(\operatorname{Re} f) \quad \text{and} \quad (\operatorname{Im} f)(x^*) = \mu(\operatorname{Im} f).$$

Conversely, if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are strongly affine, and $x^* \in B_{X^*}$ along with $\mu \in \mathcal{M}^1_{x^*}(B_{X^*})$ are given, then f is μ-measurable. (This easily follows from the fact that any open set in $\mathbb{C}$ is a countable union of rectangles.) Then

$$\mu(f) = \mu(\operatorname{Re} f) + i\mu(\operatorname{Im} f) = (\operatorname{Re} f)(x^*) + i(\operatorname{Im} f)(x^*) = f(x^*).$$

Hence f is strongly affine. □

The following theorem is due to Effros (see [3, Theorem 4.3] or [8, Theorem 5, p. 243]).

Theorem 4.5. *Let X be a complex Banach space. Then the following assertions are equivalent.*

(a) *X is a complex L_1-predual.*
(b) *If $\mu_1,\mu_2 \in \mathcal{M}^1_{x^*}(B_{X^*})$ and $\mu_1,\mu_2 \in \mathcal{M}^1_{\max}(B_{X^*})$ for some $x^* \in B_{X^*}$, then* $\hom\mu_1 = \hom\mu_2$.

The preceding theorem enables us to define the following mapping.

Definition 4.6. Let X be a complex L_1-predual. For any bounded universally measurable function f on B_{X^*} we define

$$Tf(x^*) = (\hom \mu)(f), \quad \mu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^1_{\max}(B_{X^*}), x^* \in B_{X^*}.$$

The following lemma describes a simple, yet useful, property of the mapping T.

Lemma 4.7. *Let X be a complex Banach space and $f\colon B_{X^*} \to \mathbb{C}$ be a bounded affine homogeneous function. Then there exists an element $x^{**} \in X^{**}$ such that $f(x^*) = x^{**}(x^*)$, $x^* \in B_{X^*}$. If f is moreover continuous, the element x^{**} is from X.*

Proof: Given a function f as in the premise, it is easy to check that a mapping $x^{**}\colon X^* \to \mathbb{C}$ defined as

$$x^{**}(x^*) = \begin{cases} f(x^*), & x^* \in B_{X^*}, \\ f\left(\frac{x^*}{\|x^*\|}\right), & x^* \in X^* \setminus B_{X^*}, \end{cases}$$

is a linear form on X^*. Since it is bounded on B_{X^*}, it is an element of X^{**}.

If f is moreover continuous, x^{**}, as a weak* continuous function on B_{X^*}, is an element of X (see [5, Corollary 3.94]). □

The following class of functions plays an important role in our proof. (We refer the reader to [18] for a more detailed study of descriptive classes of sets on topological spaces.)

Definition 4.8. Let K be a topological space. We call a set $H \subset K$ *resolvable* (or an *H-set*) if for every nonempty $A \subset K$ there exists a relatively open set $U \subset A$ such that $U \subset H$ or $U \subset A \setminus H$. We refer the reader to [18] for more information on resolvable sets.

Let $\Sigma_2(\mathrm{Hs}(K))$ denote the family of all countable unions of resolvable sets in K. If $f\colon K \to L$ is a function with values in a topological space L, f is $\Sigma_2(\mathrm{Hs}(K))$-*measurable* if $f^{-1}(U) \in \Sigma_2(\mathrm{Hs}(K))$ for every open $U \subset L$.

The following lemma collects some important properties of $\Sigma_2(\mathrm{Hs}(K))$-measurable functions which we will use later on.

Lemma 4.9. *Let K be a compact topological space and $f\colon K \to \mathbb{C}$.*

(a) *The function f is $\Sigma_2(\mathrm{Hs}(K))$-measurable if and only if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are $\Sigma_2(\mathrm{Hs}(K))$-measurable.*
(b) *If f is $\Sigma_2(\mathrm{Hs}(K))$-measurable, it is universally measurable.*
(c) *The family of all $\Sigma_2(\mathrm{Hs}(K))$-measurable mappings from K to $\mathbb{C}$ is a complex vector space closed with respect to uniform convergence.*
(d) *If $f\colon K \to \mathbb{R}$ is semicontinuous, it is $\Sigma_2(\mathrm{Hs}(K))$-measurable.*

Proof: (a) If f is $\Sigma_2(\mathrm{Hs}(K))$-measurable, both its real and imaginary part are clearly $\Sigma_2(\mathrm{Hs}(K))$-measurable. Conversely, let its real and imaginary part be

$\Sigma_2(\mathrm{Hs}(K))$-measurable and let $U \subset \mathbb{C}$ be an open set. We can cover U by a countable family of open rectangles, i.e., sets of the form $V_1 \times V_2$, where V_1, V_2 are open sets in $\mathbb{R}$. Since

$$f^{-1}(V_1 \times V_2) = \{x \in K \colon \operatorname{Re} f \in V_1\} \cap \{x \in K \colon \operatorname{Im} f \in V_2\},$$

and resolvable sets form an algebra, f is $\Sigma_2(\mathrm{Hs}(K))$-measurable.

(b) This follows from [14, Proposition A.118].

(c) The assertion follows by a standard argument.

(d) The proof follows from [14, Proposition A.122 and Theorem A.121]. $\square$

Definition 4.10. Let X be a complex Banach space and let $\beta \in \mathbb{T}$. We define an affine homeomorphism σ_β on B_{X^*} as

$$\sigma_\beta \colon x^* \mapsto \beta x^*, \quad x^* \in B_{X^*}.$$

Lemma 4.11. *Let X be a complex Banach space and let $\beta \in \mathbb{T}$. Then the following assertions hold.*

(a) *If $x^* \in B_{X^*}$ and $\mu \in \mathcal{M}^1_{x^*}(B_{X^*})$, then $\sigma_\beta \mu \in \mathcal{M}^1_{\beta x^*}(B_{X^*})$.*

(b) *If $\mu \in \mathcal{M}^1_{\max}(B_{X^*})$, then $\sigma_\beta \mu \in \mathcal{M}^1_{\max}(B_{X^*})$.*

PROOF: (a) If $f \colon B_{X^*} \to \mathbb{C}$ is a continuous affine function, then $f \circ \sigma_\beta$ is also affine and continuous. Thus

$$(\sigma_\beta \mu)(f) = \mu(f \circ \sigma_\beta) = (f \circ \sigma_\beta)(x^*) = f(\beta x^*).$$

(b) Let f be a convex continuous function on B_{X^*} and let f^* denote its *upper envelope* (see [8, p. 191] or [14, Definition 3.17]). We aim to show that

$$(f \circ \sigma_\beta)^* = f^* \circ \sigma_\beta. \tag{4.1}$$

To this end we consider an arbitrary point $x^* \in B_{X^*}$. By [8, Lemma 1, p. 191] (see also [14, Lemma 3.21]), there exists a measure $\nu \in \mathcal{M}^1_{x^*}(B_{X^*})$ such that

$$\nu(f \circ \sigma_\beta) = (f \circ \sigma_\beta)^*(x^*).$$

This, in combination with (a) gives us

$$(f \circ \sigma_\beta)^*(x^*) = \nu(f \circ \sigma_\beta) = (\sigma_\beta \nu)(f) \leq f^*(\beta x^*).$$

To show the other inequality find a positive measure $\nu \in \mathcal{M}^1_{\beta x^*}(B_{X^*})$ such that

$$\nu(f) = f^*(\beta x^*).$$

Using (a) we obtain

$$\begin{aligned} f^*(\beta x^*) = \nu(f) = (\sigma_\beta^{-1} \nu)(f \circ \sigma_\beta) &= (\sigma_{\beta^{-1}} \nu)(f \circ \sigma_\beta) \\ &\leq (f \circ \sigma_\beta)^*(\beta^{-1}(\sigma_\beta(x^*))) = (f \circ \sigma_\beta)^*(x^*) \end{aligned}$$

and (4.1) holds true.

By (4.1) and [8, Theorem 2, p. 193] (see also [14, Theorem 3.58 and Corollary 3.59]), we have the following equalities for the measure μ

$$\sigma_\beta\mu(f^*) = \mu(f^* \circ \sigma_\beta) = \mu((f \circ \sigma_\beta)^*) = \mu(f \circ \sigma_\beta) = \sigma_\beta\mu(f).$$

Again by [8, Theorem 2, p. 193] or [14, Theorem 3.58 and Corollary 3.59] we get that $\sigma_\beta\mu$ is maximal. This finishes the proof. $\square$

Lemma 4.12. *Let X be a complex L_1-predual. The function Tf is homogeneous and strongly affine for any function $f \in \mathcal{C}(B_{X^*}, \mathbb{C})$.*

PROOF: First we show that Tf is affine. Let $x_1^*, x_2^* \in B_{X^*}$ and $\lambda \in [0,1]$ be given. Let $\mu_i \in \mathcal{M}^1_{x_i^*}(B_{X^*}) \cap \mathcal{M}^1_{\max}(B_{X^*})$, $i = 1, 2$. It follows from [8, Theorem 2(4), p. 193] that maximal measures form a cone in $\mathcal{M}(B_{X^*}, \mathbb{R})$. Hence the measure $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ is maximal. It obviously represents the point $\lambda x_1^* + (1-\lambda)x_2^*$. Thus

$$\begin{aligned} Tf(\lambda x_1^* + (1-\lambda)x_2^*) &= (\hom \mu)(f) = \mu(\hom f) = (\lambda\mu_1 + (1-\lambda)\mu_2)(\hom f) \\ &= \lambda\mu_1(\hom f) + (1-\lambda)\mu_2(\hom f) \\ &= \lambda(\hom \mu_1)(f) + (1-\lambda)(\hom \mu_2)(f) \\ &= \lambda Tf(x_1^*) + (1-\lambda)Tf(x_2^*), \end{aligned}$$

and Tf is affine.

Furthermore, Tf is homogeneous. Let $x^* \in B_{X^*}$ and $\beta \in \mathbb{T}$ be given. If $\mu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^1_{\max}(B_{X^*})$, by Lemma 4.11 we have

$$\sigma_\beta\mu \in \mathcal{M}^1_{\beta x^*}(B_{X^*}) \cap \mathcal{M}^1_{\max}(B_{X^*}).$$

Hence

$$\begin{aligned} Tf(\beta x^*) &= (\hom \sigma_\beta\mu)(f) = (\sigma_\beta\mu)(\hom f) = \mu((\hom f) \circ \sigma_\beta) \\ &= \beta\mu(\hom f) = \beta Tf(x^*). \end{aligned}$$

Finally, we show that Tf is strongly affine. To this end it is enough to prove that Tf is $\Sigma_2(\mathrm{Hs}(B_{X^*}))$-measurable. Indeed, assuming that this is the case, both $\mathrm{Re}\, Tf$ and $\mathrm{Im}\, Tf$ are $\Sigma_2(\mathrm{Hs}(B_{X^*}))$-measurable affine functions by Lemma 4.9(a). Thus both $\mathrm{Re}\, Tf$ and $\mathrm{Im}\, Tf$ are fragmented functions by [14, Theorem A.121]. It follows from [14, Theorem 4.21] that both these functions are strongly affine. Thus by Lemma 4.4, Tf is strongly affine.

To show that Tf is $\Sigma_2(\mathrm{Hs}(K))$-measurable it is enough by Lemma 4.9(c) to construct a sequence of $\Sigma_2(\mathrm{Hs}(K))$-measurable functions converging uniformly to Tf. Let $\varepsilon > 0$ be given. We write $\hom f = a + ib$, where a, b are real continuous functions on B_{X^*}. By [1, Proposition I.1.1] (see also [14, Proposition 3.11]), for $\varepsilon > 0$ there exist convex continuous functions a_1, a_2, b_1, b_2 on B_{X^*} such that

$$\|a - (a_1 - a_2)\| + \|b - (b_1 - b_2)\| < \varepsilon.$$

Let $x^* \in B_{X^*}$ be fixed. By [1, Proposition I.3.5] (see also [14, Lemma 3.22]) there exist measures $\mu_j, \nu_j \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^1_{\max}(B_{X^*})$, $j = 1, 2$, such that

$$\mu_1(a_1) = a_1^*(x^*),\ \nu_1(b_1) = b_1^*(x^*),\ \mu_2(a_2) = a_2^*(x^*),\ \nu_2(b_2) = b_2^*(x^*).$$

Furthermore,

$$\begin{aligned}\mu_j(a) + i\mu_j(b) &= \mu_j(a + ib) = \mu_j(\hom f) = (\hom \mu_j)(f)\\ &= Tf(x^*) = (\hom \nu_j)(f) = \nu_j(\hom f)\\ &= \nu_j(a) + i\nu_j(b), \quad j = 1, 2.\end{aligned}$$

This gives us

$$\mu_1(a) = \mu_2(a) = \nu_1(a) = \nu_2(a) \quad \text{and} \quad \mu_1(b) = \mu_2(b) = \nu_1(b) = \nu_2(b). \tag{4.2}$$

Thus we get

$$\begin{aligned}&|Tf(x^*) - [(a_1^*(x^*) - a_2^*(x^*)) + i(b_1^*(x^*) - b_2^*(x^*))]|\\ &\le |\mu_1(a) - (a_1^*(x^*) - a_2^*(x^*))| + |\mu_1(b) - (b_1^*(x^*) - b_2^*(x^*))|.\end{aligned} \tag{4.3}$$

For the first term in (4.3) we have by (4.2) and [8, Lemma 1, p. 191]

$$\begin{aligned}\varepsilon + a_1^*(x^*) - a_2^*(x^*) &\ge \varepsilon + \mu_2(a_1) - \mu_2(a_2) = \varepsilon + \mu_2(a_1 - a_2) \ge \mu_2(a)\\ &= \mu_1(a) \ge \mu_1(a_1 - a_2) - \varepsilon = \mu_1(a_1) - \mu_1(a_2) - \varepsilon\\ &\ge a_1^*(x^*) - a_2^*(x^*) - \varepsilon,\end{aligned}$$

i.e., $|\mu_j(a) - (a_1^*(x^*) - a_2^*(x^*))| \le \varepsilon$, $j = 1, 2$. Since $\mu_1(b) = \nu_j(b)$, $j = 1, 2$, by (4.2), we similarly obtain

$$|\mu_1(b) - (b_1^*(x^*) - b_2^*(x^*))| = |\nu_1(b) - (b_1^*(x^*) - b_2^*(x^*))| \le \varepsilon.$$

Hence $\|Tf - ((a_1^* - a_2^*) + i(b_1^* - b_2^*))\| \le 2\varepsilon$.

Then the functions $a_1^* - a_2^*$ and $b_1^* - b_2^*$, being differences of upper semicontinuous functions, are by virtue of Lemma 4.9(d) and Lemma 4.9(c) $\Sigma_2(\mathrm{Hs}(B_{X^*}))$-measurable. Hence Tf is in the uniform closure of the space of all $\Sigma_2(\mathrm{Hs}(B_{X^*}))$-measurable functions, and thus it is also $\Sigma_2(\mathrm{Hs}(B_{X^*}))$-measurable by Lemma 4.9(c). This concludes the proof. ☐

Lemma 4.13. *Let X be a complex L_1-predual. Let $\nu_1, \nu_2 \in \mathcal{M}(B_{X^*}, \mathbb{C})$ be such that $\nu_1(f) = \nu_2(f)$ for all $f \in \mathcal{C}(B_{X^*}, \mathbb{C})$ with $f = Tf$. Then*

$$\nu_1(Tf) = \nu_2(Tf), \quad f \in \mathcal{C}(B_{X^*}, \mathbb{C}).$$

Proof: Let ν_1, ν_2 be as in the premise. We denote $\mu = \nu_1 - \nu_2$ and decompose it as $\mu = \sum_{k=0}^{3} i^k a_k \mu_k$, where, for $k = 0, \dots, 3$, $\mu_k \in \mathcal{M}^1(B_{X^*})$ and $a_k \ge 0$. Let

$x_k^* \in B_{X^*}$ be the barycenter of μ_k, $k = 0, \dots, 3$. For any $x \in X$ (considered as a function on B_{X^*}) we have $Tx = x$, and thus

$$0 = \mu(x) = \sum_{k=0}^{3} i^k a_k \mu_k(x) = \sum_{k=0}^{3} i^k a_k x(x_k^*) = x\left(\sum_{k=0}^{3} i^k a_k x_k^*\right).$$

Thus $\sum_{k=0}^{3} i^k a_k x_k^* = 0$.

Let $f \in \mathcal{C}(B_{X^*}, \mathbb{C})$ be given. By Lemma 4.12, Tf is strongly affine and homogeneous. Since it is obviously bounded, by Lemma 4.7 there exists $x^{**} \in X^{**}$ such that $x^{**} = Tf$ on B_{X^*}. Thus we obtain

$$\begin{aligned} \mu(Tf) &= \sum_{k=0}^{3} i^k a_k \mu_k(Tf) = \sum_{k=0}^{3} i^k a_k Tf(x_k^*) \\ &= x^{**}\left(\sum_{k=0}^{3} i^k a_k x_k^*\right) = x^{**}(0) = 0. \end{aligned}$$

The proof is finished. □

Proposition 4.14. *Let X be a complex L_1-predual. The mapping $\rho\colon B_{X^*} \mapsto \mathcal{M}(B_{X^*}, \mathbb{C})$ defined as*

$$\rho\colon x^* \mapsto \hom \mu, \quad x^* \in B_{X^*},\ \mu \in \mathcal{M}^1_{x^*}(B_{X^*}) \cap \mathcal{M}^1_{\max}(B_{X^*}),$$

is a complex barycentric mapping and X is isometric to A_ρ.

Proof: We need to verify the properties (a)–(c) from Definition 2.1. The property (b) follows directly from the definitions since the mapping $\mu \mapsto \hom \mu$ does not increase norm. To see that (a) holds, notice that for any $f \in \mathcal{C}(B_{X^*}, \mathbb{C})$ the function $f_\rho = Tf$ is strongly affine by Lemma 4.12, and thus universally measurable. Regarding (c), one only needs to consult Lemma 4.13.

It remains to show that X is isometric to A_ρ. Since

$$A_\rho = \{f \in \mathcal{C}(B_{X^*}, \mathbb{C})\colon Tf = f\} = \{x|_{B_{X^*}} : x \in X\}$$

by Lemma 4.12 and 4.7, the proof is finished. □

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