Mechanical oscillators with dampers defined by implicit constitutive relations

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Abstract. We study the vibrations of lumped parameter systems, the spring being defined by the classical linear constitutive relationship between the spring force and the elongation while the dashpot is described by a general implicit relationship between the damping force and the velocity. We prove global existence of solutions for the governing equations, and discuss conditions that the implicit relation satisfies that are sufficient for the uniqueness of solutions. We also present some counterexamples to the uniqueness when these conditions are not met.

Keywords: lumped parameter systems; differential-algebraic equations; Coulomb's friction; uniqueness of solutions

Classification: 34A09, 70F40, 34K32

1. Introduction

The vibrations of some lumped parameter systems (Figure 1) cannot be described by means of explicit expressions for the forces acting in the constituents of the system in terms of the kinematical variables, the constitutive description being given in terms of an implicit relationship between the force and the kinematics associated with the components. In such a situation, the problem is governed by a system of differential-algebraic equations. Recently, Rajagopal [2] and Nakshatrala et al. [1] articulated the rationale for the prescription of the kinematics in terms of the force rather than vice-versa for constitutively describing the components of the lumped parameter system as such a description is more in consonance with the demands of causality. Fully implicit prescription is thus necessary.

The above studies were not concerned with rigorous mathematical issues such as the existence or uniqueness of solutions to the governing systems. Pražák and Rajagopal [3] established existence of solutions to the governing equation when the elongation of the spring and the velocity in the dashpot are given via monotone, but not necessarily invertible functions of the corresponding forces F_s and F_d.

In this study, we consider the existence and uniqueness of solutions to the governing equations when the spring is given by the traditional linear relationship between the spring force and the elongation, the dashpot being defined through

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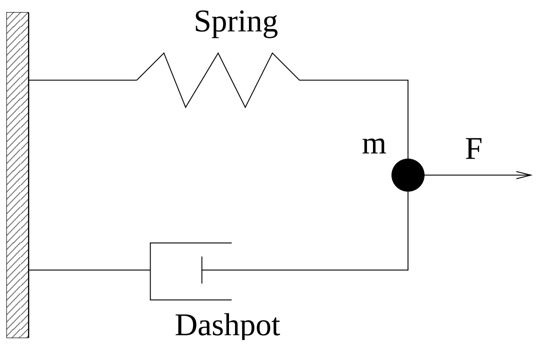

FIGURE 1. Mass-Spring-Dashpot lumped parameter system.

a fully implicit relationship between the force in the dashpot and the velocity. The problem under the study thus assumes a general form

$$m\ddot{x} + F_d + kx = F(t), \tag{1}$$

$$g(\dot{x}, F_d) = 0. \tag{2}$$

Here x is the (unknown) displacement, $m > 0$ is the mass, $k > 0$ is the spring constant, $F(t)$ is the given external force, and F_d is the damping force.

We are particularly interested in the cases where the damping law (2) is fully implicit, that is, it cannot be (globally) written either as $F_d = \hat{g}(\dot{x})$ or as $\dot{x} = \hat{g}(F_d)$. There are numerous physically meaningful examples of such dissipation; the most notable instance being the Coulomb law for friction, cf. Figures 2 and 3 below.

Such problems can usually be solved by an ad-hoc method of patching together solutions for $\dot{x}$ negative or positive, see for example [4]. While the results that are obtained are physically correct, it is still desirable to put the analysis on a firmer mathematical basis, by proving that the problem is well-posed in a suitable, clearly defined sense.

The paper is organized as follows: in Section 2 we specify the precise mathematical assumptions that are made, and collect several auxiliary results, in particular concerning maximal monotone relations. Section 3 contains the main results: we prove global existence of solution and (under slightly stronger conditions) its uniqueness. In Section 4 we provide several examples of non-unique solutions when the implicit constitutive relation does not satisfy the conditions of the general theory in Section 3.

2. Preliminaries

We will henceforth assume that (2) is specialized to

$$F_d = F_c - \gamma(\dot{x}), \tag{3}$$

$$(F_c, \dot{x}) \in \mathcal{A}, \tag{4}$$

where $\gamma(\cdot)$ is a relaxation function and F_c is a Coulomb-like force, bearing in general a monotone relationship with respect to $\dot{x}$. It is assumed that

$$\gamma \text{ is continuous,} \qquad |\gamma(u)| \leq c_1(1 + |u|) \tag{5}$$

and $\mathcal{A}$ is maximal monotone relation, meaning that

$$(F, y),\ (\tilde{F}, \tilde{y}) \in \mathcal{A} \implies (F - \tilde{F})(y - \tilde{y}) \geq 0, \tag{6}$$

$$(F_0 - F)(y_0 - y) \geq 0 \quad \forall (F, y) \in \mathcal{A} \implies (F_0, y_0) \in \mathcal{A}. \tag{7}$$

We will also need some coercivity with respect to the force:

$$(F, y) \in \mathcal{A} \implies Fy \geq c_2 F^2 - c_3. \tag{8}$$

Here and in what follows, c_1, c_2, etc. stand for positive real constants. It is easy to see that (8) (together with maximality of $\mathcal{A}$) implies that $(F, y) \in \mathcal{A}$ can be satisfied for arbitrary values of the second argument.

A typical example of maximal monotone relationship of this type is Coulomb's law for friction, given by the conditions

$$\begin{aligned} \dot{x} > 0 &\implies F_d = \phi_0, \\ \dot{x} < 0 &\implies F_d = -\phi_0, \\ \dot{x} = 0 &\implies F_d \in [-\phi_0, \phi_0] \quad \text{is arbitrary.} \end{aligned} \tag{9}$$

See Figure 2. Note that such a relationship is truly implicit, i.e., one cannot (globally) write either F_d as a function of $\dot{x}$, or $\dot{x}$ as a function of F_d.

Assuming that the relaxation function is bounded, increasing and $\gamma(0) = 0$, we arrive at a more general model (see Figure 3).

Regarding the analysis, we would like to emphasize that the continuity of γ only guarantees existence of solutions; uniqueness requires a stronger assumption, namely that γ is Lipschitz continuous. See also the last section for several examples of non well-posed problems.

Our final assumption is that

$$F(t) \in L^2(0, T). \tag{10}$$

Let us conclude this section with two results that will be instrumental in handling the monotone relationship between the force and the velocity. Firstly, we recall a useful "geometric" characterization of maximal monotone graphs.

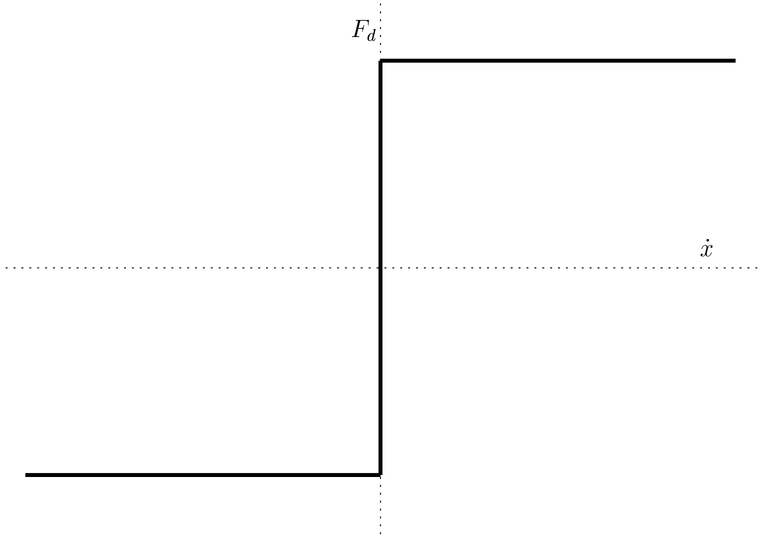

Figure 2. Coulomb friction: force-velocity relationship

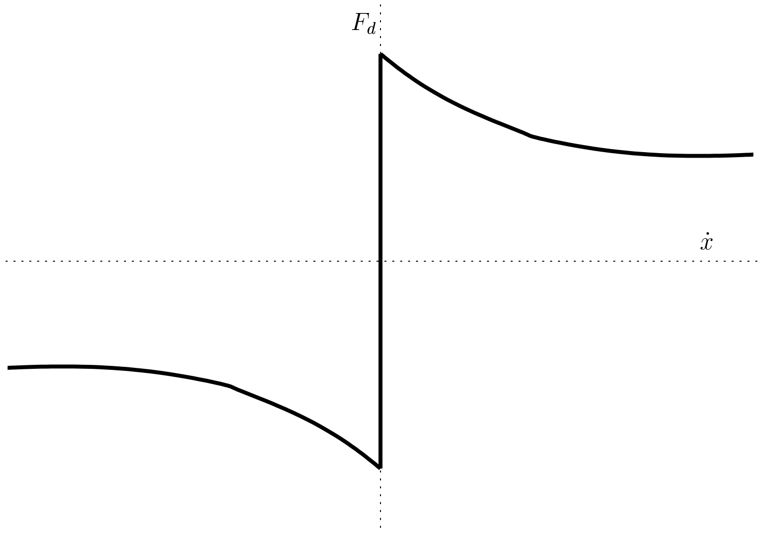

Figure 3. Coulomb friction with relaxation

Lemma 1. *Relation $\mathcal{A} \subset \mathbb{R} \times \mathbb{R}$ is maximal monotone if and only if there exists a 1-Lipschitz function $\varphi : \mathbb{R} \to \mathbb{R}$ such that*

$$(F, y) \in \mathcal{A} \iff F - y = \varphi(F + y). \tag{11}$$

Proof: See [7, Lemma 2.1]. □

The following result on weak closeness is well-known and we only give the proof for the sake of the reader's convenience.

Lemma 2. *Let F_n, y_n converge to F, y weakly in $L^2(0, T)$, and let $F_n y_n \to Fy$ in the sense of distributions. If $(F_n(t), y_n(t)) \in \mathcal{A}$ for a.e. $t \in [0, T]$, where $\mathcal{A}$ is maximal monotone, then also $(F(t), y(t)) \in \mathcal{A}$ for a.e. $t \in [0, T]$.*

Proof: Let w be an arbitrary, smooth function, compactly supported in $(0, T)$. We then have

$$|F_n - y_n - \varphi(w)|^2 \leq |F_n + y_n - w|^2$$

a.e. in $[0, T]$, where φ is provided by Lemma 1. This is equivalent to

$$-2(F_n - y_n)\varphi(w) + |\varphi(w)|^2 \leq -2(F_n + y_n)w + |w|^2 + 4F_n y_n.$$

Here we can take the limit $n \to \infty$, and backward manipulation yields

$$|F - y - \varphi(w)|^2 \leq |F + y - w|^2.$$

By appealing to the density argument, we replace w by an arbitrary square integrable function; in particular, taking $w = F + y$ concludes the proof, in view of (11). □

Theorem 1 (Parameterized Banach theorem)**.** *Let X, Y be Banach spaces, let the mapping $G = G(\phi, y) : X \times Y \to X$ be such that $G(\phi, \cdot)$ is continuous for any $\phi \in X$ fixed and uniformly contracting in ϕ, i.e., there exists $\alpha \in (0, 1)$ such that*

$$\|G(\phi_1, y) - G(\phi_2, y)\|_X \leq \alpha \|\phi_1 - \phi_2\|_X \qquad \forall \phi_1,\, \phi_2 \in X,\, y \in Y. \tag{12}$$

Then for every $y \in Y$ there exists a unique $\phi \in X$ such that $\phi = G(\phi, y)$. Moreover, the mapping $y \mapsto \phi =: g(y)$ is continuous.

Proof: The result is well-known (see e.g. [6, §1, Theorem (3.2)] for a more general statement.) For reader's convenience, we outline the proof below.

The sequence of functions $\phi_n : Y \to X$ is defined by $\phi_{n+1}(y) = G(\phi_n(y), y)$, where $\phi_0(y) \equiv \phi_0 \in X$ is arbitrary. By (12) we obtain inductively

$$\|\phi_n(y) - \phi_{n-1}(y)\|_X \leq \alpha^{n-1} \|G(\phi_0, y) - \phi_0\|_X$$

and hence for $m > n$

$$\|\phi_m(y) - \phi_n(y)\|_X \leq \sum_{j=n+1}^{m} \|\phi_j(y) - \phi_{j-1}(y)\|_X \leq \frac{\alpha^n}{1 - \alpha} \|G(\phi_0, y) - \phi_0\|_X. \tag{13}$$

It follows that $\{\phi_n(y)\}_n$ is a Cauchy sequence and its limit, denoted henceforth by $g(y)$, is the (unique by (12), and hence independent of the initial choice ϕ_0) solution of $\phi = G(\phi, y)$.

By letting $n = 0$ and $m \to \infty$ in (13), we have

$$\|g(y) - \phi_0\|_X \leq \frac{1}{1-\alpha} \|G(\phi_0, y) - \phi_0\|_X. \tag{14}$$

As the choice ϕ_0 can be arbitrary, we set $\phi_0 = g(\tilde{y}) = G(g(\tilde{y}), \tilde{y})$ to finally obtain

$$\|g(y) - g(\tilde{y})\|_X \leq \frac{1}{1-\alpha} \|G(g(\tilde{y}), y) - G(g(\tilde{y}), \tilde{y})\|_X \tag{15}$$

whence follows the continuity of $g(\cdot)$. □

3. Main results

For the sake of definiteness, we start with the definition of the concept of solution.

Definition 1. By solution we understand a couple $(F_d, x) \in L^2(0,T) \times W^{2,2}(0,T)$ such that (1), (2) hold a.e. in $[0,T]$.

Note that it follows that x and $\dot{x}$ have absolutely continuous representatives; in particular, the problem is naturally equipped with initial conditions

$$x(0) = x_0, \qquad \dot{x}(0) = x_1. \tag{16}$$

Theorem 2. *Let the assumptions* (5)–(8) *be in force; moreover let the function* γ *be Lipschitz continuous. Then the problem* (1), (3), (4) *has at most one solution, subject to the initial conditions* (16).

Proof: Let (x^1, F_d^1) and (x^2, F_d^2) be two solutions that satisfy the same initial condition. Setting $z = x^1 - x^2$, we deduce that

$$m\ddot{z} + F_d^1 - F_d^2 + kz = 0.$$

Multiplying by $2\dot{z}$, and noting that monotonicity of $\mathcal{A}$ implies

$$(F_d^1 - F_d^2)(\dot{x}^1 - \dot{x}^2) \geq -(\gamma(\dot{x}^1) - \gamma(\dot{x}^2))(\dot{x}^1 - \dot{x}^2),$$

we arrive at

$$\frac{d}{dt}\big(m(\dot{z})^2 + kz^2\big) \leq 2L(\dot{z})^2,$$

where L is the Lipschitz constant of γ. Since $z(0) = \dot{z}(0) = 0$, we conclude the result by Gronwall's lemma. □

Theorem 3. *Let the assumptions* (5)–(8) *and* (10) *be in force. Then the problem* (1), (3), (4) *has at least one solution, subject to initial conditions* (16).

PROOF: In view of Lemma 1, the relation $(F_c, \dot{x}) \in \mathcal{A}$ is equivalent to

$$F_c = \dot{x} + \varphi(F_c + \dot{x}).$$

Replacing $\dot{x}$ by $\dot{x} - F_c/n$, we obtain, after some manipulations, a sequence of approximating problems

$$F_c = G_n(F_c, \dot{x}) := \tfrac{n}{n+1}\big(\dot{x} + \varphi\big(\dot{x} + \tfrac{n-1}{n}F_c\big)\big)\,. \tag{17}$$

Observe that the function G_n satisfies the assumptions of Theorem 1 (with $\alpha = \frac{n-1}{n+1}$, and F_c, $\dot{x}$ playing the role of ϕ and y, respectively.) It follows that (17) is equivalent to

$$F_c = g_n(\dot{x}),$$

and g_n is a continuous function. We further choose continuous $F_n(t)$ such that $F_n(t) \to F(t)$ in $L^2(0,T)$. The equation

$$m\ddot{x} + g_n(\dot{x}) - \gamma(\dot{x}) + kx = F_n(t) \tag{18}$$

is locally solvable by the standard theory — see [5, Theorem 2.2.1], for example. We need a priori estimates that are independent of n. Now (17) is equivalent to

$$\big(F_c, \dot{x} - \tfrac{1}{n}F_c\big) \in \mathcal{A}.$$

By the coercivity assumption (8),

$$F_c\big(\dot{x} - \tfrac{1}{n}F_c\big) \geq c_2\big|F_c\big|^2 - c_3.$$

For n large enough

$$F_c\dot{x} \geq c_4|F_c|^2 - c_3 \tag{19}$$

with suitable positive c_4. Hence, multiplying (18) by $2\dot{x}$ yields

$$\frac{d}{dt}\big(m(\dot{x})^2 + kx^2\big) + c_5F_c^2 \leq c_6\big((\dot{x})^2 + F_n(t)^2 + 1\big). \tag{20}$$

It follows from Gronwall's lemma that $x(t)$, $\dot{x}(t)$ are bounded on $[0,T]$, and F_c is bounded in $L^2(0,T)$ by a constant only depending on the initial data x_0, x_1 and the norm of $F(t)$ in $L^2(0,T)$. In particular, the local solution can be extended to $[0,T]$; cf. [5, Theorems 2.4.2, 2.4.3].

Let now x_n be the sequence of solutions to (18) and $F_{c,n} := g_n(\dot{x}_n)$. By the above estimates and using also the equation (which gives the bound on $\ddot{x}_n$ in $L^2(0,T)$), we note that the functions x_n and $\dot{x}_n$ are uniformly equicontinuous on $[0,T]$, and so there is a subsequence (not relabelled) such that

$$F_{c,n} \to F_c,\ \ddot{x}_n \to \ddot{x} \qquad \text{weakly in } L^2(0,T), \tag{21}$$

$$\dot{x}_n \to \dot{x},\ x_n \to x \qquad \text{uniformly on } [0,T]. \tag{22}$$

Since $F_{d,n} = F_{c,n} - \gamma(\dot{x}_n) \to F_c - \gamma(\dot{x})$ and $F_n(t) \to F(t)$, the equation (1) is satisfied in the limit. It remains to verify the relationship between F_c and $\dot{x}$. We have for each n

$$\big(F_{c,n}, \dot{x}_n - \tfrac{1}{n} F_{c,n}\big) \in \mathcal{A}\,; \tag{23}$$

however, $\dot{x}_n \to \dot{x}$ strongly in $L^2(0,T)$, and thus $(F_c, \dot{x}) \in \mathcal{A}$ by Lemma 2. □

4. Some examples of nonuniqueness

The aim of this section is to exhibit some examples that show that our problem can lack the property of unique solutions, should one not meet the requirements of the assumptions of Theorem 2. In particular, neither the monotonicity of $\mathcal{A}$ nor the Lipschitz continuity of γ can be disposed of.

Consider, as a first example, a generalized Coulomb law for friction, given by

$$\begin{aligned} \dot{x} > 0 &\implies F_d = \phi_0, \\ \dot{x} < 0 &\implies F_d = -\phi_0, \\ \dot{x} = 0 &\implies F_d \in [-\phi_1, \phi_1] \quad \text{is arbitrary,} \end{aligned} \tag{24}$$

where $0 < \phi_0 < \phi_1$. This relationship between F_d and $\dot{x}$ is not monotone (see Figure 4). It corresponds to a physically meaningful situation, where the static friction is larger than the dynamic friction. However, a possible drop of the friction force at the onset of motion means that the problem is not uniquely solvable.

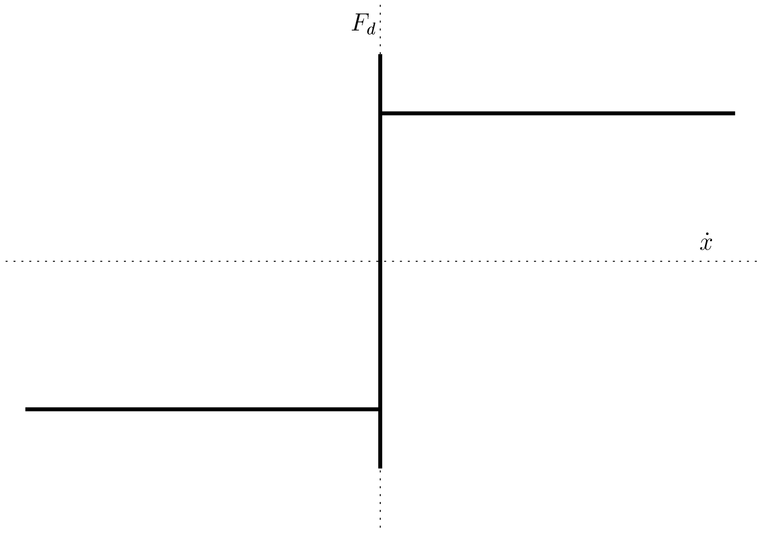

FIGURE 4. Generalized Coulomb law

Consider the data

$$x(0) = \dot{x}(0) = 0, \qquad k = m = 1,\ F(t) \equiv \phi_1. \tag{25}$$

Clearly, $F_d \equiv \phi_1$, $x \equiv 0$ is a solution on an arbitrary interval $[0, T]$. On the other hand, $F_d \equiv \phi_0$ and

$$x(t) = (\phi_1 - \phi_0)(1 - \cos t),$$

is also a solution, subject to the same data, at least on the interval $[0, \pi]$ where $\dot{x} \geq 0$.

As a second — and perhaps a more interesting — example we consider constitutive relations (3), (4), where $\mathcal{A}$ is the standard Coulomb law (9), and for the relaxation function we take

$$\gamma_N(u) = N|u|^{1-\frac{1}{N}} \operatorname{sgn} u$$

with a positive N. Note that γ_N is not Lipschitz, as it grows steeply close to 0 (see Figure 5). However, it can be made α-Hölder continuous with any $\alpha < 1$ by taking N large enough.

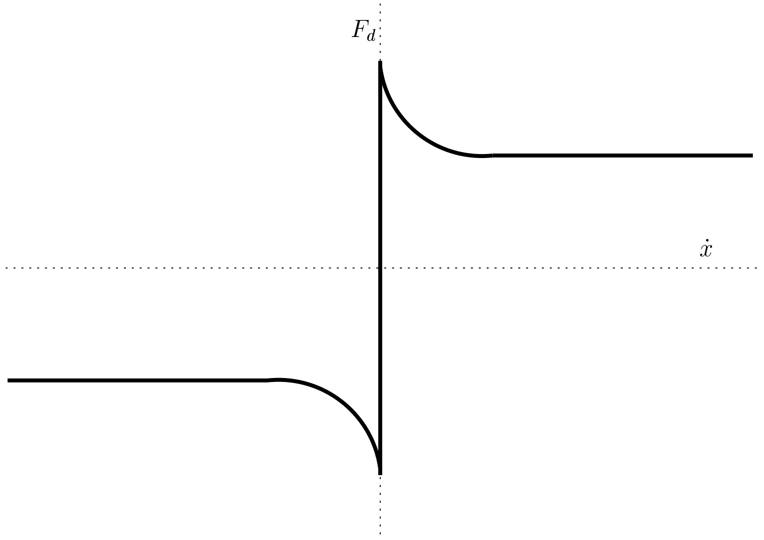

FIGURE 5. Coulomb law with steep relaxation

This lack of Lipschitz continuity gives rise to nonuniqueness of solutions, as we will presently see. Consider the following data:

$$x(0) = \dot{x}(0) = 0, \qquad k = m = 1,\ F(t) \equiv \phi_0. \tag{26}$$

Then the function $F_d = \phi_0 - \gamma_N(\dot{x})$, coupled to any x satisfying

$$\begin{aligned} \ddot{x} + x &= \gamma_N(\dot{x}), \quad \dot{x} \geq 0, \\ x(0) &= \dot{x}(0) = 0, \end{aligned} \tag{27}$$

gives rise to a solution to our problem. Thus, it remains to consider the reduced problem (27). Obviously, $x \equiv 0$ is solution on arbitrary interval $[0, T]$.

We will now construct a second, nontrivial solution by the means of a fixed point argument. Set

$$\mathcal{Y} = \left\{ x \in C^1([0,\delta]);\ x(0) = \dot{x}(0) = 0,\ at^N \leq \dot{x}(t) \leq t^N\ \forall t \in [0,\delta] \right\}.$$

The constants a, $\delta > 0$ will be determined below. We introduce the operator

$$\begin{aligned} \mathcal{T} &: \mathcal{Y} \to C^1([0,\delta]), \\ \big[\mathcal{T}x\big](t) &= y(t),\ t \in [0,\delta] \end{aligned}$$

where y solves

$$\begin{aligned} \ddot{y} &= \gamma_N(\dot{x}) - x, \\ y(0) &= \dot{y}(0) = 0. \end{aligned}$$

Let us show that an appropriate choice of constants a and δ implies $\mathcal{T}(\mathcal{Y}) \subset \mathcal{Y}$. Indeed, for any $x \in \mathcal{Y}$ one has

$$\frac{at^{N+1}}{N+1} \leq x(t) \leq \frac{t^{N+1}}{N+1} \quad t \in [0,\delta],$$

hence

$$\dot{y}(t) = \int_0^t \ddot{y}(s)\, ds = \int_0^t \gamma_N(\dot{x}(s)) - x(s) ds \leq \int_0^t N s^{N-1} ds = t^N.$$

Similarly,

$$\dot{y}(t) \geq \int_0^t N a^{1-\frac{1}{N}} s^{N-1} - \frac{s^{N+1}}{N+1} ds = a^{1-\frac{1}{N}} t^N - \frac{t^{N+2}}{(N+1)(N+2)}, \qquad t \in [0,\delta].$$

So to guarantee that $\dot{y}(t) \geq at^N$ for $t \in [0,\delta]$, it is enough to have

$$a^{-\frac{1}{N}} - \frac{a^{-1}\delta^2}{(N+1)(N+2)} \geq 1,$$

which can be satisfied for any prescribed $a \in (0,1)$ by taking a small enough $\delta > 0$.

Finally, we observe that $\mathcal{T}$ is compact in virtue of a bounded second derivative of y, and the set $\mathcal{Y}$ is convex. By Schauder's theorem [6, §6, Theorem (3.2)], $\mathcal{T}$ has a fixed point $y \in \mathcal{Y}$, which is the desired nontrivial solution to (27).

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